INTRODUCTION

For a semigroup S, we denote by E(S) the set of all idempotents of S, that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

A semigroup S is a <u>semilattice</u> if for all a, b $\in$ S, $a^2 = a$ and ab = ba.

A semigroup S is a <u>left zero semigroup</u> if ab = a for all a, b $\in$ S. A <u>right zero semigroup</u> is defined dually.

A semigroup S with zero 0 is called a <u>zero semigroup</u> if ab = 0 for all a, b $\in$ S.

Let S be a semigroup, and let 1 be a symbol not representing any element of S. The notation $S \cup \{1\}$ denotes the semigroup obtained by extending the binary operation on S to 1 by defining 1.1 = 1 and 1.a = a.1 = a for every a $\in$ S. The notation S^1 denotes the following semigroup :

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Also, the notation S^0 is defined similarly.

Let S be a semigroup. An element a of S is <u>regular</u> if a = axa for some x $\in$ S, and S is called a <u>regular semigroup</u> if every element of S is regular.

In any semigroup S, if a, x $\in$ S such that a = axa, then ax and xa are idempotents of S. Hence, if S is a regular semigroup, then E(S) $\neq \phi$.

Let a be an element of a semigroup S. An element x of S is an <u>inverse</u> of a if a = axa and x = xax. A semigroup S is an <u>inverse semigroup</u> if every element of S has a unique inverse, and the inverse of the element a in S is denoted by a^{-1}. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute with each other [1, Theorem 1.17]. Hence, if S is an inverse semigroup, then E(S) is a semilattice. For any elements a, b of an inverse semigroup S and $e \in E(S)$, we have that

$$(a^{-1})^{-1} = a, \quad (ab)^{-1} = b^{-1}a^{-1} \quad \text{and} \quad e^{-1} = e$$

[1, Lemma 1.18].

Let X be a set. By a <u>one-to-one partial transformation</u> of the set X we mean a one-to-one mapping α of a subset of X onto a subset of X. Let I_X be the set of all one-to-one partial transformations of X. For $\alpha \in I_X$, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain of α and the range of α, respectively. Note that the mapping whose domain is the empty set, is a member of I_X, which is called the <u>empty transformation</u> and will be denoted by 0. The product $\alpha\beta$ of two elements α and β of I_X is defined as follows: If $\nabla\alpha \cap \Delta\beta = \phi$, we define $\alpha\beta = 0$. For $\nabla\alpha \cap \Delta\beta \neq \phi$, let $\alpha\beta : (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \to (\nabla\alpha \cap \Delta\beta)\beta$ be the composite map; it is clear that $\nabla(\alpha\beta) = (\nabla\alpha \cap \Delta\beta)\beta$. Under this operation, I_X becomes an inverse semigroup [1] and we call it the <u>symmetric inverse semigroup</u> on the set X. It is clearly seen that the empty transformation is the zero of I_X and the identity mapping on X is the identity of I_X. Moreover,

$$E(I_X) = \{\alpha \in I_X \mid \alpha \text{ is the identity map on } \Delta\alpha\},$$

and for each $\alpha \in I_X$, the inverse map of α, α^{-1}, is the inverse element of α in I_X and $\Delta(\alpha^{-1}) = \nabla(\alpha)$, $\nabla(\alpha^{-1}) = \Delta(\alpha)$ [1].

Let S be an inverse semigroup. The relation $\leq$ defined on S by

$$a \leq b \text{ if and only if } aa^{-1} = ab^{-1}$$

is a partial order on S [2, Lemma 7.2], and this partial order is called the <u>natural partial order</u> on the inverse semigroup S. Then the restriction of the natural partial order $\leq$ on the inverse semigroup S to E(S) is as follows :

$$e \leq f \text{ if and only if } e = ef \ (= fe).$$

An equivalence relation ρ on a semigroup S is a congruence on S if for all a, b, c $\in$ S, $a\rho b$ implies $ac\rho bc$, $ca\rho cb$, equivalently, for all a, b, c, d $\in$ S, $a\rho b$ and $c\rho d$ imply $ac\rho bd$. If $i = \{(a, b) \mid a \in S\}$ and $\omega = S \times S$, then i and ω are congruences on S and we call them the <u>identity congruence</u> on S and the <u>universal congruence</u> on S, respectively.

If ρ is a congruence on a semigroup S, then the set

$$S/\rho = \{a\rho \mid a \in S\}$$

with the operation defined by $(a\rho)(b\rho) = (ab)\rho$ is a semigroup, and S/ρ under this operation is called the <u>quotient semigroup relative to the congruence</u> ρ.

