CHAPTER VI

APPLICATION TO POTENTIAL THEORY

The Theory of distributions has been studied in the previous chapters. It will be applied, in this chapter, to Potential theory.

The materials of this chapter are drawn from references [1] and [4].

6.1 Basic Concepts

6.1.1 The spherical coordinate system.

We shall be concerned primarily with some notations on the real n-dimensional Euclidean space $\mathbb{R}^n$. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are in $\mathbb{R}^n$, the inner product (x,y) is defined by $(x,y) = \sum_{j=1}^{n} x_j y_j$. The length of a vector $x \in \mathbb{R}^n$ is defined to be the positive square root of (x,x) and is denoted by $\|x\|$. The distance between two vectors x and y is defined to be $\|x-y\|$. The angle between two nonzero vectors x and y is defined to be the angle θ such that $0 \leq \theta \leq \pi$ and

$$\cos \theta = \frac{(x, y)}{\|x\| \cdot \|y\|} .$$

Many of the functions that we shall deal with are functions only of the distance from the origin $0 \in \mathbb{R}^n$. For such functions it is more convenient to use a spherical coordinate system rather than a rectangular coordinate system. The spherical coordinates of $x = (x_1, \ldots, x_n) \neq 0$ are defined as follows : if $r = \| x \|$, then

$$\Theta = \left(\frac{x_1}{r}, \ldots, \frac{x_n}{r} \right)$$

is a point of $\partial B(0,1)$, the unit sphere with centre at 0. The pair (Θ, r) uniquely determines x and are called the spherical coordinates of x. The spherical coordinates of 0 are the pair (0,0). This transformation from rectangular coordinates to spherical coordinates is essentially the mapping $(x_1, \ldots, x_n) \longmapsto (\Theta_1, \ldots, \Theta_{n-1}, r)$ where

$$\Theta_1 = \frac{x_1}{r}$$

$$\Theta_2 = \frac{x_2}{r}$$

$$\text{-------}$$

$$\Theta_{n-1} = \frac{x_{n-1}}{r}$$

$$r = (x_1^2 + \ldots + x_n^2)^{1/2} .$$

We shall also let $\Theta_n = \frac{x_n}{r}$, Θ_n is the cosine of the angle between x and the vector $(0,0,\ldots,0,1)$; that is, the angle between x and the " x_n- axis ". The Jacobian of the mapping is easily calculated and absolute value is given by

$$\left| \frac{\partial (x_1,\dots,x_n)}{\partial (\theta_1,\dots,\theta_{n-1},r)} \right| = \frac{r^{n-1}}{(1-\theta_1^2-\dots-\theta_{n-1}^2)^{1/2}} = \frac{r^{n-1}}{\|\theta_n\|} .$$

If $y = (y_1,\dots, y_n) \in \mathbb{R}^n$ and $\rho > 0$, then $\partial B(y,\rho)$ is the surface defined by the equation

$$(x_1- y_1)^2+\dots+(x_n- y_n)^2 = \rho^2 .$$

Consider a Borel set $M \subset \partial B(y,\rho) \cap \{ (x_1,\dots,x_n): x_n-y_n \geq 0 \}$. Let M_n denote the projection of M onto the subspace $\{ (x_1,\dots,x_n): x_n = 0 \}$; that is, $M_n = \{ (x_1,\dots,x_{n-1},0):(x_1,\dots,x_n) \in M \}$. For each $x \in \partial B(y,\rho)$ let $\gamma = \gamma(x)$ be the angle between the " x_n- axis " and the outer normal to $\partial B(y,\rho)$ at x. Then

$$\sec \gamma = \frac{1}{\cos \gamma} = \frac{\rho}{x_n- y_n}$$

and

$$s(M) = \int_{M_n}\!\!\dots\!\int \sec \gamma \, dx_1 \dots dx_{n-1}$$

represents the surface area of M. If $M \subset \partial B(y,\rho) \cap \{ (x_1,\dots, x_n): x_n- y_n \leq 0 \}$, the surface area of M is given by the same integral with $\sec \gamma = -\rho/(x_n- y_n)$. The integral of a Borel function f defined on $\partial B(y,\rho)$ relative to the surface area s is denoted by

$$\int_{\partial B(y,\rho)} f(x)d\, s(x).$$

Consider an extended real-valued function f with domain in $\mathbb{R}^n$. We shall take the following liberty with the functional notation. When f is considered as a function of the spherical coordinates (θ,r) of x, we shall denote the value of the composite function at (θ,r) by $f(\theta,r)$. Suppose f is integrable on $\bar{B}(0,\rho)$. Then the integral of f over $\bar{B}(0,\rho)$ can be evaluated using spherical coordinates as follows :

$$\int_{\bar{B}(0,\rho)} f(x)dx = \int_0^\rho \int\!\!\ldots\!\!\int_{\|\theta\|=1} f(\theta,r)\ \frac{r^{n-1}}{\|\theta_n\|}\, d\theta_1 \ldots d\theta_{n-1}\, dr$$

$$= \int_0^\rho r^{n-1}\left(\int_{\|\theta\|=1} f(\theta,r)d\, s(\theta)\right) dr.$$

6.1.2 Green's identity.

Let $v = (v_1,\ldots, v_n)$ be a vector-valued function whose components v_j have continuous first partial derivatives on a neighborhood of $\bar{\Omega} \subset \mathbb{R}^n$. The divergence of v is defined by

$$\operatorname{div} v = \sum_{j=1}^{n} \partial_j v_j .$$

We let n(x) denote the outer unit normal to the surface $\partial\Omega$ at the point $x \in \partial\Omega$. We shall take as our starting point the following theorem which is adequately discussed in any advanced calculus textbook.

Divergence theorem $\quad \int_{\Omega} \operatorname{div} v \, dx = \int_{\partial\Omega} (v,n) ds.$

We make the convention that whenever " n " appears in an integral over a smooth surface it is understood to be the outer unit normal to the surface.

Suppose now that u is a function defined on a neighborhood of $\bar{\Omega}$ and has continuous second partial derivatives thereon. The Laplacian of u, Δu, is defined by

$$\Delta u = \sum_{j=1}^{n} \partial_j^2 u \quad .$$

If u is a function of variables other than x and it is necessary to clasify the meaning of the Laplacian, we shall use $\Delta_{(x)}$ to signify that the Laplacian is relative to the coordinates of x. The gradient of u is defined by

$$\operatorname{grad} u = (\partial_1 u, \ldots, \partial_n u).$$

Let v be a second such function. Then

$$u \operatorname{grad} v = (u\partial_1 v, \ldots, u\partial_n v)$$

and $\qquad \operatorname{div} (u \operatorname{grad} v) = u \,\Delta v + (\operatorname{grad} u, \operatorname{grad} v).$

It follows from the divergence theorem that

$$\int_{\Omega} u \,\Delta v dx + \int_{\Omega} (\text{grad } u, \text{grad } v) dx = \int_{\partial\Omega} (u \text{ grad } v, u) ds$$

$$= \int_{\partial\Omega} u \, D_n v \, d s$$

since $(u \text{ grad } v, n) = u(\text{grad } v, n)$ and the latter inner product is just the directional derivative $D_n v$ of v in the direction n. By interchanging u and v and subtracting we obtain

Green's identity $\displaystyle\int_{\Omega} (u \,\Delta v - v \,\Delta u) dx = \int_{\partial\Omega} (u \, D_n v - v \, D_n u) ds$.

6.2 Harmonic Functions

6.2.1 Definition. Let Ω be an open set in $\mathbb{R}^n (n \geq 2)$. A function $h : \Omega \longrightarrow \mathbb{R}$ is said to be harmonic on Ω if h is continuous on Ω and for any ball $B = B(x, \rho)$ such that $\bar{B} \subset \Omega$,

$$h(x) = \frac{1}{S_n \rho^{n-1}} \int_{\partial B} h(y) \, d s \, (y)$$

$$= \mathcal{M}_h^{\rho} (x).$$

In order to prove (6.2.4), we need the following lemmas :

6.2.2 Lemma. For any $\varepsilon > 0$, $\int_0^\varepsilon S_n r^{n-1} \sigma_\varepsilon(r)dr = 1 \qquad (r < \varepsilon)$.

Proof :
$$\int_0^\varepsilon S_n r^{n-1} \sigma_\varepsilon(r)dr = \int_0^\varepsilon S_n r^{n-1} \frac{1}{\varepsilon^n} \sigma_1\left(\frac{r}{\varepsilon}\right)dr$$
$$= \int_0^1 S_n \frac{r^{n-1}}{\varepsilon^{n-1}} \sigma_1\left(\frac{r}{\varepsilon}\right) d\left(\frac{r}{\varepsilon}\right)$$
$$= \alpha S_n \int_0^1 \left(\frac{r}{\varepsilon}\right)^{n-1} \exp\left(-\frac{1}{1-\|\frac{r}{\varepsilon}\|^2}\right) d\left(\frac{r}{\varepsilon}\right)$$
$$= \alpha S_n \int_0^1 \rho^{n-1} \exp\left(-\frac{1}{1-\rho^2}\right) d\rho$$
$$= \int_{\mathbb{R}^n} \sigma_1(x)dx$$
$$= 1.$$

6.2.3 Lemma. If h is harmonic on Ω, then $h * \sigma_\varepsilon(x) = h(x)$ for all $x \in \Omega$.

Proof : Let x be a point in Ω and let $\varepsilon > 0$ be such that $B(x,\varepsilon) \subset \bar{B}(x,\varepsilon) \subset \Omega$. Then

$$h*\sigma_\varepsilon(x) = \int_\Omega h(y) \sigma_\varepsilon(x-y)dy = \int_{\|y-x\| < \varepsilon} h(y) \sigma_\varepsilon(x-y)dy$$
$$= \int_0^\varepsilon \left(\int_{\|y-x\|=r} h(y) \sigma_\varepsilon(x-y)ds(y) \right) dr$$
$$= \int_0^\varepsilon \sigma_\varepsilon(r) \left(\int_{\|y-x\|=r} h(y)ds(y) \right) dr.$$

If $\|y-x\| = r$, then y is a point of $\partial B(x,r)$. The integral within parentheses is the integral over a sphere of radius r relative to a uniformly distributed measure of total mass $S_n r^{n-1}$ and is equal to $S_n r^{n-1} h(x)$. Then

$$h * \sigma_\varepsilon(x) = \int_0^\varepsilon r^{n-1} \sigma_\varepsilon(r) S_n h(x) dr$$

$$= h(x) \int_0^\varepsilon S_n r^{n-1} \sigma_\varepsilon(r) dr.$$

By (6.2.2), we have the result.

6.2.4 <u>Theorem</u>. If h is harmonic on Ω, then $h \in C^\infty(\Omega)$.

