CHAPTER II

REAL SEMIGROUP ACTIONS:

The purpose of this chapter is to classify the analytic semigroup actions of the semigroup $\mathbb{R}$ on itself up to local isomorphism.

Let $(\mathbb{R},\cdot)$ be the usual semigroup with zero. Let ψ be the analytic semigroup action of the usual semigroup with zero on itself such that $\psi(0,x) = 0$. That is, $\psi: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ has properties:

(1) $\psi(st,x) = \psi(s,\psi(t,x)) \quad \forall\, s,t,\ x \in \mathbb{R}$

(2) $\psi(0,x) \equiv 0$

(3) $\psi(t,x) = \sum_{m,n=0}^{\infty} C_{mn}t^m x^n$ in some neighborhood of $(0,0)$.

From (3) we have

(2.1) $\psi(t,x) = C_{00}+C_{10}t+C_{01}x+C_{20}t^2+C_{11}tx+C_{02}x^2+C_{30}t^3+C_{21}t^2x$
$+ C_{12}tx^2+C_{03}x^3+ \ldots\ldots\ldots\ldots .$

Now, since $\psi(0,x) = 0$, therefore substituting $t = 0$ in (2.1) we get that $0 = \psi(0,x) = C_{00}+C_{01}x+C_{02}x^2+C_{03}x^3+ \ldots\ldots\ldots$ This implies that $C_{0n} = 0 \quad \forall\, n = 0,1,2,3,\ldots\ldots$. Because $\psi(0,x) = 0$, it follows that for any t belonging to $\mathbb{R}$, $0 = \psi(t\cdot 0,x) = \psi(t,\psi(0,x)) = \psi(t,0)$. Hence $\psi(t,0) = 0 \quad \forall\, t$.

Since $\psi(t,0) = 0 \quad \forall\, t$, the same proof as before gives us the result that $C_{m0} = 0 \quad \forall\, m = 1,2,3,4,\ldots\ldots$ Because we have $C_{0n} = 0 \quad \forall\, n = 0,1,2,3,\ldots$ and $C_{m0} = 0 \quad \forall\, m = 1,2,3,4,\ldots\ldots$, it follows that (2.1) is reduced to :

$$\psi(t,x) = C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+$$

$$C_{14}tx^4+C_{23}t^2x^3+C_{32}t^3x^2+C_{41}t^4x+\dots\dots\dots . \tag{2.2}$$

Now, we shall see what relations between the coefficients C_{mn} are determined by property (1) above.

$$\psi(st,x) = C_{11}(st)x+C_{12}(st)x^2+C_{21}(st)^2x+C_{13}(st)x^3+C_{22}(st)^2x^2$$

$$+C_{31}(st)^3x+C_{14}(st)x^4+C_{23}(st)^2x^3+C_{32}(st)^3x^2+$$

$$+C_{41}(st)^4x+\dots\dots\dots\dots .$$

$$= C_{11}stx+C_{12}stx^2+C_{21}s^2t^2x+C_{13}stx^3+C_{22}s^2t^2x^2+ \tag{2.3}$$

$$C_{31}s^3t^3x+C_{14}stx^4+C_{23}s^2t^2x^3+C_{32}s^3t^3x^2+C_{41}s^4t^4x$$

$$+\dots\dots\dots\dots .$$

We see that the coefficient of the term $s^kt^mx^n$ $(\forall k,m,n \in \mathbb{N})$ is equal to zero if $k \neq m$ and equal to C_{mn} if $k = m$.

