CHAPTER VII

EXISTENCE OF ROOM SQUARE OF ODD PRIME POWER.

7.1 Existence of Room Square of odd prime power.

Theorem 7.1.1 There exists a Room Square of side $p \neq 3, 5$ for all odd prime p.

Proof. Let p be any odd prime ; $p \neq 3$ or 5.

Since p is an odd prime, so by theorem A1 (see appendix) p can be written in the form (I) $p = 2^k t + 1$; where k is a positive integer ; t is an odd integer greater than 1, or (II) $p = 2^{2^k} + 1$; where k is a non - negative integer.

If p can be written in form (I), then by corollary 3.1.6 , there is a Room Square of side p.

If p can be written in form (II) , then by theorem 6.2.6 , there is a Room Square of side p unless k = 0,1,3 .

However, when k = 3, we have $p = 2^{2^k} + 1 = 257$, and we know from corollary 6.3.3 that a Room Square of side 257 exists.

Therefore the theorem follows.

Q.E.D.

Theorem 7.1.2 There are Room Squares of sides 3^n and 5^n for $n \neq 1$.

Proof. We first prove that there is a Room Square of side 3^n for $n \neq 1$. We shall prove by induction on $n \geq 2$.

If $n = 2$, then $3^n = 9$ and by Lemma 6.3.2, we know that a Room Square of side 9 exists.

Assume that there exists a Room Square of side 3^k.

Since $3^{k+1} = 3(3^k)$. Hence by theorem 5.1.2, there exists a Room Square of side 3^{k+1}.

Therefore, there exists a Room Square of side 3^n for all $n \geq 2$.

Next we shall show that there exists a Room Square of side 5^n for $n \neq 1$

Observe that for $n \geq 2$, we can write

$$5^n = (5^2)^\alpha (5^3)^\beta \; ; \; \alpha, \beta \text{ are non-negative integers.}$$

Since $5^2 = 2^3.3 + 1$, hence by theorem 3.1.5 there is a Room Square of side 5^2. Therefore by theorem 4.1.4, there is a Room Square of side $(5^2)^\alpha$.

Similarly we can show that Room Square of side $(5^3)^\beta$ exists by using theorem 3.2.2 and theorem 4.1.4.

Therefore by theorem 4.1.3, there is a Room Square of side $(5^2)^\alpha(5^3)^\beta = 5^n$ for $n \geq 2$.

Therefore the theorem follows

Q.E.D.

Theorem 7.1.3 There is a Room Square of side $p^n \neq 3, 5$ for all odd prime p.

Proof. Let p be any odd prime and $p^n \neq 3, 5$.

case 1. If $p \neq 3$, 5, then by theorem 7.1.1 , there is a Room Square of side p. So, by theorem 4.1.4, there is a Room Square of side p^n .

case 2. If $p = 3, 5$, then we must have $n \geq 2$, hence by theorem 7.1.2 Room Squares of sides 3^n and 5^n exist .

Q.E.D.