<u>Proof</u> : Since h is locally integrable on Ω, we have, by (4.2.2) and (6.2.3);

$$T_h * \sigma_\varepsilon(x) = h * \sigma_\varepsilon(x) = h(x) \qquad (x \in \Omega).$$

Then for every multi-index r,

$$\partial^r(T_h * \sigma_\varepsilon)(x) = \partial^r h(x). \qquad (x \in \Omega).$$

But by (4.2.3)

$$\partial^r(T_h * \sigma_\varepsilon)(x) = T_h * (\partial^r \sigma_\varepsilon)(x),$$

which is meaningful for every multi-index r (since $\varphi_\varepsilon \in C^\infty(\Omega)$).
Hence $h \in C^\infty(\Omega)$.

6.2.5 Lemma. If h is harmonic on a neighborhood of the closure of a ball $B = B(x_o, \rho)$, then

$$\int_{\partial B} D_n h \, ds = 0.$$

Proof : By (6.2.1),

$$\frac{1}{S_n \rho^{n-1}} \int_{\partial B} h(y) ds(y) = h(x_o) = \text{a constant.}$$

Then
$$\frac{\partial}{\partial r}\left(\frac{1}{S_n \rho^{n-1}} \int_{\partial B} h(y) ds(y) \right) = 0.$$

Changing into spherical coordinates relative to the pole x_o;

$$\frac{\partial}{\partial r}\left(\frac{1}{S_n \rho^{n-1}} \int_{\|\theta\|=1} h(\theta,r) \frac{\rho^{n-1}}{\|\theta_n\|} d\theta \right) = 0$$

or
$$\frac{\partial}{\partial r} \int_{\|\theta\|=1} h(\theta,r) \frac{d\theta}{\|\theta_n\|} = 0.$$

By (5.2.8),

$$\int_{\|\theta\|=1} \frac{\partial}{\partial r} h(\theta,r) \frac{d\theta}{\|\theta_n\|} = 0.$$

Thus we can get

$$\int_{\|\Theta\|=1} \frac{\partial}{\partial r} h(\Theta,r) \frac{r^{n-1}}{\|\Theta_n\|} d\Theta = 0$$

which implies that $\int_{\partial B} D_n h \, d s = 0.$

6.2.6 Theorem. If h is a harmonic on Ω, then $\Delta h = 0$ on Ω.

Proof : Let $x_o \in \Omega$, consider a ball $B = B(x_o,\rho)$ such that $\bar{B} \subset \Omega$.

By (6.2.5),

$$\int_{\partial B} D_n h \, d s = 0.$$

Putting $u = h, v = 1$ in Green's identity, we have

$$\int_B \Delta h \, d x = 0.$$

Suppose $\Delta h(x_o) \neq 0$. Without loss of generality we may assume that $\Delta h(x_o) > 0$. By the continuity of Δh at x_o, there is a ball $B(x_o,\varepsilon)$, $\varepsilon > 0$ such that $\bar{B}(x_o,\varepsilon) \subset \Omega$ and $\Delta h(x) > 0$ for all $x \in B(x_o,\varepsilon)$. Since the Lebesgue measure is positive, we have that

$$\int_{B(x_o,\varepsilon)} \Delta h \, d x > 0.$$

This is a contradiction. Therefore $\Delta h(x_o) = 0$. Since x_o is arbitrary, we conclude that $\Delta h = 0$ on Ω.

Suppose $\tau \in C^2(\Omega)$ and $\Delta \tau = 0$ on Ω, and let y be a fixed point of Ω. Then $\Delta \tau(x)$ can be regarded as a function of the spherical coordinates (θ, r) of x relative to y for which $r = \|x-y\|$ and θ is the point of intersection of the line seqment joining x to y and a unit sphere with centre at y.

Suppose that τ is a function of r alone. Then $\Delta\tau$, as a function of spherical coordinates, is easily seen to be given by

$$\Delta \tau = \frac{d^2\tau}{dr^2} + \frac{(n-1)}{r}\frac{d\tau}{dr}, \qquad r \neq 0.$$

The function only of $r = \|x-y\|$ satisfies the equation

$$\frac{d^2\tau}{dr^2} + \frac{(n-1)}{r}\frac{d\tau}{dr} = 0$$

on $\mathbb{R}^n - \{y\}$.

If $n = 2$, the general solution of this equation is $A \log r + B$ where A and B are arbitrary constants. The particular solution is $-\log r$.

If $n \geq 3$, the general solution of the above equation is $Ar^{-n+2} + B$. The particular solution is r^{-n+2}.

We shall introduce a notation for aforementioned particular solutions and, at the same time, extend their domains so as to be defined on $\mathbb{R}^n$ by putting

$$(6.2.7) \qquad \tau_y(x) = \begin{cases} +\infty & \text{if } x = y, \\ -\log \|x-y\| & \text{if } x \neq y, \text{ and } n = 2, \\ \|x-y\|^{-n+2} & \text{if } x \neq y, \text{ and } n \geq 3. \end{cases}$$

If y is the origin, we shall omit y. The function τ_y, so defined on $\mathbb{R}^n$, will also belong to $C^2(\mathbb{R}^n)$ and $\Delta\tau_y = 0$ on $\mathbb{R}^n$.

6.2.8 Theorem. If h has continuous second partial derivatives on a neighborhood of the closure of a ball $B = B(x_o, \rho)$, then

(i) for $n = 2$ and $x \in B$

$$h(x) = \frac{1}{2\pi} \int_{\partial B} \left[(-\log r)D_n h - hD_n(-\log r)\right] d\, s\, (y)$$

$$-\frac{1}{2\pi} \int_B \Delta h(-\log r)dy \; ;$$

(ii) for $n \geq 3$ and $x \in B$

$$h(x) = \frac{1}{S_n(n-2)} \int_{\partial B} (r^{-n+2}D_n h - hD_n r^{-n+2})d\, s\, (y)$$

$$-\frac{1}{S_n(n-2)} \int_B \Delta h\, r^{-n+2}\, dy,$$

where $r = \|x-y\|$, $y \in \bar{B}$.

Proof :(i) Consider a fixed $x \in B$ and let $v(y) = \tau_x(y) = -\log\|x-y\| = -\log r$ for $y \neq x$. Then $\Delta v = 0$ on $R^2 - \{x\}$. Choose $\delta > 0$ such that $\bar{B}(x,\delta) \subset B$ and let Ω be the open set $B - \bar{B}(x,\delta)$. By Green's identity

$$\int_{\Omega} (h\,\Delta v - v\,\Delta h)dy = \int_{\partial\Omega} (hD_n v - vD_n h)d\,s\,(y).$$

Since $\Delta v = 0$ on Ω and $\partial\Omega = \partial B \cup \partial B(x,\delta)$,

$$\text{(a)} \qquad -\int_{\Omega} v\,\Delta h\,dy = \int_{\partial B} (hD_n v - vD_n h)d\,s\,(y)$$

$$- \int_{\partial B(x,\delta)} (hD_n v - v\,D_n h)d\,s\,(y).$$

The minus sign precedes the second integral on the right in view of the convention about the outer unit normal vector. The outer unit normal vector for $\partial\Omega$ at a point of $\partial B(x,\delta)$ is the negative of the outer unit normal vector for $\partial B(x,\delta)$. We will now let $\delta \longrightarrow 0$ in equation (a). To show that

$$\text{(b)} \qquad \lim_{\delta\to 0} \int_{\Omega} v\,\Delta h\,d\,y = \int_{B} v\,\Delta h\,d\,y,$$

it suffices to prove that v is integrable over B since Δh is bounded on B. By transforming to spherical coordinates relative to the pole x, for $\delta < 1$

$$\int_{B(x,\delta)} |v|\, dy = \int_0^\delta r \log \frac{1}{r} \left(\int_{\|\theta\| = 1} d\, s(\theta) \right) dr$$

$$= 2\pi \int_0^\delta r \log \frac{1}{r}\, dr.$$

Since $\lim_{r \to 0} r \log \frac{1}{r} = 0$, the function $v \Delta h$ is integrable on $B(x,\delta)$. Since $v \Delta h$ is integrable on Ω, it is integrable on B and (b) holds.

Consider the right of (a). The first integral does not depend upon δ and we need only consider the second integral. Since $|D_n h| = |(n, \text{grad } h)| \leq \|n\| \|\text{grad } h\| = \|\text{grad } h\| = (\sum \partial_j h^2)^{1/2}$ and the first partial derivatives $\partial_j h$ are bounded on $\bar{B}$, $D_n h$ is bounded on $\partial B(x,\delta)$ by some constant m. For small

$$\left| \int_{\partial B(x,\delta)} v\, D_n h\, d\, s(y) \right| \leq m \int_{\partial B(x,\delta)} v\, d\, s(y)$$

$$= m \int_{\partial B(x,\delta)} \log \frac{1}{r}\, d\, s(y)$$

$$= 2\pi m\, \delta \log \frac{1}{\delta}\ .$$

Since $\delta \log \frac{1}{\delta} \to 0$ as $\delta \to 0$,

(c) $$\lim_{\delta \to 0} \int_{\partial B(x,\delta)} v\, D_n h\, d\, s(y) = 0.$$

Now cosider $\lim_{\delta \to 0} \int_{\partial B(x,\delta)} h\, D_n v\, d\, s(y)$. We can compute $D_n v(y)$

at a point $y \in \partial B(x,\delta)$ as follows. Since $v(y) = -\log r$ and the normal derivative of v is just the derivative with respect to r, $D_n v(y) = -r^{-1}$.

Therefore

$$\int_{\partial B(x,\delta)} h\, D_n v \, d\, s(y) \;=\; -\frac{1}{\delta} \int_{\partial B(x,\delta)} h \, d\, s(y).$$

The integral on the right is just the average of the continuous function h over $\partial B(x,\delta)$, except for a factor of -2π, and has a limit $-2\pi u(x)$ as $\delta \to 0$. This shows that

(a) $$\lim_{\delta \to 0} \int_{\partial B(x,\delta)} h\, D_n v \, d\, s(y) \;=\; -2\pi h(x).$$

Taking the limit as $\delta \longrightarrow 0$ in (a), we obtain

$$-\int_B v\, \Delta h \, dy \;=\; \int_{\partial B} (hD_n v - vD_n h) d\, s(y) + 2\pi h(x).$$

The proof of (ii) is basically the same with $v(y)$ being replaced by $v(y) = \|y-x\|^{-n+2}$.

The following theorem is the converse of (6.2.6).

6.2.9 <u>Theorem</u>. If $h \in C^2(\Omega)$ and $\Delta h = 0$ on Ω, then h is harmonic on Ω.

