CHAPTER IV

UNARY ALGEBRAS

Many theorems of Chapter II and III involved the construction of unary algebras. In this chapter, we will show some important properties of these kind of algebras.

The first theorem shows that for every algebra $< A; F >$, there exists a unary algebra such that every congruence relation of $< A; F>$ is a congruence relation of the corresponding unary algebra, and conversely.

4.1 <u>Theorem</u>. To every algebra $< A; F >$ there corresponds a unary algebra $< A'; F' >$ such that θ is a congruence relation of $< A; F >$ if and only if it is a congruence relation of $< A'; F' >$.

<u>Proof</u>. Let $< A; F >$ be an algebra of type τ. Construct a unary algebra as follows : Set $A' = A$. For each γ, $0 \leq \gamma < O(\tau)$, let $a_1, \ldots, a_{n_\gamma -1}$ be any $n_\gamma -1$ elements of A, and define unary operations on A by

$$\mathcal{U}^{(i)}_{a_1, \ldots, a_{n_\gamma -1}}(a) = f_\gamma(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_{n_\gamma -1})$$

for all $a \in A$, $f_\gamma \in F$, $i = 1, 2, \ldots, n_\gamma$. Set

$$F' = \{\mathcal{U}^{(i)}_{a_1, \ldots, a_{n_\gamma -1}} \mid a_1, \ldots, a_{n_\gamma -1} \in A,\ i=1, \ldots, n_\gamma,\ 0 \leq \gamma < O(\tau)\}.$$

Consider the unary algebra $< A'; F' >$.

Assume that θ is a congruence relation on $< A; F >$. To show that it is a congruence relation on $< A'; F' >$, let $a_i \equiv b_i\ (\theta)$ for $a_i, b_i \in A$, $0 \leq i < n_\gamma$, $0 \leq \gamma < 0(\tau)$. Let $c_1, \ldots, c_{n_\gamma - 1} \in A$. Then $c_k \equiv c_k\ (\theta)$ for all $0 < k \leq n_\gamma - 1$. Therefore, for any i, $0 < i \leq n_\gamma$,

$$\begin{aligned} \mathcal{U}^{(i)}_{c_1, \ldots, c_{n_\gamma - 1}}(a_i) &= f_\gamma(c_1, \ldots, c_{i-1}, a_i, c_{i+1}, \ldots, c_{n_\gamma - 1}) \\ &\equiv f_\gamma(c_1, \ldots, c_{i-1}, b_i, c_{i+1}, \ldots, c_{n_\gamma - 1}) \\ &= \mathcal{U}^{(i)}_{c_1, \ldots, c_{n_\gamma - 1}}(b_i) \qquad (\theta) . \end{aligned}$$

Hence, θ is a congruence relation on $< A'; F' >$.

Conversely, assume that θ be a congruence relation on $< A'; F' >$. Let $a_i \equiv b_i\ (\theta)$ for $a_i, b_i \in A'$, $0 < \gamma < 0(\tau)$, $0 \leq i < n_\gamma$. Then

$$\begin{aligned} f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) &= \mathcal{U}^{(1)}_{a_1, \ldots, a_{n_\gamma - 1}}(a_0) \\ &\equiv \mathcal{U}^{(1)}_{a_1, \ldots, a_{n_\gamma - 1}}(b_0) \\ &= f_\gamma(b_0, a_1, \ldots, a_{n_\gamma - 1}) \qquad (\theta). \end{aligned}$$

Suppose that $f_\gamma(a_0, \ldots, a_{n_\gamma - 1}) \equiv f_\gamma(b_0, b_1, \ldots, b_i, a_{i+1}, \ldots, a_{n_\gamma - 1})\ (\theta)$. Therefore,

$$\begin{aligned} & f_\gamma(b_0, b_1, \ldots, b_i, a_{i+1}, \ldots, a_{n_\gamma - 1}) \\ &= \mathcal{U}^{(i+1)}_{b_0, b_1, \ldots, b_i, a_{i+2}, \ldots, a_{n_\gamma - 1}}(a_{i+1}) \\ &\equiv \mathcal{U}^{(i+1)}_{b_0, b_1, \ldots, b_i, a_{i+2}, \ldots, a_{n_\gamma - 1}}(b_{i+1}) \\ &= f_\gamma(b_0, \ldots, b_{i+1}, a_{i+2}, \ldots, a_{n_\gamma - 1})\ (\theta) . \end{aligned}$$

By transitivity of θ,

$$f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \equiv f_\gamma(b_0, \ldots, b_{i+1}, a_{i+2}, \ldots, a_{n_\gamma-1}) \; (\theta)$$

Hence, by induction of finite steps, we can have

$$f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \equiv f_\gamma(b_0, \ldots, b_{n_\gamma-1}) \quad (\theta),$$

so that θ is a congruence relation on $< A; F >$. This completes the proof of the theorem. #

The next proposition shows that a simple unary idempotent algebra can have only one idempotent. And, any 2-idempotent unary entire algebra has at least 4 congruence relations.

4.2 Proposition. Let $< A; F >$ be a unary algebra such that $|A| > 2$. Then the followings hold :

(1) If $< A; F >$ is simple, then it has at most one idempotent element.

(2) If $< A; F >$ is 2-idempotent entire algebra, then it has at least 4 congruence relations.

