CHAPTER V

CONTINUOUS SOLUTION OF $f(xy)+f(xy^{-1}) = 2f(x)f(y\theta)$ ON A TOPOLOGICAL GROUP

5.1 Introduction

In [3], T.M. Flett found all continuous complex-valued functionson $\mathbb{R}^2$ satisfying

$$f(z_1+ z_2) + f(z_1- z_2) = 2f(z_1)f(z_2) ,$$

to be $f \equiv 0$ or of the form

$$f(z) = f(x + iy) = \cosh (\alpha x+\beta y) = \frac{e^{(\alpha x+\beta y)} + e^{-(\alpha x + \beta y)}}{2} ,$$

where α, β are some complex numbers.

In this chapter, we characterize the continuous solution of

(S) $$f(xy) + f(x y^{-1}) = 2f(x)f(y\theta) ,$$

where f is a function from a topological group G into $\mathbb{C}$. This result is applied to the case $G = \mathbb{R}^n$ to obtain all continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that

$$f(x + y) + f(x - y) = 2f(x)f(y + \theta).$$

This includes the equation studied in [3] as a special case.

5.2 Continuous Solutions of $f(x y) + f(x y^{-1}) = 2f(x)f(y\theta)$ on a Topological Group.

5.2.1 Theorem Let G be a topological group. Any continuous function not identically zero on G satisfies

(S) $$f(xy) + f(xy^{-1}) = 2f(x)f(y\theta)$$

and

(A) $$f(xyz) = f(xzy)$$

for every x, y, z in G, if and only if there exists a continuous homomorphism h from G into $\mathbb{C}^*$ such that

$$f(x) = \frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2} .$$

<u>Proof</u> Assume that $f : G \longrightarrow \mathbb{C}$ is continuous and satisfies (S) and (A).

By theorem 4.3.1, we have

$$f(x) = \frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2} ,$$

for all x in G, where h is a homomorphism from G to $\mathbb{C}^*$.

It remains to be proved that h is continuous.

Since $f(x) = \dfrac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2}$, for every x in G.

Then $f(\theta x) = \dfrac{h(x) + h(x^{-1})}{2}$, for every x in G.

Thus
$$f(\theta xy) = \frac{h(xy) + h(y^{-1}x^{-1})}{2} ,$$

$$= \frac{h(x)h(y) + h(y^{-1})h(x^{-1})}{2} \qquad (5.2.1.1)$$

and
$$f(\theta x)f(\theta y) = \frac{h(x) + h(x^{-1})}{2} \cdot \frac{h(y) + h(y^{-1})}{2} ,$$

$$= \frac{1}{4}\left[h(x)h(y)+h(x^{-1})h(y)+h(x)h(y^{-1})+h(x^{-1})h(y^{-1})\right] \qquad (5.2.1.2)$$

Substracting (5.2.1.2) from (5.2.1.1), we get

$$f(\theta xy)-f(\theta x)f(\theta y) = \frac{1}{4}\left[h(x)h(y)+h(x^{-1})h(y^{-1})-h(x^{-1})h(y)-h(x)h(y^{-1})\right],$$

$$= \frac{h(x) - h(x^{-1})}{2} \cdot \frac{h(y) - h(y^{-1})}{2} .$$

But $$f(\theta x) = \frac{h(x) + h(x^{-1})}{2},$$

and hence $$h(x) - f(\theta x) = \frac{h(x) - h(x^{-1})}{2} .$$

Thus $$f(\theta xy) - f(\theta x)f(\theta y) = \left[h(x)-f(\theta x)\right]\left[h(y) - f(\theta y)\right] \quad (5.2.1.3)$$

Case I Suppose $h(x) = f(\theta x)$, for all x in G.

Then clearly, h is continuous .