Proof : Only the $n \geq 3$ case will be proved, since the $n = 2$ case is similar. By (6.2.8 (ii)), for any ball $B = B(x_o, \rho)$ such that $\bar{B} \subset \Omega$,

$$h(x_o) = \frac{1}{S_n(n-2)} \int_{\partial B} \left[\| x_o - y \|^{-n+2} D_n h - h \, D_n \| x_o - y \|^{-n+2} \right] d\, s(y),$$

since $\Delta h = 0$ on $\bar{B}$. For $y \in \partial B$, $\| x_o - y \|^{-n+2} = \rho^{-n+2}$ and

$D_n \| x_o - y \|^{-n+2} = D_n r^{-n+2} \big|_{r=\rho} = -(n-2) \rho^{-n+1}$. After substituting in the above integral

$$h(x_o) = \frac{\rho^{-n+2}}{S_n(n-2)} \int_{\partial B} D_n h \; d\, s(y) + \frac{\rho^{-n+1}}{S_n} \int_{\partial B} h(y) d\, s(y) ,$$

The first integral on the right is zero by (6.2.5). Then

$$h(x_o) = \frac{1}{S_n \rho^{n-1}} \int_{\partial B} h(y) \; d\, s\, (y) .$$

Hence the result (6.2.1).

The above theorem is the another definition of harmonic functions. The function h is said to be harmonic on the open set $\Omega \subset \mathbb{R}^n$ if h has continuous second partial derivatives on Ω and $\Delta h = 0$ on Ω.

Thus the function τ_y (see 6.2.7) is harmonic on $\mathbb{R}^n$ and so called the <u>fundamental harmonic function</u> for $\mathbb{R}^n$ with pole y.

6.2.10 Theorem. The class of functions harmonic on an open set $\Omega \subset \mathbb{R}^n$ is a linear vector space over the reals which contains the constant functions.

Proof : It follows from the fact that the Laplacian is a linear differential operator.

6.2.11 Theorem. If τ is the fundamental harmonic function for $\mathbb{R}^n$ with pole at the origin (see 6.2.7), then

$$\Delta T_{\tau} = - k_n T_{\delta} ,$$

when $k_n = 2\pi$ for $n = 2$ and $k_n = S_n(n-2)$ for $n \geq 3$.

Proof : τ is not continuous at the origin, but we can define a distribution T_{τ} , since it is improperly integrable. For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\Delta T_{\tau}(\varphi) = T_{\tau}(\Delta\varphi) = \int_{\mathbb{R}^n} \tau \, \Delta \varphi \, dx .$$

Since $\Delta\tau = 0$, we can get

$$\Delta T_{\tau}(\varphi) = \int_{\mathbb{R}^n} (\tau \Delta \varphi - \varphi \Delta \tau) dx$$

$$= \lim_{\substack{\varepsilon \to 0 \\ R \to +\infty}} \int_{\varepsilon < \|x\| < R} (\tau \Delta \varphi - \varphi \Delta \tau) dx.$$

By Green's identity,

$$\Delta T_{\tau}(\varphi) = \lim_{\varepsilon \to 0} \int_{\|x\| = \varepsilon} (\tau D_n \varphi - \varphi D_n \tau) ds$$

$$= \lim_{\varepsilon \to 0} \int_{\|x\| = \varepsilon} \tau D_n \varphi \, ds - \lim_{\varepsilon \to 0} \int_{\|x\| = \varepsilon} \varphi D_n \tau \, ds.$$

The first integral on the right is zero, by (6.2.8 (c)), and the second integral is equal to $k_n \varphi(0)$ when $k_n = 2\pi$ for $n = 2$ and $k_n = S_n(n-2)$ for $n \geq 3$, by applying (6.2.8 (d)). Then by (3.1.2 (i)) we can conclude that

$$\Delta T_{\tau} = - k_n T_{\delta} ,$$

when $k_n = 2\pi$ for $n = 2$ and $k_n = S_n(n-2)$ for $n \geq 3$.

6.2.12 Theorem. If T is a distribution on Ω such that $\Delta T = 0$ on Ω, there exists a harmonic function h on Ω such that $T = T_h$.

Proof : Suppose first that T is a distribution on $\mathbb{R}^n$ and $\Delta T = 0$ on $\mathbb{R}^n$. Then for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $T * \varphi \in C^{\infty}(\mathbb{R}^n)$ (4.2.3), and $\Delta(T*\varphi) = (\Delta T) * \varphi = 0$ which implies that $T * \varphi$ is harmonic on $\mathbb{R}^n$.

Let δ_1 and $\delta_{1/m}$, where m is a positive integer, be the Schwartz functions. Then by ([7],p.23),(4.1.2(i)),and (6.2.3),

$$(T*\sigma_1)*\sigma_{1/m} = T*(\sigma_1*\sigma_{1/m}) = T*(\sigma_{1/m}*\sigma_1)$$

$$= (T*\sigma_{1/m})*\sigma_1 = T*\sigma_{1/m} .$$

Let $h_1 = T*\sigma_1$. Since $h_1*\sigma_{1/m}$ and $T*\sigma_{1/m}$ have bounded supports, we can define distributions $T_{h_1*\sigma_{1/m}}$ and $T_{T*\sigma_{1/m}}$, and

(a) $$T_{h_1*\sigma_{1/m}} = T_{T*\sigma_{1/m}} .$$

Since h_1 and $h_1*\sigma_{1/m}$ are locally integrable, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have, by (4.2.2),

$$T_{h_1*\sigma_{1/m}}*\varphi = (h_1*\sigma_{1/m})*\varphi = (T_{h_1}*\sigma_{1/m})*\varphi .$$

Then $$T_{h_1*\sigma_{1/m}} = T_{h_1}*\sigma_{1/m} .$$

Now we claim that $T_{T*\sigma_{1/m}} = T*\sigma_{1/m}$. Since $T*\sigma_{1/m}$ is locally integrable, we have, by (4.2.2),

$$T_{T*\sigma_{1/m}}*\check{\varphi}(0) = (T*\sigma_{1/m})*\check{\varphi}(0) \qquad (\varphi \in \mathcal{D}(\mathbb{R}^n))$$

or $$T_{T*\sigma_{1/m}}(\varphi) = (T*\sigma_{1/m})(\varphi).$$

Therefore $\quad T_{T*\delta_{1/m}} \quad = \quad T*\delta_{1/m}$.

So by (a), $\quad T_{h_1}*\delta_{1/m} \quad = \quad T*\delta_{1/m}$.

But $T_{\delta_{1/m}} \to T_\delta$ as $m \to +\infty$. Then

$$T_{h_1} * T_\delta \quad = \quad T * T_\delta .$$

Finally by (4.2.16) we have

$$T_{h_1} \quad = \quad T .$$

If now T is defined on an open set $w \subset \mathbb{R}^n$. By (3.1.6) There exists a distribution T' on $\mathbb{R}^n$ such that

$$T'\big|_{\mathcal{D}(w_0)} \quad = \quad T\big|_{\mathcal{D}(w_0)}$$

whenever $w_0 \subset \bar{w}_0 \subset w$. For $w_1 \subset \bar{w}_1 \subset w_0$, by the above proof, there exists a harmonic function h_2 on w_1 such that $T' = T'_{h_2}$ on w_1 and so $T_{h_2} = T$ on w_1, since $T' = T$ on w_1. But w_1 is arbitrary, and thus we get the result.

6.3 The Poisson Integrals

6.3.1 Definition. For a function f integrable relative to surface area measure on a sphere $\partial B(0,\rho) \subset \mathbb{R}^n$. The Poisson integral I_f of f in $B = B(0,\rho)$ is given by

$$I_f(x) = \frac{1}{S_n \rho} \int_{\partial B} \frac{\rho^2 - \|x\|^2}{\|x-y\|^n} f(y)\,d\,s(y) \qquad (x \in B).$$

6.3.2 Theorem. I_f defined as above is harmonic on $B = B(0,\rho)$.

Proof : Using (5.2.8), we can show that I_f has continuous second partials and that

$$\Delta I_f(x) = \frac{1}{S_n \rho} \int_{\partial B} \Delta_{(x)} \frac{\rho^2 - \|x\|^2}{\|y-x\|^n} f(y)\,d\,s(y).$$

A tedious, but straightforward, differentiation shows that the integrand is zero for $x \in B$. Hence the result.

Let $B = B(0,\rho)$ be a ball in $\mathbb{R}^n$. For any $x \in B$ and $x \neq 0$, choose $x^* \notin \bar{B}$ on the radial line joining 0 to x so that $\|x\| \cdot \|x^*\| = \rho^2$. Then

$$x^* = \frac{\rho^2}{\|x\|^2} \cdot x$$

and is called the <u>inverse</u> of x relative to the sphere ∂B .

6.3.3 <u>Theorem.</u> If f is harmonic on a neighborhood of the closure of the ball $B = B(0,\rho)$ and $x \in B$, then

$$f(x) = I_f(x) = \frac{1}{S_n \rho} \int_{\partial B} \frac{\rho^2 - \|x\|^2}{\|y-x\|^n} f(y)d\,s(y).$$

In order to prove the theorem we need

6.3.4 <u>Lemma.</u> If h is harmonic on a neighborhood of the closure of the ball $B = B(0,\rho)$, $x \in B$, and $x \neq 0$, then

(i) for $n = 2$

$$h(x) = -\frac{1}{2\pi} \int_{\partial B} h\, D_n\left(\log \frac{\|x\|}{\rho} \frac{\|y-x^*\|}{\|y-x\|}\right) d\,s(y) \;;$$

(ii) for $n \geq 3$

$$h(x) = -\frac{1}{S_n(n-2)} \int_{\partial B} h\, D_n\left(\frac{1}{\|y-x\|^{n-2}} - \frac{\rho^{n-2}}{\|x\|^{n-2}} \cdot \frac{1}{\|y-x^*\|^{n-2}}\right) d\,s(y),$$

where x^* is the inverse of x relative to ∂B.

