CHAPTER IV

CONVOLUTIONS

In this chapter we characterize the convolution of functions and of distributions, which we will use repeatedly in the last chapter.

The materials of this chapter are drawn from references [5], [7], [10], [11] and [12].

4.1 Convolution of Functions

4.1.1 <u>Definition</u>. Let f, g be two continuous functions in $\mathbb{R}^n$ with one having a compact support. The convolution $f*g$ of f and g is defined by

$$f*g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

4.1.2 <u>Remarks</u>. (i) $f*g = g*f$.

(ii) $(f*g)*h = f*(g*h)$(at least one has a compact support).

Proof :

$$\text{(ii)} \quad (f*g)*h(x) = \int \left[(f*g)(x-z)\right] h(z)dz$$
$$= \iint f(y)g(x-z-y)dyh(z)dz$$
$$= \iint f(y)g(x-y-z)h(z)dydz.$$
$$f*(g*h)(x) = \int f(y)\left[(g*h)(x-y)\right]dy$$
$$= \int f(y) \int g(x-y-z)h(z)dzdy$$
$$= \iint f(y)g(x-y-z)h(z)dzdy.$$

Since fgh is continuous and has bounded support, we can change the order of integration (apply Fubini's Theorem). Hence the result.

4.1.3 Theorem. Let f, g be as in (4.1.1). We have

$$\text{Supp}(f*g) \subseteq \text{Supp}(f) + \text{Supp}(g) \qquad \text{(vector sum).}$$

Moreover, if both Supp(f) and Supp(g) are compact, Supp(f*g) is also compact.

Proof : Let x belong to the complement of Supp(f)+Supp(g) ; then, for any $y \in$ Supp(g), x-y belongs to the complement of Supp(f), hence

$$f*g(x) = \int f(x-y)g(y)dy = 0.$$

Since both Supp(f) and Supp(g) are closed, their sum is closed and so must contain Supp(f*g).

Clearly, if both Supp(f) and Supp(g) are compact, so is Supp(f*g).

4.2 Convolution of Distributions

4.2.1 Definition. Let $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ (or $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$. The convolution of T and φ, denoted by $T*\varphi$, is given by

$$T*\varphi(x) = T_y(\varphi(x-y)) \qquad (x \in \mathbb{R}^n).$$

The notation T_y means that the distribution T acts on a function $\varphi(x-y)$ when the latter is regarded as a function of the variable y.

4.2.2 Remark. If f is a locally integrable function on $\mathbb{R}^n$, we have

$$T_f*\varphi(x) = (T_f)_y(\varphi(x-y)) = \int_{\mathbb{R}^n} f(y)\varphi(x-y)dy = f*\varphi(x).$$

4.2.3 Theorem. If $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ (or $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$), then for every multi-index r,

$$\partial^r(T*\varphi) = (\partial^r T)*\varphi = T*(\partial^r\varphi),$$

and $T * \varphi \in C^{\infty}(\mathbb{R}^n)$.

<u>Proof</u> : For any $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$, by Tayler's formula applied to the function $y \longmapsto \partial_y^r \varphi(x-y)$, we get :

$$\|h\|^{-1}\left\{\partial_y^r\varphi(x+h-y)-\partial_y^r\varphi(x-y)-\sum_{j=1}^{n} h_j \partial_j(\partial_y^r\varphi(x-y))\right\}$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \|h\|^{-1}\partial_{ij}^2(\partial_y^r\varphi(x+\gamma h-y)), \text{ where } 0 < \gamma < 1.$$

By commuting the order of differentiation (which is admissible by the infinite differentiability of φ)

$$\|h\|^{-1}\left\{\partial_y^r\varphi(x+h-y)-\partial_y^r\varphi(x-y)-\sum_{j=1}^{n} h_j \partial_y^r(\partial_j\varphi(x-y))\right\}$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \|h\|^{-1}\partial_{ij}^2(\partial_y^r\varphi(x+\gamma h-y))$$