Case II Otherwise, there is an x_o in G such that

$$\delta = h(x_o) - f(\theta x_o) \neq 0.$$

With x_o for y in (5.2.1.3), we get

$$f(\theta xx_o) - f(\theta x)f(\theta x_o) = \left[h(x) - f(\theta x)\right]\left[h(x_o) - f(\theta x_o)\right],$$

$$h(x) = f(\theta x) + \frac{1}{\delta}\left[f(\theta xx_o)-f(\theta x)f(\theta x_o)\right]. \quad (5.2.1.4)$$

Hence h is continuous.

Conversely, if $h : G \longrightarrow \mathbb{C}^*$ is a continuous homomorphism from G to $\mathbb{C}^*$, then it is clear that

$$f(x) = \frac{h(\theta^{-1} x) + h(x^{-1}\theta)}{2},$$

is continuous. Furthermore, it follows from theorem 4.3.1 that f satisfies (S) and (A).

5.2.2 Corollary Let G be a commutative topological group. Then $f : G \longrightarrow \mathbb{C}$ is continuous and satisfies

(S) $$f(x\,y) + f(xy^{-1}) = 2f(x)f(y\theta)$$

if and only if there exists a continuous homomorphism h from G into the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers such that

$$f(x) = \frac{h(\theta^{-1}x) + h(x^{-1}\theta)}{2} .$$

Proof Since G is a commutative group, hence the condition

(A) $$f(x\,yz) = f(x\,zy) ,$$

holds for every x , y, z in G and for all functions $f : G \longrightarrow \mathbb{C}$. Hence the class of all continuous functions that satisfies (S) coincides with the class of all continuous functions that satisfies (S) and (A).

5.3 Continuous Solution of $f(x\,y) + f(x\,y^{-1}) = 2f(x)f(y\theta)$ on $\mathbb{R}^n$

The following theorem gives all continuous solutions of

(S_1) $$f(x + y) + f(x - y) = 2f(x)f(y + \theta)$$

on $\mathbb{R}^n$.

5.3.1 Theorem Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ be continuous and not identically zero on $\mathbb{R}^n$. Then f satisfies (S_1) if and only if there exists $r_1, \ldots, r_n \in \mathbb{C}$ such that

$$f(x_1,\ldots,x_n) = \frac{e^{r_1(x_1-\theta_1)+\ldots+r_n(x_n-\theta_n)} + e^{r_1(\theta_1-x_1)+\ldots+r_n(\theta_n-x_n)}}{2}.$$

Proof By corollary 5.2.2, any continuous function f satisfies (S_1) if and only if f is of the form

$$f(x) = \frac{h(x-\theta) + h(\theta - x)}{2},$$

where $x = (x_1,\ldots,x_n)$ and h is a continuous homomorphism from $\mathbb{R}^n$ to $\mathbb{C}^*$.

By theorem 3.3.5, we have

$$h(x) = e^{r_1x_1+\ldots+r_nx_n},$$

where $x = (x_1,\ldots,x_n)$ and $r_j \in \mathbb{C}$, $j = 1,\ldots,n$.

Thus every continuous solution of (S_1) on $\mathbb{R}^n$ must be of the form

$$f(x_1,\ldots,x_n) = \frac{e^{r_1(x_1-\theta_1)+\ldots+r_n(x_n-\theta_n)} + e^{r_1(\theta_1-x_1)+\ldots+r_n(\theta_n-x_n)}}{2}.$$

5.3.2 Corollary Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous and not identically zero on $\mathbb{R}^n$. Then f satisfies (S_1) if and only if f is of the form

$$(5.3.2.1)\quad f(x_1,\ldots,x_n) = \frac{1}{2}\Big[e^{r_1(x_1-\theta_1)+\ldots+r_n(x_n-\theta_n)} + e^{r_1(\theta_1-x_1)+\ldots+r_n(\theta_n-x_n)}\Big],$$

where all $r_1,\ldots,r_n$ are real numbers or all $r_1,\ldots,r_n$ are pure imaginary numbers.

Proof Assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous and satisfies (S_1). Since $\mathbb{R} \subseteq \mathbb{C}$. We may consider f to be a function on $\mathbb{R}^n$ into $\mathbb{C}$.