<u>Proof</u> :(i) By (6.2.8) and since $\Delta h = 0$ on B,

(a) $$h(x) = \frac{1}{2\pi} \int_{\partial B} \left[(-\log \|y-x\|)D_n h - hD_n(-\log \|y-x\|)\right] d\,s(y).$$

Consider the integral on the right as a function of x. Since $-\log\|y-x^*\|$ is harmonic on a neighborhood of $\bar{B}$ and by Green's identity

$$(b)\qquad 0 = \frac{1}{2\pi}\int_{\partial B}\Big[(-\log\|y-x^*\|)D_n h - hD_n(-\log\|y-x^*\|)\Big]d\,s(y).$$

Although it is not essential for the following argument, we shall modify (b) to incorporate some constants for later use. By (6.2.5),

$$(c)\qquad \frac{1}{2\pi}\int_{\partial B}\Big[-\log\Big(\frac{\|x\|}{\rho}\Big)\Big]D_n h\,d\,s(y) = 0$$

since the factor in brackets is a constant. Moreover,

$$D_n\Big[-\log(\|x\|\,\|y-x^*\|/\rho)\Big] = D_n\Big[-\log\|y-x^*\|\Big],$$

since $\log(\|x\|/\rho)$ is a constant. It follows from (c),(b),and this remark that

$$(d)\qquad 0 = \frac{1}{2\pi}\int_{\partial B}\Big[-\log\Big(\frac{\|x\|\,\|y-x^*\|}{\rho}\Big)D_n h - hD_n\Big(-\log\frac{\|x\|\,\|y-x^*\|}{\rho}\Big)\Big]d\,s(y).$$

It follows from (a) and (d) that

$$(e)\qquad h(x) = \frac{1}{2\pi}\int_{\partial B}\Big[\log\Big(\frac{\|x\|}{\rho}\cdot\frac{\|y-x^*\|}{\|y-x\|}\Big)D_n h - hD_n\Big(\log\Big(\frac{\|x\|}{\rho}\cdot\frac{\|y-x^*\|}{\|y-x\|}\Big)\Big)\Big]d\,s(y).$$

Let θ be the angle at the origin between y and x. Then

$$\text{(f)} \qquad \frac{(y,x)}{\|y\|\,\|x\|} \quad = \quad \cos\theta \quad = \quad \frac{(y,x^*)}{\|y\|\,\|x^*\|}\ .$$

Since $(y,x) = \sum_{j=1}^{n} y_j x_j = \frac{1}{2}(\|y\|^2 + \|x\|^2 - \|y-x\|^2)$ and by replacing $\|x^*\| = \rho^2/\|x\|$ and $\|y\| = \rho$ in (f), we obtain

$$\text{(g)} \qquad \frac{\|x\|}{\rho} \cdot \frac{\|y-x^*\|}{\|y-x\|} \quad = \quad 1.$$

Substituting (g), independently of $y \in \partial B$, in (e), we get the result.

(ii) We follow the same steps as in proving (i) with $-\log\|y-x\|$ replaced by $\|y-x\|^{-n+2}$ in (a) and with the appropriate constant before the integral. If x^* is defined as before, $-\log\|y-x^*\|$ is also replaced by $\|y-x^*\|^{-n+2}$ in (b), and the constant before the integral in (b) is adjusted, then upon multiplying both sides of the equation corresponding to (b) by α and subtracting the result from the equation corresponding to (a) we obtain

$$\text{(h)} \qquad h(x) = \frac{1}{s_n(n-2)} \int_{\partial B} \left[\left(\frac{1}{\|y-x\|^{n-2}} - \alpha \frac{1}{\|y-x^*\|^{n-2}} \right) D_n h \right.$$
$$\left. - h D_n \left(\frac{1}{\|y-x\|^{n-2}} - \alpha \frac{1}{\|y-x^*\|^{n-2}} \right) \right] d\,s(y).$$

If we choose $\alpha = (\rho/\|x\|)^{n-2}$, then by (g)

$$\alpha = \frac{\rho^{n-2}}{\|x\|^{n-2}} = \frac{\|y-x^*\|^{n-2}}{\|y-x\|^{n-2}}$$

for all $y \in \partial B$. As before, the quetient on the right is independent of $y \in \partial B$. With this choice of α, (h) reduces to the equation in (ii).

<u>Proof of the theorem</u> : Consider the n = 2 case. Recall that $D_n f = (n, \text{grad } f)$. The outer unit normal to the surface $\partial B(0,\rho)$ at $y \in \partial B$ is simply $y/\|y\| = y/\rho$. With $x \in B$ fixed, $x \neq 0$, and x^* the inverse of x,

$$\text{grad} \log \|y-x\| = \frac{y-x}{\|y-x\|^2} \qquad \text{for} \quad y \in \partial B \quad \text{and}$$

$$\text{grad} \log\left(\frac{\|x\|}{\rho}\cdot\frac{\|y-x^*\|}{\|y-x\|}\right) = \text{grad} \log\left(\frac{\|x\|}{\rho}\right) + \text{grad} \log\|y-x^*\| - \text{grad} \log\|y-x\|$$

$$= \frac{y-x^*}{\|y-x^*\|^2} - \frac{y-x}{\|y-x\|^2}$$

Therefore at $y \in \partial B$

$$D_n\left(\log\left(\frac{\|x\|}{\rho}\cdot\frac{\|y-x^*\|}{\|y-x\|}\right)\right) = \left(\frac{y}{\rho}, \frac{y-x^*}{\|y-x^*\|^2} - \frac{y-x}{\|y-x\|^2}\right).$$

Substituting $x^* = \rho^2 x/\|x\|^2$ into the right side of this equation, and also using (6.3.4 (g)), we obtain

$$D_n\left(\log\left(\frac{\|x\|}{\rho}\cdot\frac{\|y-x^*\|}{\|y-x\|}\right)\right) = -\frac{1}{\rho}\,\frac{\rho^2-\|x\|^2}{\|y-x\|^2}$$

for $y \in \partial B$. Applying this result to (i) of (6.3.4),

$$f(x) = \frac{1}{2\pi\rho}\int_{\partial B} f(y)\,\frac{\rho^2-\|x\|^2}{\|y-x\|^n}\,d\,s(y) \qquad (x \in B-\{0\}),$$

under the conditions of the theorem. Note also that this equation holds when $x = 0$ since it reduces to $f(0) = \mathcal{M}_f^{\rho}(0)$. Then

$$f(x) = I_f(x) = \frac{1}{2\pi\rho}\int_{\partial B}\frac{\rho^2-\|x\|^2}{\|y-x\|^n}\,f(y)\,d\,s(y) \qquad (x \in B).$$

Exactly the same procedure is used to estiblish the representation in the $n \geq 3$ case.

6.3.5 <u>Corollary</u>. If $f = 1$ on the ball $B = B(0,\rho)$, then $I_f = 1$ on B.

<u>Proof</u> : It follows from the fact that the constant function 1 is harmonic on a neighborhood of $\bar{B}$ and by (6.3.3),

$$I_f(x) = f(x) = 1 \qquad (x \in B).$$

6.3.6 <u>Lemma</u>. If f is integrable relative to surface area measure on $\partial B(0,\rho)$ and if there is a constant k such that $f \leq k$ a.e.(s) in a neighborhood of $x_0 \in \partial B$, then $\limsup_{x\to x_0} I_f(x) \leq k$.

Proof : We first show that we can assume $k > 0$. If this is not the case, then consider $f-k+1$. By hypothesis, $f-k+1 \leq 1$ a.e.(s) in a neighborhood of x_o and $\lim_{x \to x_o} \sup I_{f-k+1}(x) = \lim_{x \to x_o} \sup I_f(x)-k+1$. Thus, if we can prove the $k > 0$ case, then the general case can be reduced to it. Assuming $k > 0$, choose $\varepsilon > 0$ such that $f(y) \leq k$ a.e.(s) for $y \in B_o(x_o, \varepsilon) \cap \partial B$. Then

$$I_f(x) = \frac{1}{S_n \rho} \int_{\{\|y-x_o\| < \varepsilon\} \cap \partial B} \frac{\rho^2 - \|x\|^2}{\|y-x\|^n} f(y)\, d\, s(y)$$

$$+ \frac{1}{S_n \rho} \int_{\{\|y-x_o\| \geq \varepsilon\} \cap \partial B} \frac{\rho^2 - \|x\|^2}{\|y-x\|^n} f(y)\, d\, s(y).$$

If $J_1(x)$ and $J_2(x)$ denote the first and second terms, respectively, then

$$J_1(x) \leq \frac{k}{S_n \rho} \int_{\{\|y-x_o\| < \varepsilon\} \cap \partial B} \frac{\rho^2 - \|x\|^2}{\|y-x\|^n}\, d\, s(y)$$

$$\leq \frac{k}{S_n \rho} \int_{\partial B} \frac{\rho^2 - \|x\|^2}{\|y-x\|^n}\, d\, s(y) = k\, I_1(x) = k \ (6.3.5).$$

Consider now $|J_2(x)|$. Suppose $x \in B(x_o, \varepsilon/2)$. Then $\|y-x\| \geq \varepsilon/2$ when $\|y-x_o\| > \varepsilon$ for if not we would have $\|y-x_o\| \leq \|y-x\| + \|x-x_o\| \leq \varepsilon$. Then for $x \in B(x_o, \varepsilon/2)$

$$|J_2(x)| \leq \frac{1}{S_n \rho} \int_{\{\|y-x_0\| \geq \epsilon\} \cap \partial B} \frac{\rho^2 - \|x\|^2}{(\epsilon/2)^n} |f(y)| \, d\, s(y)$$

$$\leq \frac{\rho^2 - \|x\|^2}{S_n \rho (\epsilon/2)^n} \int_{\partial B} |f(y)| \, d\, s(y).$$

Since $\|x\| \to \rho$ as $x \to x_0$, $|J_2(x)| \to 0$ as $x \to x_0$. Therefore

$$\limsup_{x \to x_0} I_f(x) \leq \limsup_{x \to x_0} J_1(x) + \limsup_{x \to x_0} J_2(x) \leq k.$$

6.3.7 Lemma. If f is integrable relative to surface area measure on $\partial B(0,\rho)$, then for $x_0 \in \partial B$

$$\limsup_{\substack{x \to x_0 \\ x \in B}} I_f(x) \leq \limsup_{\substack{x \to x_0 \\ x \in \partial B}} f(x).$$

Proof : We can assume that the right side is not $+\infty$, for otherwise there is nothing to prove. If k is any number greater than the right number of the above inequality, then $f(x) < k$ for all $x \in \partial B$ in a neighborhood of $x_0 \in \partial B$. By (6.3.6)

$$\limsup_{\substack{x \to x_0 \\ x \in B}} I_f(x) \leq k,$$

but, since k is any number greater than $\limsup_{\substack{x \to x_0 \\ x \in \partial B}} f(x)$, the lemma is proved.

6.3.8 <u>Theorem</u>. If f is integrable relative to surface area measure on $\partial B(0,\rho)$ and u.s.c. at $x_o \in \partial B$, then

$$\limsup_{\substack{x \to x_o \\ x \in B}} I_f(x) \leq f(x_o).$$

<u>Proof</u> : By (6.3.7),

$$\limsup_{\substack{x \to x_o \\ x \in B}} I_f(x) \leq \limsup_{\substack{x \to x_o \\ x \in \partial B}} f(x).$$

Since $\limsup_{\substack{x \to x_o \\ x \in \partial B}} f(x) = f(x_o)$ (5.1.1),

we have the result.