(the subscribt y signifies that the partial derivatives of the function with respect to y). From this and continuity of $y \longmapsto \partial_y^r \partial_{ij}^2 \varphi(x-y)$, we have that for any compact set K, there is positive integer M_r such that

$$\|h\|^{-1}\left\{\partial_y^r\varphi(x+h-y)-\partial_y^r\varphi(x-y)-\sum_{j=1}^{n} h_j \partial_y^r(\partial_j\varphi(x-y))\right\}$$

$$= \frac{1}{2} M_r \sum_{i,j=1}^{n} h_i h_j \|h\|^{-1}$$

which tends to zero as $h \to 0$. Thus for every multi-index r,

$$\text{(a)}\quad y \longmapsto \|h\|^{-1}\left\{\partial_y^r\varphi(x+h-y)-\partial_y^r\varphi(x-y)-\sum_{j=1}^{n} h_j \partial_y^r(\partial_j\varphi(x-y))\right\}$$

converges uniformly to zero for $y \in K$ as $h \to 0$, Note that for a fixed x, if H is the support of $y \mapsto \varphi(x-y)$, then for all h with $\|h\| < 1$, and for every multi-index r, the supports of $y \mapsto \partial_y^r\varphi(x+h-y)$

are contained in the compact set $H_1 = \left\{ x: \text{dist.}(x,H) \leq 1 \right\}$.

Thus, by (a) with $H_1 = K$ and by (2.1.9), we have that for fixed x

$$y \longmapsto \|h\|^{-1}\left\{ \varphi(x+h-y) - \varphi(x-y) - \sum_{j=1}^{n} h_j \partial_j \varphi(x-y) \right\}$$

converges to zero in $\mathcal{D}(\mathbb{R}^n;H_1)$ (and hence in $\mathcal{D}(\mathbb{R}^n)$) as $h \longrightarrow 0$.

It follows that

$$\|h\|^{-1}\left\{ (T*\varphi)(x+h) - (T*\varphi)(x) - \sum_{j=1}^{n} h_j (T* \partial_j \varphi)\ (x) \right\}$$

$$= \|h\|^{-1}\left\{ T_y(\varphi(x+h-y)) - T_y(\varphi(x-y)) - \sum_{j=1}^{n} h_j T_y(\partial_j \varphi(x-y)) \right\}$$

$$= T_y\left\{ \|h\|^{-1}(\varphi(x+h-y) - \varphi(x-y) - \sum_{j=1}^{n} h_j \partial_j \varphi(x-y)) \right\}$$

converges to zero as $h \longrightarrow 0$. Thus $T*\varphi$ is differentiable and its partial derivatives are $T*(\partial_j \varphi)$. They can be expressed in another way, because

$$T*(\partial_j \varphi)(x) = T_y(\frac{\partial}{\partial x_j}\varphi(x-y)) = T_y(-\frac{\partial}{\partial y_j}\varphi(x-y))$$
$$= \frac{\partial T_y}{\partial y_j}(\varphi(x-y)) = (\partial_j T)*\varphi(x) .$$

Now this process can be repeated as often as we please and so by mathematical induction, we have

$$\partial^r (T*\varphi) = (\partial^r T)*\varphi = T*(\partial^r \varphi),$$

for every multi-index r and hence $T*\varphi \in C^\infty(\mathbb{R}^n)$.

4.2.4 Theorem. If $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ (or $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$), then

$$\mathrm{Supp}(T*\varphi) \subseteq \mathrm{Supp}(T)+\mathrm{Supp}(\varphi) \qquad \text{(vector sum).}$$

Moreover, if both Supp(T) and Supp(φ) are compact, so is Supp($T*\varphi$).

Proof : Let x belong to the complement of Supp(T)+Supp(φ) ; then, for any $y \in$ Supp(T), x-y belongs to the complement of Supp(φ), hence

$$T*\varphi(x) = T_y(\varphi(x-y)) = 0.$$

Since one of Supp(T), Supp(φ) is compact and the other closed, their sum is closed and so must contain Supp($T*\varphi$).

Clearly, if both Supp(T) and Supp(φ) are compact, so is Supp($T*\varphi$).

4.2.5 Definition. For any function φ, define $\check{\varphi}(x) = \varphi(-x)$.

4.2.6 Lemma. If $T \in \mathcal{D}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$(or $T \in \mathcal{E}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{E}(\mathbb{R}^n)$), then $T(\varphi) = T*\check{\varphi}(0)$.

Proof : $T*\check{\varphi}(0) = T_y(\check{\varphi}(-y)) = T_y(\varphi(y)) = T(\varphi)$.