Proof. (1) Assume that $< A; F >$ is a simple unary algebra. Suppose that $< A; F >$ has more than one idempotent. Let P_1, P_2 be idempotents of $< A; F >$. Then $f_\gamma(P_i) = P_i$ for all $f_\gamma \in F$, $i = 1, 2$. Define a binary θ on A by

$$x \equiv y \; (\theta) \longleftrightarrow x, y \in \{P_1, P_2\} \text{ or } x = y$$

for all $x, y \in A$. Then θ is a congruence relation on A such that $\theta \neq \iota$ and $\theta \neq \omega$. It is a contradiction.

To prove (2), assume that $< A; F >$ has two idempotents, say P_1, P_2, and P_1 is zero. Since $< A; F >$ is an entire algebra, $\bar{\theta}$ is a congruence relation on A. Define a binary relation θ' on A by

$$x \equiv y \quad (\theta') \longleftrightarrow x, y \in \{P_1, P_2\} \quad \text{or} \quad x = y$$

for all $x, y \in A$. Obviously, θ' is a congruence relation on A. Hence, $< A; F >$ has at least 4 congruence relations, completing the proof. #

Thus the algebra in any proof of the Theorem 3.13, cannot be a unary algebra.

The next theorem gives the reason why an algebra in the Theorem 3.7 [2] cannot be constructed to be a unary algebra.

4.3 <u>Theorem</u>. Let $< A; F >$ be a simple unary algebra and let $< G; \cdot >$ be the automorphism group of $< A; F >$. Then $< G; \cdot >$ is a cyclic group of prime order P or $< G; \cdot >$ is a group consisting of the identity alone.

<u>Proof</u>. If $|A| = 1$, then the statement is trivial. We may assume that $|A| > 1$.

Let $< G; \cdot >$ be the automorphism group of $< A; F >$ such that $G \neq \{\varepsilon\}$, where ε denotes the identity of G. For each $\alpha \in G$, $\alpha \neq \varepsilon$, define a binary relation on A by the rule :

$$x \equiv y \quad (\theta_\alpha) \longleftrightarrow x\alpha^n = y \quad \text{for some } n \in \mathbf{Z},$$

for all $x, y \in A$, where $\mathbf{Z}$ denotes the set of all integers. Then θ_α

is an equivalence relation. Since every $f_\gamma \in F$ is unary, θ_α is also a congruence relation on A.

Let H be a nontrivial subgroup of G. Then there is a β in H such that $\beta \neq \varepsilon$. Let G_1 denote a subgroup of H generated by β. Since $\beta \neq \varepsilon$, we have $\theta_\beta \neq \omega$. Using that $< A; F >$ is simple, we conclude that $\theta_\beta = \iota$.

Let us note that for any γ in G, $a \in A$, $a = a\gamma$ implies $\gamma = \varepsilon$. Indeed, if $a = a\gamma$ then $a = a\gamma^n$ for all integers n, thus for any $b \in A$, $b \equiv a\ (\theta_\gamma)$; i.e. $b = a\gamma^m$ for some m Z; thus a=b this implies that $\theta_\gamma = \omega$. Therefore, $\gamma = \varepsilon$.

Now, let $\gamma \in G$ and $x \in A$. Since $\theta_\beta = \iota$, $x \equiv x\gamma\ (\theta_\beta)$ so that $x\gamma = x\beta^n$ for some $n \in \mathbb{Z}$. Since $\gamma \in G$, we have $x = (x\beta^n)\gamma^{-1} = x(\beta^n\gamma^{-1})$. Hence, by note, $\beta\gamma^{-1} = \varepsilon$. This implies that $\gamma = \beta^n \in G_1$; i.e. G = H. Therefore, G has no nontrivial proper subgroup. Hence, G is a group of prime order P, completing the proof of the theorem. #

The following is an example of a semigroup which is not isomorphic with the endomorphism semigroup of any simple unary algebra.

Example : Let $< S; \cdot > = < \mathbb{Z}/4; + >$. Then $< S; \cdot >$ is a group of order 4. Let $< A; F >$ be a simple unary algebra such that $S \cong E(A; F)$. Then E(A; F) is a group; i.e. E(A; F) = Aut (A; F). Therefore $|E(A; F)| = P$ for some prime P. Thus E(A; F) is not isomorphic to S which is a contradiction.

4.4 Definition. Let < A; F > and < A'; F' > be unary algebras belonging to the same similarity class $K(\tau)$. A mapping $\psi : A \longrightarrow A'$ such that for each $0 < \gamma < 0(\tau)$,

$$(f_\gamma(a))\psi = f'_\gamma(a\psi)$$

for all $a \in A$, is called a unary algebra-homomorphism of < A; F > into < A'; F' >.

Let $\mathcal{U}$ be a category of unary algebras whose objects are unary algebras belonging to the same similarity class $K(\tau)$ and whose morphisms are unary algebra-homomorphisms.

The following theorem shows that a monoid, as a small category, can be fully embedded into the category of unary algebras.

4.5 Theorem. A monoid M can be fully embedded into a category of unary algebras of type $\tau = |M|$.

Proof. Let $\mathcal{U}$ be a category of unary algebras all of type $\tau = |M|$. Consider a monoid M as a small category such that $a \circ b = ab$. We construct a unary algebra < A; F > as in Theorem 2.1, so that $E(A; F) \cong M$.

Let ψ be an isomorphism of M onto E(A; F). Define $\theta : M \rightarrow \mathcal{U}$ by

$$\theta(M) = \langle A; F \rangle ,$$

and

$$\theta(a) = a\psi$$

for all $a \in M$. Obviously, θ is full, faithful functor and one-one on object. #