6.3.9 <u>Theorem</u>. Let $S = \partial B(0,\rho)$. If $f \in C(S)$, then I_f is harmonic on B and

$$\lim_{x \to y} I_f(x) = f(y) \qquad (y \in S).$$

<u>Proof</u> : Since f is continuous on S, f is integrable relative to surface area measure on S. Then I_f is harmonic on B (6.3.2). Since $\limsup_{x \to y} f(x) = f(y)$ and by (6.3.7), we have

$$\limsup_{x \to y} I_f(x) \leq f(y).$$

Since $I_{-f}(x) = -I_f(x)$, we also have

$$\limsup_{x \to y} -I_f(x) \leq -f(y)$$

or

$$\liminf_{x \to y} I_f(x) \geq f(y).$$

This shows that $\lim_{x \to y} I_f(x)$ exists and is equal to $f(y)$.

6.4 Superharmonic Functions

6.4.1 Definition. If Ω is an open subset of $\mathbb{R}^n$, $\mathcal{L}(\Omega)$ will denote the class of extended real-valued function on Ω satisfying

(i) u is not identically $+\infty$ on any component of Ω,

(ii) $u > -\infty$ on Ω,

(iii) u is l.s.c. on Ω.

Note that for any $u \in \mathcal{L}(\Omega)$, $\mathcal{M}_u^\rho(x)$ is defined whenever $\bar{B}(x,\rho) \subset \Omega$ since u is bounded below on ∂B by the l.s.c. of u.

6.4.2 Definition. An extended real-valued function u on an open set Ω is superharmonic on Ω if $u \in \mathcal{L}(\Omega)$ and for every ball $B = B(x,\rho)$ such that $\bar{B} \subset \Omega$ u is integrable relative to surface area measure on ∂B and

$$u(x) \geq \mathcal{M}_u^{\rho}(x) = \frac{1}{S_n \rho^{n-1}} \int_{\partial B} u(y)\,d\,s(y).$$

6.4.3 Definition. A function u defined on an open connected set Ω obeys the maximum principle if $\sup_{x \in \Omega} u(x)$ is not attained on Ω unless u is a constant. There is a corresponding minimum principle if $\inf_{x \in \Omega} u(x)$ is not attained unless u is a constant.

6.4.4 Theorem. If Ω is an open connected set in $\mathbb{R}^n$, $u \in \mathcal{S}(\Omega)$, and for each $x \in \Omega$ there is a ρ_x such that $\bar{B}(x, \rho_x) \subset \Omega$ and $u(x) \geq \mathcal{M}_u^{\rho}(x)$ whenever $\rho < \rho_x$, then u satisfies the minimum principle on Ω.

Proof : Suppose there is a point $x_o \in \Omega$ such that $u(x_o) = \inf_{x \in \Omega} u(x)$. Since u is superharmonic on Ω, $-\infty < u(x_o) = \inf_{x \in \Omega} u(x) < +\infty$. Let $M = \{ y : u(y) = \inf_{x \in \Omega} u(x) \}$, which is a relatively closed subset of Ω by the l.s.c. of u. We shall show that M is also open. Consider any $y \in M$. Then there is a ρ_y such that $\bar{B}(y, \rho_y) \subset \Omega$ and $u(y) \geq \mathcal{M}_u^{\rho}(y)$ whenever $\rho < \rho_y$. Suppose there is a point $z \in B(y, \rho_y) - M$. Let $\delta = \| y - z \|$. Since $y \in M$, $u(y) \leq \mathcal{M}_u^{\rho}(y)$. Therefore $u(y) = \mathcal{M}_u^{\delta}(y)$ or $\mathcal{M}_{u-u(y)}^{\delta}(y) = 0$. Since $u - u(y) \geq 0$ on $\partial B(y, \delta)$, $u - u(y) = 0$ a.e.(s) on $\partial B(y, \delta)$. But since $u(z) > u(y)$, there is an α such that $u(z) > \alpha > u(y)$. By the l.s.c. of u there is a neighborhood U of z such that $u > \alpha > u(y)$ on $U \cap \partial B(y,\delta)$. Since the latter set has positive surface area, $u-u(y) > 0$ on a set of positive area, a contradiction. This show that $B(y,\rho_y) \subset M$; that is, M is open. Therefore $M = \emptyset$ or $M = \Omega$ by the connectedness of Ω with the first possibility obviously being excluded. It follows that $M = \Omega$ and that u is constant if it attains its infimum.

6.4.5 Corollary. If u is superharmonic on an open connected set $\Omega \subset \mathbb{R}^n$, then u satisfies the minimum principle on Ω.

Proof : It follows immediately from (6.4.2) and (6.4.4).

6.4.6 Theorem. If u is superharmonic on a bounded open set $\Omega \subset \mathbb{R}^n$ and $\liminf_{z \to x} u(z) \geq 0$ for all $x \in \partial\Omega$, then $u \geq 0$ on Ω.

Proof : It suffices to prove that $u \geq 0$ on each component of Ω; that is, we can assume that Ω is connected. Suppose there is a point $y \in \Omega$ such that $u(y) < 0$. Then u is not a constant function. Define a function v on $\bar{\Omega}$ by

$$v(x) = \liminf_{z \to x} u(z) \qquad (x \in \bar{\Omega}).$$

u is l.s.c. on Ω since u is superharmonic. By (5.1.1), we have

$$\liminf_{z \to x} u(z) = u(x) \qquad (x \in \Omega).$$

Therefore $\quad v(x) = \liminf_{z \to x} u(z) = u(x) \qquad (x \in \Omega),$

and so

$$v(x) = \liminf_{z \to x} v(z) \qquad (x \in \bar{\Omega}).$$

That is, v is l.s.c. on $\bar{\Omega}$, and we have $v \geq 0$ on $\partial\Omega$ and $v(y) = u(y) < 0$. Then it attains a negative minimum on $\bar{\Omega}$, infact on Ω (since $v \geq 0$ on $\partial\Omega$); but this contradicts the minimum principle. Therefore $u \geq 0$ on Ω.

6.4.7 <u>Theorem</u>. Let u be superharmonic on an open set Ω and let w be an open subset of Ω with compact closure $\bar{w} \subset \Omega$. If h is continuous on $\bar{w}$, harmonic on w and $u \geq h$ on ∂w, then $u \geq h$ on w.

<u>Proof</u> : If h has the above properties, then it has the same properties on each component of w. By considering the components of w, we can assume that w is connected. Consider the function $u-h$ on $\bar{w}$. On ∂w, $u-h \geq 0$. Since h is harmonic on w, $u-h$ is superharmonic on w and cannot attain its infimum on w by (6.4.6). As a l.s.c. function on the compact set $\bar{w}$, $u-h$ attains its infimum on $\bar{w}$ and, in fact, on ∂w. Since $u-h \geq 0$ on ∂w, $u-h \geq 0$ on w.

6.4.8 Theorem. An extended real-valued function $u \in \mathcal{L}(\Omega)$ is superharmonic on Ω if and only if it satisfies the following property :

(*) If w is an open subset of Ω with compact closure $\bar{w} \subset \Omega$, h is continuous on $\bar{w}$, h is harmonic on w, and $u \geq h$ on ∂w, then $u \geq h$ on w.

Proof : The necessity is (6.4.7). Let us prove the sufficiency. We let $\mathcal{L}_o(\Omega)$ be the class of all functions $u \in \mathcal{L}(\Omega)$ and satisfying (*). Consider any $u \in \mathcal{L}_o(\Omega)$ and any $\bar{B}(x,\rho) \subset \Omega$. Since u is l.s.c. on ∂B, there is a sequence (f_m) of continuous function on ∂B such that $f_m \uparrow u$ on ∂B (5.1.11). Let

$$h_m = \begin{cases} I_{f_m} & \text{on } B , \\ f_m & \text{on } \partial B . \end{cases}$$

Then h_m is continuous on $\bar{B}$, harmonic on B (6.3.2) and $u \geq f_m \geq h_m$ on ∂B. Since u satisfies (*), $u \geq h_m$ on B. Therefore

$$u(x) \geq h_m(x) = I_{f_m}(x) = \mathcal{M}^{\rho}_{f_m}(x)$$

with the latter equality holding since x is the center of the ball B (6.2.1). Since $f_m \uparrow u$ on ∂B, $u(x) \geq \lim_{m \to +\infty} \mathcal{M}^{\rho}_{f_m}(x) = \mathcal{M}^{\rho}_{\lim_{m \to +\infty} f_m}(x) = \mathcal{M}^{\rho}_{u}(x)$ by the Lebesgue monotone convergence theorem (5.2.6). Hence the result.

6.4.9 Theorem. If u is superharmonic on an open set $\Omega \subset \mathbb{R}^n$, then for every ball $B = B(x,\rho)$ such that $\bar{B} \subset \Omega$.

$$u(x) \geq \mathcal{A}_u^{\rho}(x) = \frac{n}{\rho^n}\int_0^{\rho} r^{n-1}\mathcal{M}_u^{r}(x)\,dr.$$

Proof : Suppose u is superharmonic on Ω. For any ball $B = B(x,\rho)$ such that $\bar{B} \subset \Omega$, if $u(x) = +\infty$, the inequality is trivially true. Assume that $u(x) < +\infty$. Since $u(x) \geq \mathcal{M}_u^r(x)$ for all $0 < r < \rho$,

$$u(x) \geq \mathcal{M}_u^r(x) = \frac{1}{S_n r^{n-1}}\int_{\partial B(x,r)} u(y)\,d\,s(y) \qquad (0 < r < \rho)$$

$$\text{or} \quad S_n r^{n-1} u(x) \geq \int_{\partial B(x,r)} u(y)\,d\,s(y) \qquad (0 < r < \rho).$$

By integrating with respect to r over $(0,\rho)$,

$$\frac{S_n \rho^n}{n}\,u(x) \geq \int_0^{\rho}\left(\int_{\partial B(x,r)} u(y)\,d\,s(y)\right)dr = \int_{B(x,\rho)} u(y)\,dy$$

$$u(x) \geq \frac{n}{\rho^n}\int_0^{\rho}\left(\frac{1}{S_n}\int_{\partial B(x,r)} u(y)\,d\,s(y)\right)dr$$

$$= \frac{n}{\rho^n}\int_0^{\rho} r^{n-1}\,\mathcal{M}_u^r(x)\,dr$$

$$= \mathcal{A}_u^{\rho}(x).$$

6.4.10 Theorem. If u is superharmonic on an open set $\Omega \subset \mathbb{R}^n$, then u is finite a.e. on Ω relative to Labesgue measure and locally integrable (with respect to Labesgue measure) on Ω.