A nonempty subset A of a semigroup S is an <u>ideal</u> of S if $SA \subseteq A$ and $AS \subseteq A$.

Let A be an ideal of a semigroup S. The relation ρ_A defined on S by $(x, y) \in \rho_A$ if and only if either $x, y \in A$ or $x = y$ is a congruence on the semigroup S, and ρ_A is called the

<u>Rees congruence on S induced by A</u>. The semigroup S/ρ_A is called the <u>Rees quotient semigroup relative to A</u>. The semigroup S/ρ_A is a semigroup with zero and for $x \in S$, $x\rho_A$ is the zero of S/ρ_A if and only if $x \in A$.

Let ρ be a congruence on an inverse semigroup S. Then S/ρ is also an inverse semigroup, and for any $a \in S$,

$$(a\rho)^{-1} = a^{-1}\rho,$$

and hence for $a, b \in S$

$$a\rho b \text{ if and only if } a^{-1}\rho b^{-1}.$$

Moreover, for any $a\rho \in E(S/\rho)$, there exists $e \in E(S)$ such that $a\rho = e\rho$. Hence

$$E(S/\rho) = \{e\rho \mid e \in E(S)\}.$$

Let S be a semigroup. The relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ on S are defined as follow :

$$a\mathcal{L}b \iff S^1a = S^1b,$$

$$a\mathcal{R}b \iff aS^1 = bS^1,$$

and

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

Note that $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ are equivalence relations on S and $\mathcal{H} \subseteq \mathcal{L}$, $\mathcal{H} \subseteq \mathcal{R}$. These relations are called <u>Green's relations</u> on S. For $a \in S$, L_a denotes the $\mathcal{L}$-class of S containing a; and R_a and H_a are defined similarly.

A subset G of a semigroup S is a <u>subgroup</u> of S if under the operation of S, G is a group.

Let S be a semigroup and e be an idempotent of S. Then H_e is the maximum subgroup of S having e as its identity.

Let S be a semigroup with identity 1. An element a of S is called a <u>unit</u> of S if there exists $b \in S$ such that $ab = ba = 1$. The set of units of S forms the maximum subgroup of S having 1 as its identity and we call it the <u>group of units</u> or the <u>unit group of</u> S. Hence, H_1 is the group of units of S and $H_1 = \{a \in S \mid aa' = a'a = 1$ for some $a' \in S\}$.

An ideal A of a semigroup S is said to be <u>completely prime</u> if for all $a, b \in S$, $ab \in A$ implies $a \in A$ or $b \in A$.

An ideal M of a semigroup S is called a <u>maximal ideal</u> of S if there is no ideal lies properly between M and S and $M \neq S$.

A subsemigroup T of a semigroup S is called a <u>filter</u> of S if for all $a, b \in S$, $ab \in T$ implies $a, b \in T$.

Let T be a nonempty subset of a semigroup S. Then T is a filter of S if and only if $S \setminus T$ is either an empty set or $S \setminus T$ is a completely prime ideal of S.

A semigroup S is called <u>simple</u> if S is the only ideal of S.

A semigroup S with zero 0 is called a <u>0-simple semigroup</u> if $S^2 \neq \{0\}$, and $\{0\}$ and S are the only ideals of S.

Let S be a semigroup with zero 0. If S has an identity which is different from the zero of S, then S is 0-simple if and only if $\{0\}$ and S are the only ideals of S.

A semigroup S is called a <u>congruence-free semigroup</u> if the identity congruence and the universal congruence are the only congruences on S.

Let ρ be a congruence on a semigroup S. Then ρ is said to be a congruence-free congruence if S/ρ is a congruence-free semigroup.

For any set X, let the notation $|X|$ denote the cardinality of the set X.

General properties of congruence-free congruences on semigroups are introduced in the first chapter. Including in this chapter, some remarks on congruence-free Rees congruences are also given.

The characterizations of all congruence-free congruences on the semigroup of integers under multiplication, the semigroup of nonnegative integers under addition and the semigroup of nonnegative real numbers which are less than or equal to 1 under multiplication are studied in the second chapter.

Trotter has characterized congruence-free inverse semigroups with and without zero. In the last chapter, we use Trotter's work of characterizing congruence-free inverse semigroup with zero to determine all congruence-free congruences on any symmetric inverse semigroup on countable set. It is proved that the symmetric inverse semigroup on a countably nonempty set has exactly one nonuniversal congruence-free congruence, and the explicit form of such congruence is also given.