CHAPTER II

ENDOMORPHISMS OF SEMIGROUPS

George Grätzer [1] has shown that for every semigroup S with identity, there exists an algebra such that the endomorphism semigroup of the algebra is isomorphic to the semigroup S.

In this chapter, we will show that we can choose the algebra to be a semigroup.

2.1 <u>Theorem</u>. A semigroup $< S; \cdot >$ is isomorphic to the endomorphism semigroup of some algebra $< A; F >$ if and only if $< S; \cdot >$ has an identity element.

<u>Proof</u>. See [[1], Theorem 3], page 68.

The next theorem shows that if we consider a monoid M to be a category then M can be fully-embedded into the functor category Ens^M.

2.2 <u>Lemma</u>. A monoid M can be seen as a small category whose only object is M and whose morphisms are elements of M.

<u>Proof</u>. Define the composition on M by

$$a \circ b \quad = \quad ab$$

for all $a, b \in M$. Obviously, CAT 1, 2, 3 are satisfied. #

2.3 <u>Theorem</u>. A monoid M can be fully embedded into the functor category Ens^M, where Ens is the category of sets.

Proof. Consider a monoid M as a small category such that for any a, b $\in$ M, a$\circ$b = ab. Construct a functor F of M into Ens as follows :

$$F(M) = [M, M] = M,$$

and for a $\in$ M, F(a) $\in$ [F(M), F(M)], denoted by F_a, is defined by

$$F_a(x) = ax \qquad (x \in M).$$

Then, by the example (2) on page 6. F is a functor of M into Ens, (i.e. for any a, b, x $\in$ M, $F_e(a) = ea = a = 1_{F(M)}(a)$ and $F_{ab}(x) = (ab)x = a(bx) = F_a(bx) = (F_a \circ F_b)(x)$, and so $F_{ab} = F_a \circ F_b$); and by Definition 1.10, F_a is an inner left translation induced by a.

Next, we want to construct a natural transformation of F into F as follows : For each a $\in$ M, let $\eta^a : F \to F$ such that $\eta^a_M : M \to M$ be defined by

$$x\eta^a_M = xa \qquad (x \in M).$$

Then by Definition 1.10, η^a_M is an inner right translation induced by a. Since M is globally idempotent, F_a and η^b_M are permutable for all a, b in M (i.e. $(F_a(x))\eta^b_M = F_a(x\eta^b_M)$ for all a, b, x $\in$ M). Hence, η^a is a natural transformation of F into F.

Define $\theta : M \longrightarrow Ens^M$ by

$$\theta(M) = F,$$

and

$$\theta(x) = \eta^x \qquad (x \in M).$$

(i) θ is a functor of M into Ens^M.

Indeed, $x\eta^e_M = xe = x$ for all x $\in$ M. Hence, η^e_M is the identity mapping on M. Since $(1_F)_M = 1_{F(M)} = 1_M = \eta^e_M$, we have

$\theta(1_M) = \theta(e) = \eta^e = 1_F = 1_{\theta(M)}$. Let a, b, $x \in M$. Then $x\eta_M^{ab} = x(ab)$ $= (xa)b = (x\eta_M^a)\eta_M^b = x(\eta_M^a o \eta_M^b)$. Hence, $\theta(a \circ b) = \theta(ab) = \eta^{ab} = \eta^a o \eta^b$ $= \theta(a) o \theta(b)$.

(ii) θ is full.

Since for any natural transformation η of F into F and $x \in M$, $x\eta_M$ $= (xe)\eta_M = (F_x(e))\eta_M = F_x(e\eta_M) = x(e\eta_M) = x\eta_M^a$, where $a = e\eta_M$, we have $\eta = \eta^a$.

(iii) θ is faithful.

Indeed, for any a, $b \in M$ such that $\eta^a = \eta^b$, we can get $a = ea = e\eta_M^a$ $= e\eta_M^b = eb = b$. Hence, $\eta^a = \eta^b$ is equivalent to $a = b$.