4.2.7 Lemma. If $T \in \mathcal{D}'(\Omega)$, then there are regular distributions on Ω such that they approximate to T.

<u>Proof</u> : Let Θ_ε be a function of $\mathcal{D}(\Omega)$ that is equal to 1 for $\|x\| \leq \frac{1}{\varepsilon}$ and consider $\psi_\varepsilon = (\Theta_\varepsilon T) * \sigma_\varepsilon$. For all $\varphi \in \mathcal{D}(\Omega)$,

$$T_{\psi_\varepsilon}(\varphi) = \int \psi_\varepsilon(x)\varphi(x)dx = \int \psi_\varepsilon(x)\check{\varphi}(-x)dx$$

$$= \psi_\varepsilon * \check{\varphi}(0) = ((\Theta_\varepsilon T) * \sigma_\varepsilon) * \check{\varphi}(0)$$

$$= ((\Theta_\varepsilon T) * (\sigma_\varepsilon * \check{\varphi}))(0) = (\Theta_\varepsilon T)((\sigma_\varepsilon * \check{\varphi})^\vee)$$

$$= (\Theta_\varepsilon T)(\check{\sigma}_\varepsilon * \varphi) = T(\Theta_\varepsilon(\check{\sigma}_\varepsilon * \varphi)).$$

Now $\lim_{\varepsilon \to 0^+} \Theta_\varepsilon(x)(\check{\sigma}_\varepsilon * \varphi)(x) = \lim_{\varepsilon \to 0^+} \Theta_\varepsilon(x) \int \check{\sigma}_\varepsilon(y)\varphi(x-y)dy$

$$= \lim_{\varepsilon \to 0^+} \Theta_\varepsilon(x) \int \sigma_\varepsilon(y)\varphi(x-y)dy \quad (\text{since } \check{\sigma}_\varepsilon(y) = \sigma_\varepsilon(y))$$

$$= \lim_{\varepsilon \to 0^+} \Theta_\varepsilon(x)\ \varphi_\varepsilon(x)$$

$$= \lim_{\varepsilon \to 0^+} \Theta_\varepsilon(x) . \lim_{\varepsilon \to 0^+} \varphi_\varepsilon(x)$$

$$= \lim_{\varepsilon \to 0^+} \varphi_\varepsilon(x) = \varphi(x) .$$

Then $T_{\psi_\varepsilon} \longrightarrow T$ as $\varepsilon \to 0^+$.

4.2.8 <u>Theorem</u>. For any $T \in \mathcal{D}'(\Omega)$ there is a sequence (ψ_m) of functions of $\mathcal{D}(\Omega)$ such that, for all $\varphi \in \mathcal{D}(\Omega)$,

$$T(\varphi) = \lim_{m \to +\infty} \int \psi_m(x)\ \varphi(x)\ dx.$$

Proof : For any positive integer m, choose $\varepsilon_m = \frac{1}{m}$. Let $\psi_m = \eta_{1/m} = (\theta_{1/m} T) * \delta_{1/m}$. Then by (4.2.7), for every $\varphi \in \mathcal{D}(\Omega)$,

$$\lim_{m \to +\infty} \int \psi_m(x)\varphi(x)dx = \lim_{1/m \to 0^+} \int \eta_{1/m}(x)\varphi(x)dx = T(\varphi).$$

The above theorem is sometimes taken as part of an alternative definition of a distribution (generalized function).

4.2.9 Theorem. If $T \in \mathcal{D}'(\mathbb{R}^n)$ (or $T \in \mathcal{E}'(\mathbb{R}^n)$), then the convolution $\varphi \mapsto T * \varphi$ is a continuous linear map of $\mathcal{D}(\mathbb{R}^n)$ (or $\mathcal{E}(\mathbb{R}^n)$) into $\mathcal{E}(\mathbb{R}^n)$.

Proof : It will suffice to prove that, for every compact subset K of $\mathbb{R}^n$, $\varphi \mapsto T * \varphi$ is a continuous linear map of $\mathcal{D}(\mathbb{R}^n;K)$ into $\mathcal{E}(\mathbb{R}^n)$. The topology of $\mathcal{E}(\mathbb{R}^n)$ is defined by the seminorms

$$\varphi \longmapsto \max_{|p| \leq m} \sup_{x \in H} \left|\partial^p \varphi(x)\right|, \quad H, \text{ compact subset of } \mathbb{R}^n,\ m \geq 0.$$

For any $x \in H$ and for any φ such that $\mathrm{Supp}(\varphi) \subseteq K$, the function $y \mapsto \partial_x^p \varphi(x-y)$ varies in $\mathcal{D}(\mathbb{R}^n;H-K)$. The restriction of T to $\mathcal{D}(\mathbb{R}^n;H-K)$ is a continuous linear form on this space. Therefore, there is a constant $\beta > 0$ and an integer $m_0 \geq 0$ such that we have, for all $\psi \in \mathcal{D}(\mathbb{R}^n;H-K)$,

$$|T(\psi)| \leq \beta \max_{|q| \leq m_0} \sup_{y \in H-K} \left|\partial_y^q \psi(y)\right|$$

$$\left| T(\psi) \right| \leq \beta \max_{|q| \leq m_o} \sup_{y \in H-K} \left| \partial_y^q \psi(y) \right|$$

$$\leq \beta \max_{|q| \leq m_o} \sup_{y \in \mathbb{R}^n} \left| \partial_y^q \psi(y) \right| .$$