Proof : It suffices to show that u is finite a.e. on each of the components of Ω. We might as well assume that Ω is connected. Since u is not identically $+\infty$ on Ω, there is at least one point of Ω where u is finite. Let

$$M = \{x \in \Omega : \text{u is finite a.e. on } B(x,\rho) \subset \bar{B}(x,\rho) \subset \Omega \text{ for some } \rho > 0\}.$$

M is nonempty since there is at least one point of Ω where u is finite and, according to (6.4.9), u is finite a.e. on each ball in Ω having this point as its center and its closure is in Ω. We first show that M is open. Suppose $x \in M$. Then u is finite a.e. on $B(x,\rho) \subset \bar{B}(x,\rho) \subset \Omega$ for some $\rho > 0$. Consider any $y \in B(x,\rho)$ and the ball $B(y,\delta)$, where $3\delta = \min\{\|y-x\|, \rho - \|y-x\|\}$. Then $B(y,\delta) \subset B(x,\rho)$ and u is finite a.e. on $B(y,\delta)$; that is, $y \in M$. This shows that $B(x,\rho) \subset M$ and that M is open. We next show that M is relatively closed in Ω. Let (x_j) be a sequence in M with $x_j \to x \in \Omega$ as $j \to +\infty$. Since Ω is open, there is an $\varepsilon > 0$ such that $B(x,\varepsilon) \subset \bar{B}(x,\varepsilon) \subset \Omega$. Choose j_o so that $x_{j_o} \in B(x,\varepsilon/2)$. Since $x_{j_o} \in M$, there is a $\rho > 0$ such that u is finite a.e. on $B(x_{j_0}, \rho)$. In particular, u is finite a.e. on $B(x_{j_0}, \rho) \cap B(x, \varepsilon/2)$ which has positive Lebesgue measure. It follows that there is a point $z \in B(x_{j_0}, \rho) \cap B(x, \varepsilon/2)$ such that $u(z) < +\infty$. By (6.4.9) u is finite a.e. on $B(z, \varepsilon/2) \subset \bar{B}(z, \varepsilon/2) \subset \Omega$. Since $x \in B(z, \varepsilon/2)$, there is a ball $B(x, \delta) \subset \bar{B}(x, \delta) \subset B(z, \varepsilon/2)$ on which u is finite a.e.. This show that $x \in M$ and that M is relatively closed. Since $M \neq \emptyset$, $M = \Omega$ by the connectedness of Ω. By definition of M to each $x \in M = \Omega$ there corresponds a $B(x, \rho_x) \subset \bar{B}(x, \rho_x) \subset \Omega$ on which u is finite a.e.. Since u is finite a.e. on each element of a countable covering of Ω, u is finite a.e. on Ω.

Next, we will show that u is locally integrable on Ω. Consider any compact set K in Ω, there is a finite number of balls with centres x_j and radii ρ_j such that $\bar{B}(x_j, \rho_j) \subset \Omega, j=1,\ldots,m$ and $K \subset \bigcup_{j=1}^{m} B(x_j, \rho_j)$. Since u is finite a.e. on arbitrarily small balls containing each x_j, we can assume that $u(x_j) < +\infty$. Then $+\infty > u(x_j) \geq \mathcal{A}_u^{\rho_j}(x_j)$ for each j. Since u is bounded below on $\bigcup_{j=1}^{m} \bar{B}(x_j, \rho_j)$ we can assume that $u \geq 0$ on this set. Then

$$-\infty < \int_K u(y)dy \leq \sum_{j=1}^{m} \int_{B(x_j, \rho_j)} u(y)dy \leq \sum_{j=1}^{m} \frac{S_n \rho_j^n}{n} u(x_j) < +\infty$$

and u is locally integrable on Ω.

6.4.11 Theorem. If u is superharmonic on an open set $\Omega \subset \mathbb{R}^n$ and $B = B(x,\rho)$ is a ball with $\bar{B} \subset \Omega$, then I_u is harmonic on B and $u \geq I_u$ on B.

Proof : We can assume that $u \geq 0$ on ∂B. Since u is l.s.c. on ∂B, there is a sequence (f_m) of non-negative continuous functions on ∂B such that $f_m \uparrow u$ on ∂B (5.1.11). Let

$$v_m = \begin{cases} I_{f_m} & \text{on } B, \\ f_m & \text{on } \partial B. \end{cases}$$

Since $u \geq v_m = f_m$ on ∂B, v_m is continuous on $\bar{B}$ and v_m is harmonic on B (6.3.2), $u \geq v_m$ on B by (6.4.8). Now (v_m) is an increasing sequence of functions harmonic on B and $v = \lim_{m \to +\infty} v_m$ is either identically $+\infty$ or harmonic on B (see [4], p.33). Since u is finite a.e. on Ω (6.4.10) and $u \geq v$, v is harmonic on B. It also follow from the Lebesgue monotone convergence theorem (5.2.6) and $f_m \uparrow u$ that $\lim_{m \to +\infty} I_{f_m} = I_u$ on B, i.e., $v = I_u$ on B.

6.4.12 Theorem. If u is superharmonic on an open set $\Omega \subset \mathbb{R}^n$ and $B = B(x,\rho)$ is a ball with $\bar{B} \subset \Omega$, then $\mathcal{M}_u^{\delta}(x)$ is a monotone decreasing function of δ on $(0,\rho]$ and

$$\lim_{\delta \to 0^+} \mathcal{M}_u^{\delta}(x) = u(x).$$

Proof : Suppose $0 < \delta \leq \rho$. Define a function v on $B(x,\rho)$ by $v = I_u$ on $B(x,\rho)$. Then $u \geq v$ on $B(x,\rho)$ by (6.4.11). Since $\bar{B}(x,\delta) \subseteq \bar{B}(x,\rho) \subset \Omega$ and v is harmonic on the latter ball,

$$\mathcal{M}_u^{\delta}(x) \geq \mathcal{M}_v^{\delta}(x) = v(x) = \mathcal{M}_u^{\rho}(x) .$$

This shows that $\mathcal{M}_u^{\delta}(x)$ is monotone decreasing on $(0,\rho]$. Since u is l.s.c. at x, $\{y : u(y) > u(x)-\varepsilon\}$ is a neighborhood of x for each $\varepsilon > 0$ and

$$\mathcal{M}_u^{\delta}(x) \geq \mathcal{M}_{u(x)-\varepsilon}^{\delta}(x) = u(x)-\varepsilon$$

for all sufficiently small δ. This shows that

$$\mathcal{M}_u^{\delta}(x) \uparrow u(x) \quad \text{as} \quad \delta \downarrow 0^+.$$

6.4.13 Theorem. If u is superharmonic on an open set $\Omega \subset \mathbb{R}^n$ and w is a relatively compact open set in Ω, then there exists an increasing sequence (u_m) of C^{∞} superharmonic functions on w such that $u = \lim_{m \to +\infty} u_m$ on w.

Proof : Take another relatively compact open set Ω_1, such that $w \subset \bar{w} \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$. Without loss of generality we can assume that $u \geq 0$ on $\bar{\Omega}_1$. Let $0 < \epsilon < \text{dist}\,(\bar{w}, \Omega_1^c)$. For every $x \in w$ and for every positive integer m, we define

$$(a) \qquad u_m(x) = u * \sigma_{\epsilon/m}(x) = \int_{\|y-x\| < \frac{\epsilon}{m}} u(y)\, \sigma_{\frac{\epsilon}{m}}(x-y)dy,$$

where $\sigma_{\frac{\epsilon}{m}}$ is the Schwartz function.

Step I $\quad u_m \in C^\infty(w)$.

Using (5.2.8), we can show that for every multi-index r,

$$\partial^r u_m(x) = \int_{\|y-x\| < \frac{\epsilon}{m}} u(y)\, \partial_x^r \sigma_{\frac{\epsilon}{m}}(x-y)dy \qquad (x \in w).$$

Hence $u_m \in C^\infty(w)$.

Step II $\quad u_m$ is superharmonic on w.

It is enough to prove that $u_m(x_0) \geq \mathcal{M}_{u_m}^{\rho}(x_0)$ for all m, whenever $\bar{B}(x_0,\rho) \subset w$. For any ball $B(x_0,\rho)$ such that $\bar{B}(x_0,\rho) \subset w$, we have

$$\int_{\|z\|=\rho} u_m(z+x_o)ds(z) = \int_{\|z\|=\rho}\left(\int_{\|y\|<\frac{\varepsilon}{m}} u(z+x_o-y)\,\sigma_{\frac{\varepsilon}{m}}(y)dy\right)ds(z)$$

$$= \int_{\|y\|<\frac{\varepsilon}{m}}\left(\int_{\|z\|=\rho} u(z+x_o-y)ds(z)\right)\sigma_{\frac{\varepsilon}{m}}(y)dy$$

$$\leq \int_{\|y\|<\frac{\varepsilon}{m}} S_n\,\rho^{n-1}u(x_o-y)\,\sigma_{\frac{\varepsilon}{m}}(y)dy,$$

according to (a),(5.2.9) and (6.4.2), respectively. Therefore

$$\mathcal{M}^{\rho}_{u_m}(x_o) = \frac{1}{S_n\rho^{n-1}}\int_{\|z\|=\rho} u_m(z+x_o)ds(z) \leq \int_{\|y\|<\frac{\varepsilon}{m}} u(x_o-y)\,\sigma_{\frac{\varepsilon}{m}}(y)dy = u_m(x_o).$$

<u>Step III</u> $u_m \leq u$ for all m.

Consider any $x_o \in w$. If $u(x_o) = +\infty$, then there is nothing to prove. Assume that $u(x_o) < +\infty$. Then

$$u_m(x_o) = \int_{\|y\|<\frac{\varepsilon}{m}} u(x_o-y)\,\sigma_{\frac{\varepsilon}{m}}(y)dy = \int_0^{\frac{\varepsilon}{m}} dr\left(\int_{\|y\|=r} u(x_o-y)\,\sigma_{\frac{\varepsilon}{m}}(y)ds(y)\right)$$

$$= \int_0^{\frac{\varepsilon}{m}} \sigma_{\frac{\varepsilon}{m}}(r)dr\left(\int_{\|y\|=r} u(x_o-y)ds(y)\right)$$

$$= \int_0^{\frac{\varepsilon}{m}} S_n r^{n-1} \sigma_{\frac{\varepsilon}{m}}(r)dr \left(\frac{1}{S_n r^{n-1}} \int_{\|y\|=r} u(x_o-y)ds(y) \right).$$

Therefore,

$$\text{(b)} \quad u_m(x_o) = \int_0^{\frac{\varepsilon}{m}} S_n r^{n-1} \mathcal{M}_u^r (x_o) \; \sigma_{\frac{\varepsilon}{m}}(r)dr$$

$$\leq \int_0^{\frac{\varepsilon}{m}} S_n r^{n-1} \sigma_{\frac{\varepsilon}{m}}(r)dr \cdot u(x_o)$$

$$= u(x_o),$$

where the inequality and the final equality follow from (6.4.11) and (6.2.2), respectively.