(iv) Clearly, θ is one - one on object, completing the proof of the theorem. #

2.4 <u>Theorem</u> [5]. Let A be a small category. Then Ens^A admits a full embedding into the category of semigroups.

Now, we will show that for a given monoid M, there exists a semigroup $< S; \cdot >$ such that $M \cong E(S; \cdot)$. Before we prove this theorem, we have the following theorem.

2.5 <u>Theorem</u>. Every monoid can be fully embedded into the category of semigroups.

<u>Proof</u>. Let M be a monoid. Then by Theorem 2.3, let γ be a full embedding functor of M into the functor category Ens^M. And, as M is a small category, then by Theorem 2.4, let δ be a full embedding functor of Ens^M into the category of semigroups S. So, we have

the following diagram

$$M \xrightarrow{\gamma} Ens^M \xrightarrow{\delta} S .$$

Define $\theta : M \longrightarrow S$ by

$$\theta(M) = \delta o \gamma(M) ,$$

and

$$\theta(x) = \delta o \gamma(x)$$

for all $x \in M$. Since the composition of functors is a functor and the composition of onto functions is an onto function, we have θ is a full, faithful functor. #

2.6 Theorem. A semigroup < S; • > is isomorphic to the endomorphism semigroup of some semigroup < S'; * > if and only if < S; • > has an identity element.

Proof. The 'only if' part follows from [1]. To prove the 'if' part, assume that < S; • > has an identity element. Therefore S is a monoid which can be considered as a small category.

Let θ be a full embedding functor of S into the category of semigroups $\mathcal{S}$. Then the function $\theta : S \longrightarrow [\theta(S), \theta(S)]$ is 1 - 1, onto and for any $a, b \in S$, $\theta(ab) = \theta(a) \bullet \theta(b)$. Hence, S is isomorphic to $[\theta(S), \theta(S)]$ which is E(S'; *), where $\theta(S) = S'$. #

The following remarks state some results which explain why the above theorem was not phrased more strongly. (i.e. why "semigroup < S'; * >" cannot be replaced by 'monoid', 'finite semigroup', 'group', 'lattice' and 'commutative semigroup'.)

2.7 Remarks.

(1) A nontrivial group is not isomorphic with E(M) for any monoid M.

Proof. Let M be a monoid with an identity e, and let $< G; \cdot>$ be a nontrivial group. Suppose that $E(M) \cong G$. Let the mapping $f : M \longrightarrow M$ be defined by

$$f(x) = e \qquad (x \in M).$$

Then f is a constant map which is an endomorphism of M. Hence, E(M) is not a group which contradicts to $E(M) \cong G$. #

(2) A nontrivial group is not isomorphic to the endomorphism semigroup of any group, or finite semigroup, or lattice.

Proof. Indeed, a group is a monoid, every finite semigroup has an idempotent and every element of lattice is idempotent. Then similarly to the proof of (1), we get the results. #

(3) The cyclic group of order 2 is not isomorphic with the endomorphism semigroup of any commutative semigroup.

Proof. Assume that S is a commutative semigroup such that $E(S; \cdot) \cong C_2$, where C_2 denotes a cyclic group of order 2. Then S must have at least two elements. The mapping $f : a \longrightarrow a^2$ is an endomorphism of S, and therefore either $f = 1_S$ or $f^2 = 1_S$ and $f \neq 1_S$, where 1_S is the identity automorphism of S. In the first case, every element of S is idempotent, hence all constant mappings are endomorphisms. This contradicts the assumption that $E(S; \cdot)$ is a group. In the second case, $a^4 = f(f(a)) = a$, for all $a \in S$. But $a^3 \cdot a^3 = a^4 \cdot a^2 = a^3$, hence a^3 is idempotent. Therefore the constant mapping with value a^3 is an endomorphism which is not an automorphism. #