$$\psi(s,\psi(t,x)) = C_{11}s\psi(t,x)+C_{12}s(\psi(t,x))^2+C_{21}s^2\psi(t,x)+C_{13}s(\psi(t,x))^3+$$

$$+C_{22}s^2(\psi(t,x))^2+C_{31}s^3\psi(t,x)+C_{14}s(\psi(t,x))^4$$

$$+C_{23}s^2(\psi(t,x))^3+C_{32}s^3(\psi(t,x))^2+C_{41}s^4\psi(t,x)+\dots\dots$$

$$\dots\dots\dots\dots\dots\dots$$

$$= C_{11}s(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\dots)+$$

$$C_{12}s(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\dots)^2+$$

$$C_{21}s^2(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\dots)+$$

$$C_{13}s(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\dots)^3+$$

$$C_{22}s^2(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\dots)^2+ \tag{2.4}$$

$$C_{31}s^3(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\dots)+$$

$$C_{14}s(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\dots)^4+$$

$$C_{23}s^2(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\ldots)^3+$$

$$C_{32}s^3(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\ldots)^2+$$

$$C_{41}s^4(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\ldots)+\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= C_{11}C_{11}stx + C_{11}C_{12}stx^2 + C_{11}C_{21}st^2x + C_{21}C_{11}s^2tx$$

$$C_{21}C_{21}s^2t^2x + C_{21}C_{12}s^2tx^2 + (C_{12}C_{11}^2 + C_{11}C_{22})st^2x^2 + \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

If $m \leqslant n$, the coefficient of the term $s^k t^m x^n$ in (2.4) equals:

$$C_{k1}C_{mn}$$

$$+\ C_{k2}\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+\ C_{k3}\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\ C_{km}\Big(\sum_{\substack{i_1+i_2+\ldots+i_m=m.\\ j_1+j_2+\ldots+j_m=n.}} C_{i_1j_1}C_{i_2j_2}\ldots\ldots\ldots\ldots C_{i_mj_m}\Big).$$

If $m > n$, the coefficient of the term $s^k t^m x^n$ in (2.4) equals:

$$C_{k1}C_{mn}$$

$$+\ C_{k2}\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+\ C_{k3}\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$+ \dots\dots\dots\dots\dots\dots\dots\dots$

$+ \dots\dots\dots\dots\dots\dots\dots\dots$

$$+ C_{kn}\Big(\sum_{\substack{i_1+i_2+\dots+i_n=m. \\ j_1+j_2+\dots+j_n=n.}} C_{i_1j_1} C_{i_2j_2} \cdots\cdots C_{i_nj_n} \Big).$$

Since (2.3) and (2.4) are equal, we get that the coefficients of the term $s^k t^m x^n (k,m,n \in \mathbb{N})$ in (2.3) and (2.4) must be equal.

With this as a background we are now ready to prove the following theorem:

Theorem: Let ψ be an analytic semigroup action of $(\mathbb{R},\cdot)$ with zero on $\mathbb{R}$. Then ψ is identically zero or ψ is locally analytically isomorphic to the analytic semigroup action ϕ , where $\phi(t,x) = t^{m_o}x$, for some $m_o \in \mathbb{N}$.

Proof: Now, considering (2.2), we have two cases:

case 1. We assume that $C_{m1} = 0 \quad \forall\, m = 1,2,3,4,\dots$. In this case we want to prove that $C_{mn} = 0 \quad \forall\, m, \forall\, n$. We shall prove this by induction on n.

Consider the following diagram:

$$\begin{array}{cccccccc}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & \cdots\cdots & C_{1n} & \cdots\cdots \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & \cdots\cdots & C_{2n} & \cdots\cdots \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & \cdots\cdots & \vdots & \cdots\cdots \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & \cdots\cdots & \vdots & \cdots\cdots \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & \cdots\cdots & \vdots & \cdots\cdots \\
\vdots & & & & & & \vdots & \\
C_{m1} & - - - & - - - & - - - & - - - & - - - & C_{mn} & - - - \\
\vdots & & & & & & \vdots & \\
1^{\underline{st}} \text{ column } \dots & & & & & & n^{\underline{th}} \text{ column.} &
\end{array}$$

Induction on n. (We prove induction on the columns.)

We assume that the first column is zero and we want to show that the $n^{\underline{th}}$ column is zero $\forall n \in \mathbb{N}$. By assumption $C_{m1} = 0 \; \forall m$. Now, we assume this true for all natural numbers less than n. We want to show $C_{mn} = 0 \;\; \forall m$. Consider the coefficients of the term $s^m t^m x^n$ in (2.3) and (2.4), respectively.