We replace $\psi(y)$ by $\partial_x^p \varphi(x-y)$ with $x \in H$ and $\mathrm{Supp}(\varphi) \subseteq K$. This yields

$$\left| T_y(\partial_x^p \varphi(x-y)) \right| \leq \beta \max_{|q| \leq m_o} \sup_{y \in \mathbb{R}^n} \left| \partial_y^q (\partial_x^p \varphi(x-y)) \right| ,$$

and by (3.4.1), we have

$$\left| \partial_x^p (T_y(\varphi(x-y)) \right| \leq \beta \max_{|r| \leq m_o+|p|} \sup_{z \in \mathbb{R}^n} \left| \partial^r \varphi(z) \right|$$

which implies that

$$\max_{|p| \leq m} \sup_{x \in H} \left| \partial^p (T*\varphi(x)) \right| \leq \beta \max_{|r| \leq m_o+|p|} \sup_{z \in \mathbb{R}^n} \left| \partial^r \varphi(z) \right| .$$

This proves the asserted continuity.

The other case can easily follow by the above one.

4.2.10 <u>Definition</u>. Let S and T be two distributions on $\mathbb{R}^n$ of which at least one has compact support. The convolution $S*T$ is defined by

$$(S*T)(\varphi) = (S*(T*\check{\varphi}))(0) = S_z(T_y(\varphi(y+z)))$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

4.2.11 <u>Theorem</u>. The convolution $S*T$ defined above is a distribution on $\mathbb{R}^n$.

<u>Proof</u> : We can assume that T has a compact support, so that $T \in \mathcal{E}'(\mathbb{R}^n)$. Now we first show that the convolution $S*T$ is a distribution on $\mathbb{R}^n$. By (4.2.9), the map $\varphi \longmapsto T*\varphi$ is continuous linear on $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{D}(\mathbb{R}^n)$, since Supp(T) is compact. Also, by (4.2.9), the map $T*\varphi \longmapsto S*(T*\varphi)$ is continuous linear on $\mathcal{D}(\mathbb{R}^n)$ into $\mathcal{E}(\mathbb{R}^n)$. So the composite map obtained from the sequence of mapping $\varphi \longmapsto \check{\varphi} \longmapsto T*\check{\varphi} \longmapsto S*(T*\check{\varphi}) \longmapsto (S*(T*\check{\varphi}))(0)$ is continuous linear on $\mathcal{D}(\mathbb{R}^n)$ into $\mathbb{R}$. Hence by the definition (4.2.10), the convolution $S*T$ is a distribution on $\mathbb{R}^n$.

Similarly we can show that the composite map obtained from the sequence of mapping $\varphi \mapsto \check{\varphi} \mapsto S*\check{\varphi} \mapsto T*(S*\check{\varphi}) \mapsto (T*(S*\check{\varphi}))(0)$ is continuous linear on $\mathcal{D}(\mathbb{R}^n)$ into $\mathbb{R}$. Hence by the definition (4.2.10), the convolution $T*S$ is a distribution on $\mathbb{R}^n$.

4.2.12 <u>Theorem</u>. If $S \in \mathcal{D}'(\mathbb{R}^n)$ and $T \in \mathcal{E}'(\mathbb{R}^n)$, then for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$(S*T)*\varphi = S*(T*\varphi)$$

and

$$(T*S)*\varphi = T*(S*\varphi).$$

Proof : Since $S \in \mathcal{D}'(\mathbb{R}^n)$ and $T*\varphi \in \mathcal{D}(\mathbb{R}^n)$, the convolution $S*(T*\varphi)$ makes sense. By (4.2.1)

$$S*(T*\varphi)\ (x) = S_z(T*\varphi(x-z)) = S_z(T_y(\varphi(x-y-z))).$$

Since S*T is a distribution on $\mathbb{R}^n$ (4.2.10) and by (4.2.11),

$$(S*T)*\varphi(x) = (S*T)_t(\varphi(x-t)) = S_z(T_y(\varphi(x-y-z))).$$

Hence $(S*T)*\varphi = S*(T*\varphi)$.