<u>Step IV</u> $u_m \leq u_{m+1}$ for all m.

Put $\rho = \frac{m}{m+1} r$. Therefore

$$u_m(x) = \int_0^{\frac{\varepsilon}{m+1}} S_n \left(\frac{m+1}{m}\right)^{n-1} \rho^{n-1} \mathcal{M}_u^{\frac{(m+1)\rho}{m}} (x) \; \sigma_{\frac{\varepsilon}{m}}\left(\frac{m+1}{m}\rho\right)\left(\frac{m+1}{m}\right) d\rho$$

$$= \int_0^{\frac{\varepsilon}{m+1}} S_n \rho^{n-1} \mathcal{M}_u^{\frac{(m+1)\rho}{m}} (x) \; \sigma_{\frac{\varepsilon}{m+1}}(\rho) \, d\rho \;,$$

since $\sigma_{\frac{\varepsilon}{m}}\left(\frac{m+1}{m}\rho\right) = \left(\frac{m}{m+1}\right)^n \sigma_{\frac{\varepsilon}{m+1}}(\rho)$. According to (6.4.11) and (b), we get

$$u_m(x) \leq \int_0^{\frac{\varepsilon}{m+1}} s_n \rho^{n-1} \mathcal{M}_u^{\rho}(x)\, \sigma_{\frac{\varepsilon}{m+1}}(\rho)\, d\rho$$

$$= u_{m+1}(x).$$

Step V $\quad u_m \longrightarrow u$ as $m \to +\infty$.

It is enough to prove that for any $x \in w$, if $k < u(x)$, then there exists a positive integer m_o such that $u_{m_o}(x) > k$. If $k < u(x)$, by the l.s.c. of u there exists a neighborhood U of x such that $u(y) > k$ for all $y \in U$, which imples that $u(y) \geq \mathcal{M}_u^{\rho}(y) > k$ for all ρ such that $B(y,\rho) \subset \bar{B}(y,\rho) \subset U$. For any $\varepsilon > 0$, let m_o be such that $\varepsilon/m_o < \rho$. Then by (b) and (6.2.2),

$$u_{m_o}(y) = \int_0^{\varepsilon/m_o} s_n \rho^{n-1} \mathcal{M}_u^{\rho}(y)\, \sigma_{\varepsilon/m_o}(\rho)\, d\rho$$

$$> k \int_0^{\varepsilon/m_o} s_n \rho^{n-1} \sigma_{\varepsilon/m_o}(\rho)\, d\rho$$

$$= k.$$

Since for any $x \in w$, the $\lim_{m \to +\infty} u_m(x)$ exists, we have $\lim_{m \to +\infty} u_m(x) = u(x)$ for all $x \in w$.

6.4.14 <u>Theorem</u>. Let $u \in C^2(\Omega)$. Then u is superharmonic on Ω if and only if $\Delta u \leq 0$ on Ω.

<u>Proof</u> : Let $x_0 \in \Omega$ and $\rho_0 > 0$ be such that $\bar{B}(x_0, \rho_0) \subset \Omega$. Then for all $\rho \leq \rho_0$, we have by Green's identity that

$$\int_{\|x-x_0\| < \rho} \Delta u(x)dx = \int_{\|y-x_0\| = \rho} D_n u(y) d\, s(y)$$

$$= \int_{\|t\|=1} \rho^{n-1} D_n u(x_0+\rho t) d\, s(t)$$

$$= \rho^{n-1} \int_{\|t\|=1} \frac{\partial u}{\partial \rho} (x_0+\rho t) d\, s(t)$$

$$= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{\|t\|=1} u(x_0+\rho t) d\, s(t)$$

$$\text{(a)} \qquad = S_n \rho^{n-1} \frac{\partial}{\partial \rho} \mathcal{M}_u^{\rho}(x_0).$$

Now, if u is superharmonic on Ω, then by (6.4.12) we have that $\mathcal{M}_u^{\rho}(x_0)$ is a monotone decreasing function of ρ and hence by (a),

$$\int_{\|x-x_0\|<\rho} \Delta u(x)dx \quad \leq \quad 0 \qquad\qquad (\rho \leq \rho_0).$$

It then follows from the continuity of Δu that $\Delta u(x_0) \leq 0$.

Suppose conversely that $\Delta u \leq 0$ on Ω. Then by (a) we get that $\mathcal{M}_u^\rho(x_0)$ is a monotone decreasing function of ρ on $(0,\rho_0]$. Since continuity of u alone implies $\mathcal{M}_u^\rho(x_0) \to u(x_0)$ as $\rho \to 0^+$. Hence we conclude that

$$\mathcal{M}_u^\rho(x_0) \quad \leq \quad u(x_0) \qquad\qquad (0 < \rho \leq \rho_0).$$

6.4.15 <u>Theorem</u>. If u is superharmonic on Ω, then $\Delta T_u \leq 0$ on Ω.

In order to prove the theorem we need

6.4.16 <u>Lemma</u>. If $u \in C^2(\Omega)$ and superharmonic on Ω, then $\Delta T_u \leq 0$.

<u>Proof</u> : u is locally integrable on Ω, since $u \in C^2(\Omega)$. Then T_u exists. Take any $\varphi \in \mathcal{D}(\Omega)$ and $\varphi \geq 0$, we have

$$\Delta T_u(\varphi) \quad = \quad T_u(\Delta\varphi) \quad = \quad \int_\Omega u\,\Delta\varphi\,dx.$$

Since $\mathrm{Supp}(\varphi)$ is compact, there exists $r > 0$ such that $\mathrm{Supp}(\varphi) \subset B(0,r)$. Thus

$$\Delta T_u(\varphi) = \int_{\|x\|<r} u\,\Delta\varphi\,dx$$

$$= \int_{\|x\|<r} (u\,\Delta\varphi - \varphi\,\Delta u)dx + \int_{\|x\|<r} \varphi\,\Delta u dx$$

$$\text{(a)} \qquad = \int_{\partial B(0,r)} (u\,D_n\varphi - \varphi D_n\,u)ds + \int_{\|x\|<r} \varphi\,\Delta u dx\,.$$

Since ∂B is outside the support of φ, we have that the first integral in (a) must vanish. Therefore

$$\Delta T_u(\varphi) = \int_{\|x\|<r} \varphi\,\Delta u dx \leq 0, \quad \text{since} \quad \Delta u \leq 0\,.$$

That is, $\Delta T_u \leq 0$.

<u>Proof of the theorem</u> : It is enough to prove that $\Delta T_u \leq 0$ in an open set w which is relatively compact in Ω. By (6.4.12) there exists an increasing sequence (u_m) of C^∞ superharmonic functions on w such that $u = \lim_{m\to+\infty} u_m$ on w. Then for all $\varphi \in \mathcal{D}(w)$, we have

$$T_{u_m}(\varphi) = \int_w u_m(x)\,\varphi(x)dx \qquad \text{for all } m$$

and $$T_u(\varphi) = \int_w u(x)\,\varphi(x)dx\,.$$

Since $u_m \uparrow u$, and u_m and u are integrable on w, we have, by the Lebesgue's monotone convergence theorem (5.2.6),

$$\lim_{m \to +\infty} T_{u_m}(\varphi) = T_u(\varphi) \qquad (\varphi \in \mathcal{D}(w)).$$

Replacing φ by $\Delta\varphi$, we get

$$\lim_{m \to +\infty} T_{u_m}(\Delta\varphi) = T_u(\Delta\varphi)$$

or equivalently by (3.4.1)

$$\lim_{m \to +\infty} \Delta T_{u_m}(\varphi) = \Delta T_u(\varphi) \qquad (\varphi \in \mathcal{D}(w)).$$

Since $u_m \in C^2(w)$ and superharmonic on w, we have, by (6.4.16), that $\Delta T_{u_m} \leq 0$ for all m.

Hence $\Delta T_u \leq 0$.

6.4.17 <u>Lemma</u>. If u, v are superharmonic functions on Ω and $T_u = T_v$ on Ω, then u = v on Ω.

<u>Proof</u> : For any point $x_0 \in \Omega$, let $\varepsilon_0 > 0$ be such that $B(x_0, \varepsilon_0) \subset \Omega$. Take $0 < \varepsilon < \varepsilon_0$, we can see that

$$\int_{\Omega} u(y)\,\sigma_{\varepsilon}(y-x_o)\,dy = \int_{\|z\|<\varepsilon} u(x_o+z)\,\sigma_{\varepsilon}(z)\,dz$$

$$= \int_0^{\varepsilon} dr\,\sigma_{\varepsilon}(r) \int_{\|z\|=r} u(x_o+z)\,ds(z)$$

$$= \int_0^{\varepsilon} S_n r^{n-1}\,\sigma_{\varepsilon}(r)\,\mathcal{M}_u^r(x_o)\,dr.$$

By (6.4.12) and (6.2.2), we obtain

$$\mathcal{M}_u^{\varepsilon}(x_o) \leq \int_0^{\varepsilon} S_n r^{n-1}\,\sigma_{\varepsilon}(r)\,\mathcal{M}_u^r(x_o)\,dr$$

$$\leq u(x_o) \int_0^{\varepsilon} S_n r^{n-1}\,\sigma_{\varepsilon}(r)\,dr$$

$$= u(x_o),$$

i.e.,

$$\mathcal{M}_u^{\varepsilon}(x_o) \leq \int_{\Omega} u(y)\,\sigma_{\varepsilon}(y-x_o)\,dy \leq u(x_o).$$

Similarly for v,

$$\mathcal{M}_v^{\varepsilon}(x_o) \leq \int_{\Omega} v(y)\,\sigma_{\varepsilon}(y-x_o)\,dy \leq v(x_o).$$

Since $T_u = T_v$, we have

$$\int_{\Omega} u(y)\,\delta_{\varepsilon}(y-x_0)\,dy = \int_{\Omega} v(y)\,\delta_{\varepsilon}(y-x_0)\,dy.$$

Hence $\mathcal{M}_v^{\varepsilon}(x_0) \leq u(x_0)$,

and $\mathcal{M}_u^{\varepsilon}(x_0) \leq v(x_0)$.

Taking $\varepsilon \to 0^+$, by (6.4.12) we have

$$v(x_0) \leq u(x_0) \quad \text{and} \quad u(x_0) \leq v(x_0).$$

Then we conclude that $u(x_0) = v(x_0)$.