In (2.3) the coefficient of this term is C_{mn}.

In (2.4) we have two cases to consider:

<u>subcase 1.1</u> For $m \leq n$, the coefficient of this term is

$$
\begin{aligned}
& C_{m1}C_{mn} \\
& + C_{m2}\Big(\sum_{\substack{i_1+i_2=m. \\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2} \Big) \\
& + C_{m3}\Big(\sum_{\substack{i_1+i_2+i_3=m. \\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3} \Big) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + C_{mm-1}\Big(\sum_{\substack{i_1+i_2+\dots+i_{m-1}=m \\ j_1+j_2+\dots+j_{m-1}=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{m-1}j_{m-1}} \Big) \qquad (2.5) \\
& + C_{mm}\Big(\sum_{\substack{i_1+i_2+\dots+i_m=m. \\ j_1+j_2+\dots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_mj_m} \Big).
\end{aligned}
$$

Claim that the terms involving $C_{m\,m+1}$, $C_{m\,m+2}, \ldots\ldots$ do not appear. To see this, suppose that we have

$$
C_{mm+1}\Big(\sum_{\substack{i_1+i_2+\dots+i_{m+1}=m. \\ j_1+j_2+\dots+j_{m+1}=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{m+1}j_{m+1}} \Big).
$$

Because $i_\alpha \geq 1 \ \forall \alpha$ and $m = i_1+i_2+\ldots+i_{m+1}$ (m+1 terms) $\geq 1+1+\ldots+\ldots+1 = m+1$. This implies that $m \geq m+1$, a contradiction. Using the same proof as above, the other terms do not appear.

If $m < n$ then by assumption $C_{m1}, C_{m2}, C_{m3}, \ldots\ldots, C_{mm}$ are all zero. Now, assume that $m = n$. By the induction hypothesis, $C_{m1}, C_{m2}, C_{m3}, \ldots\ldots C_{m,m-1}$ are all zero. We want to show that the term

$$C_{mm}\Big(\sum_{\substack{i_1+i_2+\ldots+i_m=m.\\ j_1+j_2+\ldots+j_m=n.}} C_{i_1j_1} C_{i_2j_2} \ldots\ldots\ldots C_{i_mj_m}\Big) = 0 \text{ also.}$$

We see that $j_\beta = 1 \ \forall \beta$. Therefore, $C_{i_\alpha 1} = 0 \ \forall \alpha$. Hence the sum of the product is zero.

Therefore, the coefficient of the term $s^m t^m x^n$ $(m \leq n)$ in (2.4) equals zero.

Hence $C_{mn} = 0$.

<u>Subcase 1.2</u> For $m > n$, the coefficient of the term $s^m t^m x^n$ in (2.4) is

$$\begin{aligned}
& C_{m1} C_{mn} \\
&+ C_{m2}\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1} C_{i_2j_2}\Big) \\
&+ C_{m3}\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1} C_{i_2j_2} C_{i_3j_3}\Big) \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(2.6)\quad &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&+ C_{m\,n-1}\Big(\sum_{\substack{i_1+i_2+\ldots+i_{n-1}=m.\\ j_1+j_2+\ldots+j_{n-1}=n.}} C_{i_1j_1} C_{i_2j_2} \ldots C_{i_{n-1}j_{n-1}}\Big) \\
&+ C_{mn}\Big(\sum_{\substack{i_1+i_2+\ldots+i_n=m.\\ j_1+j_2+\ldots+j_n=n.}} C_{i_1j_1} C_{i_2j_2} \ldots C_{i_nj_n}\Big).
\end{aligned}$$

Claim that the terms involving $C_{mn+1}, C_{mn+2}, \ldots$ do not appear. To see this, we consider the term

$$C_{mn+1}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{n+1}=m.\\ j_1+j_2+\cdots+j_{n+1}=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{n+1}j_{n+1}}\Big).$$

Since $j_\beta \geqslant 1 \ \forall\beta$ and $n = j_1+j_2+\cdots+j_{n+1}$ (n+1 terms) $\geqslant 1+1+\cdots+1 = n+1$, which a contradiction. Similarly we see that the other terms do not appear.