Similarly $T \in \mathcal{E}'(\mathbb{R}^n)$ and $S*\varphi \in \mathcal{E}(\mathbb{R}^n)$, the convolution $T*(S*\varphi)$ makes sense. By (4.2.1)

$$T*(S*\varphi)(x) = T_z(S*\varphi(x-z)) = T_z(S_y(\varphi(x-y-z))).$$

Since T*S is a distribution on $\mathbb{R}^n$ (4.2.10) and by (4.2.1),

$$(T*S)*\varphi(x) = (T*S)_t(\varphi(x-t)) = T_z(S_y(\varphi(x-y-z))).$$

Hence $(T*S)*\varphi = T*(S*\varphi)$.

4.2.13 Theorem. If $S \in \mathcal{D}'(\mathbb{R}^n)$ and $T \in \mathcal{E}'(\mathbb{R}^n)$, then $S*T = T*S$.

Proof : For any $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$,

$$\left[(S*T)*\varphi\right]*\psi = \left[S*(T*\varphi)\right]*\psi = S*\left[(T*\varphi)*\psi\right]$$

$$= S*\left[\psi*(T*\varphi)\right] = (S*\psi)*(T*\varphi)$$

$$= (T*\varphi)*(S*\psi) = T*[\varphi*(S*\psi)]$$

$$= T*[(S*\psi)*\varphi] = T*[S*(\psi*\varphi)]$$

$$= T*[S*(\varphi*\psi)] = T*[(S*\varphi)*\psi]$$

$$= [T*(S*\varphi)]*\psi = [(T*S)*\varphi]*\psi,$$

according to ([7], p.23), (4.2.12) and (4.1.2 (i)). Hence

$$(S*T)*\varphi = (T*S)*\varphi \qquad (\varphi \in \mathcal{D}(\mathbb{R}^n)).$$

Replacing φ by $\check{\varphi}$ and taking $x = 0$, we get

$$(S*T)*\check{\varphi}(0) = (T*S)*\check{\varphi}(0)$$

or equivalently by (4.2.6)

$$(S*T)(\varphi) = (T*S)(\varphi) \qquad (\varphi \in \mathcal{D}(\mathbb{R}^n)),$$

whence the result.

4.2.14 <u>Theorem</u>. If $S \in \mathcal{D}'(\mathbb{R}^n)$ and $T \in \mathcal{E}'(\mathbb{R}^n)$, then

$$\mathrm{Supp}(S*T) \subsetneq \mathrm{Supp}(S) + \mathrm{Supp}(T).$$

Moreover, if both S and T belong to $\mathcal{E}'(\mathbb{R}^n)$, then Supp(S*T) is compact.

<u>Proof</u> : If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is zero on the closed set $H = \mathrm{Supp}(S)+\mathrm{Supp}(T)$, then, for each $x \in \mathrm{Supp}(S)$, $T_y(\varphi(x+y))$ is zero, because the function $y \longmapsto \varphi(x+y)$ vanishes on Supp(T). Hence $S*T(\varphi) = S_x(T_y(\varphi(x+y))) = 0$.

Clearly, if both S and T belong to $\mathcal{E}'(\mathbb{R}^n)$, then Supp(S*T) is compact.

4.2.15 <u>Theorem</u>. If $S \in \mathcal{D}'(\mathbb{R}^n)$ and $T \in \mathcal{E}'(\mathbb{R}^n)$, then for every multi-index r

$$\partial^r(S*T) = (\partial^r S)*T = S*(\partial^r T).$$

<u>Proof</u> : For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\partial^r(S*T)*\varphi = (S*T)*(\partial^r\varphi) = S*(T*\partial^r\varphi)$$

$$= S*((\partial^r T)*\varphi) = (S*(\partial^r T))*\varphi,$$

according to (4.2.11),(4.2.3) and (4.2.12), whence the result.

4.2.16 <u>Theorem</u>. If $T \in \mathcal{D}'(\mathbb{R}^n)$, then $T*T_\delta = T_\delta*T = T$ and T_δ is called the convolution identity.

<u>Proof</u> : For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$T*T_\delta(\varphi) = (T*(T_\delta * \check{\varphi}))(0) = T_z((T_\delta)_y(\varphi(y+z))).$$

But by (3.1.2 (i)),

$$(T_\delta)_y\,(\varphi(y+z)) = \varphi(z).$$

Hence $\quad T*T_\delta(\varphi) = T_z(\varphi(z)) = T(\varphi)$

which implies the result.

4.2.17 Corollary. If $T \in \mathcal{D}'(\mathbb{R}^n)$, then for every multi-index r

$$\partial^r T = (\partial^r T_\delta) * T.$$

Proof : $\partial^r T = \partial^r (T_\delta * T) = (\partial^r T_\delta) * T.$