6.5 Potentials

6.5.1 <u>Definition</u>. Let m be a positive integer and let $\tau_m = \min(\tau, m)$ where τ is the fundamental harmonic function (Note that τ_m is real continuous and superharmonic on $\mathbb{R}^n$). Let μ be a positive Radon measure on $\mathbb{R}^n$ with compact support. We define

$$U_m^{\mu}(x) = \int_{\mathbb{R}^n} \tau_m(x-y)\,d\mu(y) \qquad (x \in \mathbb{R}^n).$$

6.5.2 Lemma. (U_m^{μ}) is an increasing sequence of continuous superharmonic functions on $\mathbb{R}^n$.

Proof : Since $\tau_m \leq \tau_{m+1}$, we have that $U_m^{\mu} \leq U_{m+1}^{\mu}$. If K is a compact subset of $\mathbb{R}^n$, then for all $x \in K$ and for all $y \in \mathrm{Supp}(\mu)$, $\tau_m(x-y)$ is bounded. And since μ has a compact support, we conclude that U_m^{μ} is continuous on $\mathbb{R}^n$. Finally

$$\mathcal{M}^{\rho}_{U_m^{\mu}}(x_o) = \frac{1}{S_n \rho^{n-1}} \int_{\|t\|=1} U_m^{\mu}(x_o+\rho t)ds(t)$$

$$= \frac{1}{S_n \rho^{n-1}} \int_{\|t\|=1} \int_{\mathbb{R}^n} \tau_m(x_o+\rho t-y)d\mu(y)ds(t).$$

By (5.2.9) and (6.4.2), we obtain

$$\mathcal{M}^{\rho}_{U_m^{\mu}}(x_o) = \int_{\mathbb{R}^n} \frac{1}{S_n \rho^{n-1}} \int_{\|t\|=1} \tau_m(x_o+\rho t-y)ds(t)d\mu(y)$$

$$\leq \int_{\mathbb{R}^n} \tau_m(x_o-y)d\mu(y).$$

Then

$$\mathcal{M}^{\rho}_{U_m^{\mu}}(x_o) \leq U_m^{\mu}(x_o).$$

This is true for all $\rho > 0$ and for all $x_o \in \mathbb{R}^n$. Hence U_m^{μ} is superharmonic on $\mathbb{R}^n$ (6.4.2).

6.5.3 Lemma. If $x_o \notin \mathrm{Supp}(\mu)$, then $\sup_m U_m^{\mu}(x_o) < +\infty$.

Proof : We first note that dist.$(x_o, \mathrm{Supp}(\mu)) > 0$. Let ε be a positive real number such that $\varepsilon <$ dist.$(x_o, \mathrm{Supp}(\mu))$. Choose $m > \tau((\varepsilon,0,\dots,0))$. Then for all $y \in \mathrm{Supp}(\mu)$,

$$\tau_m(x_o-y) = \min(\tau(x_o-y),m) = \tau(x_o-y)$$

$$\leq \tau((\varepsilon,0,\dots,0))$$

for all m. Therefore

$$U_m^{\mu}(x_o) = \int_{\mathbb{R}^n} \tau_m(x_o-y)d\mu(y) \leq \tau((\varepsilon,0,\dots,0))\mu(\mathrm{Supp}(\mu))$$

for all m. So

$$\sup_m U_m^{\mu}(x_o) \leq \tau((\varepsilon,0,\dots,0))\mu(\mathrm{Supp}(\mu)).$$

Since the right side is less than $+\infty$, we have the result.

6.5. 4 Theorem. $\sup_m U_m^{\mu}$ is superharmonic on $\mathbb{R}^n$.

Proof : From (6.5.2) and ([4], p.68), $\sup_m U_m^{\mu}$ is either identical to $+\infty$ or superharmonic on $\mathbb{R}^n$. By (6.5.3), we conclude that $\sup_m U_m^{\mu}$ is superharmonic on $\mathbb{R}^n$.

6.5.5 Definition. Let μ be a positive Radon measure on $\mathbb{R}^n$ with compact support. The potential U^{μ} of μ on $\mathbb{R}^n$ is given by

$$U^{\mu}(x) = \lim_{m \to +\infty} U_m^{\mu}(x) \qquad (x \in \mathbb{R}^n).$$

By the Lebesgue monotone convergence theorem,

$$\lim_{m \to +\infty} U_m^{\mu}(x) = \int_{\mathbb{R}^n} \tau(x-y)\,d\mu(y) \qquad (x \in \mathbb{R}^n).$$

So U^{μ} can be defined by

$$U^{\mu}(x) = \int_{\mathbb{R}^n} \tau(x-y)\,d\mu(y) \qquad (x \in \mathbb{R}^n).$$

6.5.6 Definition. Let S be a distribution on $\mathbb{R}^n$ with compact support. The distributional potential U^S of S is

$$U^S = S * T_{\tau}.$$

Note that the definition makes sense, since S has a compact support and T_{τ} is a distribution on $\mathbb{R}^n$ (τ is locally integrable on $\mathbb{R}^n$).

6.5.7 Theorem. $U^{T_{\mu}} = T_{U^{\mu}}$.

Proof : Since U^{μ} is superharmonic on $\mathbb{R}^n$, it is locally integrable on $\mathbb{R}^n$ (6.4.10). Therefore $T_{U^{\mu}}$ is a distribution on $\mathbb{R}^n$. For all

$\varphi \in \mathcal{D}(\mathbb{R}^n)$, by (4.2.3), we know that $T_{U^{\mu}} * \varphi$ and $U^{T_{\mu}} * \varphi$ are in $C^{\infty}(\mathbb{R}^n)$. Further by (4.2.2),

$$T_{U^{\mu}} * \varphi(x) = \int_{\mathbb{R}^n} U^{\mu}(y)\, \varphi(x-y)dy$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tau(y-z)d\mu(z)\varphi(x-y)dy.$$

By (5.2.9), we can see that

$$T_{U^{\mu}} * \varphi\,(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tau(y-z)\varphi(x-y)dyd\mu(z)\;.$$

Changing the variable, we have

$$T_{U^{\mu}} * \varphi\,(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tau(\xi)\varphi(x-\xi-z)d\xi\, d\mu(z)$$

$$= (T_{\mu})_z\;(\int_{\mathbb{R}^n} \varphi(x-\xi-z)\tau(\xi)d\xi\,)$$

$$= (T_{\mu})_z\;((T_{\tau})_{\xi}(\varphi(x-\xi-z)))$$

$$= (T_{\mu})_z\;((T_{\tau}*\varphi)(x-z))$$

$$= (T_{\mu}*(T_{\tau}*\varphi))(x).$$

Finally by (4.2.12) and (6.5.6), we conclude that

$$T_{U^{\mu}} * \varphi(x) = ((T_{\mu} * T_{\tau}) * \varphi)(x)$$

$$= T_{U^{\mu}} * \varphi(x).$$

This is true for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$. Replacing φ by $\check{\varphi}$ and taking $x = 0$, we get

$$T_{U^{\mu}} * \check{\varphi}(0) = T_{U^{\mu}} * \check{\varphi}(0)$$

or equivalently

$$T_{U^{\mu}}(\varphi) = T_{U^{\mu}}(\varphi) \qquad (\varphi \in \mathcal{D}(\mathbb{R}^n)),$$

whence the result.

6.5.8 Theorem. (Schwartz). Let T be a distribution on an open set $\Omega \subset \mathbb{R}^n$ such that $\Delta T \leq 0$ on Ω and let w be a relatively compact set in Ω. Then there exist a positive Radon measure μ on $\mathbb{R}^n$ with compact support and a harmonic function h on w such that

$$T = T_{U^{\mu}} + T_h = T_{U^{\mu}+h} .$$

<u>Proof</u> : The last equality follows immediately from (6.5.7). Let us prove the first one. Since $\Delta T \leq 0$ on Ω, we have $-\Delta T/k_n \geq 0$, where k_n is a positive constant (see 6.2.11). Then by (3.2.8), there is a positive Radon measure μ on $\mathbb{R}^n$ with compact support in Ω such that

$$T_\mu = -\Delta T/k_n \qquad \text{in } w.$$

Since U^{T_μ} is a distribution, we can take the laplacian

$$\Delta(U^{T_\mu}) = \Delta(T_\mu * T_\tau) = T_\mu * \Delta T_\tau .$$

Substituting ΔT_τ by $-k_n T_\delta$ (see 6.2.11), we have

$$\Delta(U^{T_\mu}) = T_\mu * (-k_n T_\delta) = -k_n (T_\mu * T_\delta)$$

$$= -k_n T_\mu .$$

Hence $\Delta(U^{T_\mu}) = \Delta T \qquad \text{in } w.$

or $\Delta(T - U^{T_\mu}) = 0 \qquad \text{in } w.$

By (6.2.12), there exists a harmonic function h on w such that

$$T - U^{T_\mu} = T_h \qquad \text{in } w.$$

Hence the first equality.

6.5.9 Theorem. (Riesz Decomposition Theorem). Let u be a superharmonic function on an open set $\Omega \subset \mathbb{R}^n$ and let w be a relatively compact set in Ω. Then

$$u(x) = \int_w \tau(x-y)\,d\mu(y) + h(x) \qquad (x \in w),$$

where μ is a positive Radon measure on $\mathbb{R}^n$ with compact support such that

$$T_\mu = -\frac{\Delta T_u}{k_n} \text{ in } w \ (k_n = \text{a constant, see (6.2.11)})$$

and h is a harmonic function on w.

Proof : By (6.4.16), $\Delta T_u \leq 0$, and hence by (6.5.8) we can get

$$T_u = T_{(U^\mu + h^*)} \qquad \text{on } w,$$

where h^* is a harmonic function on w and μ is as stated in the theorem. Since $U^\mu + h^*$ is superharmonic on w and by (6.4.17), we conclude that

$$u = U^\mu + h^* \qquad \text{on } w,$$

i.e.,

$$u(x) = \int_{\mathbb{R}^n} \tau(x-y)d\mu(y) + h^*(x)$$

$$= \int_{w} \tau(x-y)d\mu(y) + \int_{w^c} \tau(x-y)d\mu(y) + h^*(x)$$

$$= \int_{w} \tau(x-y)d\mu(y) + h(x),$$

where

$$h(x) = \int_{w^c} \tau(x-y)d\mu(y) + h^*(x) \qquad (x \in w).$$

The proof will be complete, if we show that h is harmonic on Ω. Since for any $y \notin \mathrm{Supp}(\mu)$, the function $x \longmapsto \tau(x-y)$ is harmonic on w. Hence by applying (5.2.8), we have

$$\Delta_{(x)} \int_{w^c} \tau(x-y)d\mu(y) = \int_{w^c} \Delta_{(x)} \tau(x-y)d\mu(y)$$

$$= 0 .$$

That is, $\int_{w^c} \tau(x-y)d\mu(y)$ is harmonic on w.