By Theorem 5.3.1 , f must be of the form

$$f(x_1,\ldots,x_n) = \frac{1}{2}\Big[e^{r_1(x_1-\theta_1)+\ldots+r_n(x_n-\theta_n)} + e^{r_1(\theta_1-x_1)+\ldots+r_n(\theta_n-x_n)} \Big] ,$$

where $r_1, \ldots, r_n \in \mathbb{C}$.

Since f maps $\mathbb{R}^n$ into $\mathbb{R}$, hence the imaginary part of $f(x_1,\ldots,x_n)$ must be zero for all $x_1,\ldots, x_n$, i.e. we have

$$(5.3.2.2)\quad \operatorname{Im}\left(\frac{e^{r_1(x_1-\theta_1)+\ldots+r_n(x_n-\theta_n)} + e^{r_1(\theta_1-x_1)+\ldots+r_n(\theta_n-x_n)}}{2}\right) = 0,$$

for all $(x_1,\ldots,x_n) \in \mathbb{R}^n$.

For convenience, for $j = 1,\ldots, n$, let

$$r_j = r_j' + i r_j'' ,$$

where $i^2 = -1$, r_j' , r_j'' are real numbers, and we denote $x_j - \theta_j$ by y_j.

With this notation, it follows from (5.3.2.2) that

$$(5.3.2.3)\quad \Big[e^{r_1' y_1+\ldots+r_n' y_n} - e^{-(r_1' y_1+\ldots+r_n' y_n)}\Big]\sin(r_1'' y_1+\ldots+r_n'' y_n) = 0,$$

for all $x_1,\ldots, x_n$.

If $r_{j_o}'' \neq 0$ for some j_o, we can choose $x_1,\ldots,x_n$ so that

$$\sin(r_1'' y_1+\ldots+r_n'' y_n) \neq 0 .$$

Therefore, (5.3.2.3) implies

$$e^{r_1' y_1+\ldots+r_n' y_n} = e^{-(r_1' y_1+\ldots+r_n' y_n)} ,$$

for all $x_1,\ldots, x_n$.

Hence, we have

$$r_1'y_1 + \dots + r_n'y_n = -(r_1'y_1 + \dots + r_n'y_n).$$

Thus $\quad r_1'y_1 + \dots + r_n'y_n = 0,\quad$ for all $x_1, \dots, x_n$.

We see that if $r_j' \neq 0$ for any j, we can choose $x_1, \dots, x_n$ so that

$$r_1'y_1 + \dots + r_n'y_n \neq 0 .$$

Therefore, we must have $r_j' = 0$ for all $j = 1, \dots, n$.
Hence, if not all r_j'' are zero, then all r_j' must be zero.
Therefore, all of $r_1, \dots, r_n$ must be real or all must be pure imaginary.

Conversely, if f is of the form (5.3.2.1), then it is clear that f is real-valued, and a straightforward verification shows that f satisfies (S_1).

5.4 Existence of Discontinuous Solution of $f(xy)+f(xy^{-1}) = 2f(x)f(y\theta)$.

5.4.1 Theorem There exists discontinuous function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that f satisfies

$(S_1)\qquad f(x + y) + f(x - y) = 2f(x)f(y + \theta).$

Proof By corollary 5.2.2, every continuous solution of (S_1) must be of the form

$$(5.4.1.1)\qquad f(x) = \frac{h(x - \theta) + h(\theta - x)}{2} ,$$

where h is a continuous homomorphism from $\mathbb{R}^n$ to $\mathbb{C}^*$. Therefore, by taking h to be a discontinuous homomorphism from $\mathbb{R}^n$ to $\mathbb{C}^*$, we can get a discontinuous solution of (S_1) of the form (5.4.1.1). The existence of discontinuous homomorphisms from $\mathbb{R}^n$ to $\mathbb{C}^*$ is already demonstrated in section 3.4.