# CHAPTER II

# PRELIMINARIES

## 2.1 Hypergraphs

A hypergraph H is an ordered pair $(V, \mathcal{E})$, where V is a finite set and $\mathcal{E}$ is a set of non-empty subsets of V such that $\cup\mathcal{E} = V$. Any element v in V is called a vertex and any element E in $\mathcal{E}$ is called an edge. We allow the set V to be empty. When this is this case, such a hypergraph H will be called the empty hypergraph. The other hypergraphs will be called a non-empty hypergraph. For any positive integer r, an r-uniform hypergraph we mean a hypergraph in which every edge has cardinality r. A 2-uniform hypergraph is called a graph.

## 2.2 Isomorphisms

Let $H = (V,\mathcal{E})$ and $H_1 = (V_1,\mathcal{E}_1)$ be hypergraphs. A one-to-one mapping $\psi$ from V onto $V_1$ is called an isomorphism from H to $H_1$ if for each subset E of V,

$$E \text{ belongs to } \mathcal{E} \text{ if and only if } \psi[E] \text{ belongs to } \mathcal{E}_1.$$

Here, and in the sequal, $\psi[E]$ denotes the set $\{\psi(v)/v \in E\}$, the image of E under $\psi$. If there is an isomorphism from H to $H_1$, then we say that H is isomorphic to $H_1$ and we write $H \cong H_1$. When this is the case, we have $|V| = |V_1|$ and $|\mathcal{E}| = |\mathcal{E}_1|$. Here, and in the sequal, the notation $|S|$ will be used to denote the cardinality of the set S. If $\psi$ is an isomorphism from H into itself, then $\psi$ is called an automorphism of H.

2.2.1 Proposition Let $H = (V,\mathcal{E})$ be a hypergraph, $V_1$ be a set, f be a bijection from V to $V_1$, and $\mathcal{E}_1 = \{f[E] / E \in \mathcal{E}\}$. Then f is an isomorphism from H to $H_1$, where $H_1 = (V_1,\mathcal{E}_1)$.

Proof. This is clear from the definition of $\mathcal{E}_1$. #

## 2.3 Degree and Degree Sequence.

Let $H = (V,\mathcal{E})$ be a hypergraph. For each vertex v in V the degree of a vertex v, written $d_H(v)$, is the cardinality of the set $\{E \in \mathcal{E} / v \in E\}$, i.e.

$$d_H(v) = |\{ E \in \mathcal{E} / v \in E\}| .$$

2.3.1 Proposition. Let $\psi$ be an isomorphism from $H = (V,\mathcal{E})$ to $H_1 = (V_1,\mathcal{E}_1)$. Let v be any vertex in H. Then $d_H(v) = d_{H_1}(\psi(v))$.

Proof. Let $\psi$ be an isomorphism from $H = (V,\mathcal{E})$ to $H_1 = (V_1,\mathcal{E}_1)$. Let v be any vertex in H. Observe that

$$\begin{aligned} d_H(v) &= |\{ E \in \mathcal{E} / v \in E \}|, \\ &= |\{ \psi[E] / E \in \mathcal{E} \text{ and } v \in E \}| , \\ &= |\{ \psi[E] / E \in \mathcal{E} \text{ and } \psi(v) \in \psi[E]\}|, \end{aligned}$$

$$= |\{ E_1 / E_1 \in \mathcal{E}_1 \text{ and } \psi(v) \in E_1 \}| ,$$

$$= d_{H_1}(\psi(v)).$$

Therefore $d_H(v) = d_{H_1}(\psi(v))$. #

2.3.2 Proposition. Let $H = (V, \mathcal{E})$ be an r-uniform hypergraph. Then $\sum_{v \in V} d_H(v) = r \cdot |\mathcal{E}|$ .

Proof. Let $H = (V, \mathcal{E})$ be an r-uniform hypergraph. For each v in V, let

$$\mathcal{L}(v) = \{(v,E)/E \in \mathcal{E} \text{ and } v \in E \}.$$

For each E in $\mathcal{E}$, let

$$\mathcal{S}(E) = \{(v,E)/v \in E \}.$$

Observe that

(1) If $v \neq v'$, then $\mathcal{L}(v) \cap \mathcal{L}(v') = \emptyset$,

(2) If $E \neq E'$, then $\mathcal{S}(E) \cap \mathcal{S}(E') = \emptyset$,

(3) $\bigcup_{v \in V} \mathcal{L}(v) = \bigcup_{E \in \mathcal{E}} \mathcal{S}(E)$ ,

(4) $d_H(v) = |\mathcal{L}(v)|$

and

(5) $|\mathcal{S}(E)| = r$.

Hence

$$\sum_{v \in V} d_H(v) = \sum_{v \in V} |\mathcal{L}(v)| ,$$

$$= \Big| \bigcup_{v \in V} \mathcal{L}(v) \Big| ,$$

$$= \left| \bigcup_{E \in \mathcal{E}} \mathcal{S}(E) \right| ,$$

$$= \sum_{E \in \mathcal{E}} |\mathcal{S}(E)| ,$$

$$= \sum_{E \in \mathcal{E}} r ,$$

$$= r \cdot |\mathcal{E}| .$$

Therefore $\sum_{v \in V} d_H(v) = r \cdot |\mathcal{E}|$ . #

Let H be a hypergraph with vertices $v_1, v_2, \ldots, v_p$. If $d_H(v_i) = d_i$, $i = 1,2,3,\ldots,p$, we say that $\Pi = (d_1, d_2, \ldots, d_p)$ is a <u>degree sequence of</u> H. A finite sequence of non-negative integers $\Pi = (d_1, d_2, \ldots, d_p)$ is said to be an <u>r-degree sequence</u> if there exists an r-uniform hypergraph H such that $\Pi$ is a degree sequence of H.

2.3.3 <u>Proposition.</u> If $\Pi = (d_1, d_2, \ldots, d_p)$ is an r-degree sequence, then $\sum_{i=1}^{p} d_i$ is divisible by r.

<u>Proof.</u> Let $\Pi = (d_1, d_2, \ldots, d_p)$ be an r-degree sequence. Hence there exists an r-uniform hypergraph $H = (V, \mathcal{E})$ such that $\Pi$ is an r-degree sequence of H. By Proposition 2.3.2 , we have

$$\sum_{i=1}^{p} d_i = \sum_{v \in V} d_H(v) ,$$

$$= r \cdot |\mathcal{E}| .$$

Hence $\sum_{i=1}^{p} d_i$ is divisible by r. #