Now, (2.6) is reduced to

$$C_{mn}\Big(\sum_{\substack{i_1+i_2+\cdots+i_n=m.\\ j_1+j_2+\cdots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big)$$ because by the induction hypothesis $C_{m1}, C_{m2}, \ldots\ldots C_{m,n-1}$ are all zero.

We need to show that $C_{mn}\Big(\sum_{\substack{i_1+i_2+\cdots+i_n=m.\\ j_1+j_2+\cdots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_nj_n}\Big) = 0.$

Since $j_\beta \geqslant 1 \ \forall\beta$ and $n = j_1+j_2+\cdots+j_n$ (n terms), this implies that $j_\beta = 1 \ \forall\beta$. But for all α, $C_{i_\alpha j_\beta} = C_{i_\alpha 1} = 0$, therefore the sum of the products is zero implying that

$$C_{mn}\Big(\sum_{\substack{i_1+i_2+\cdots+i_n=m.\\ j_1+j_2+\cdots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big) = 0.$$

Hence $C_{mn} = 0$. Therefore in any case we have that $C_{mn} = 0 \ \forall m \ \forall n$.

<u>Case 2</u>. We assume that there exists an m such that $C_{m1} \neq 0$. Let m_o be the smallest natural number such that $C_{m_o1} \neq 0$. Define $\Phi : R \times R \to R$ by

$$\phi(t,x) = t^{m_o}x \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}.$$

We see that Φ is analytic. Now, we need to show that Φ is an analytic semigroup action of $(\mathbb{R},\cdot)$ with zero on $\mathbb{R}$, that is, $\Phi(st,x) = \Phi(s,\Phi(t,x))$ and $\Phi(0,x) = 0 \quad \forall s,t \in \mathbb{R}, \forall x \in \mathbb{R}$. To see this, let s,t,x belong to $\mathbb{R}$ then $\Phi(st,x) = (st)^{m_o}x = s^{m_o}t^{m_o}x$ $s^{m_o}(t^{m_o}x) = s^{m_o}\Phi(t,x) = \Phi(s,\Phi(t,x))$ and $\Phi(0,x) = 0^{m_o}x = 0$. Therefore Φ is a semigroup action of $\mathbb{R}$ on $\mathbb{R}$. Next, we claim that the analytic semigroup action ψ is locally analytically isomorphic to the analytic semigroup action Φ. In order to prove this, we have to find an analytic map $\eta : U \to U'$ where U and U' are neighborhoods of 0 in $\mathbb{R}$, such that η is one-to-one, onto and satisfies the following property:

$$\eta(\psi(t,x)) = \Phi(t,\eta(x)) \qquad t, x \in \mathbb{R}.$$

We have $\psi(t,x) = C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+C_{14}tx^4+C_{23}t^2x^3+C_{32}t^3x^2+C_{41}t^4x+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ and $\phi(t,x) = t^{m_o}x$.

Define $\eta : \mathbb{R} \to \mathbb{R}$ by

$$\eta(x) = d_0+d_1x+d_2x^2+d_3x^3+\ldots\ldots\ldots+d_nx^n+\ldots\ldots\ldots$$

where $d_k = \dfrac{C_{m_ok}}{C_{m_o1}}$. Since $C_{m_o0} = 0$, it follows that $d_0 = 0$, that is $\eta(0) = 0$. Therefore, $\eta(x) = \sum_{k=1}^{\infty} \dfrac{C_{m_ok}}{C_{m_o1}} x^k$. Since $\sum_{m,n=1}^{\infty} C_{mn}t^nx^n$ converges in a neighborhood of (0,0), by section (0.5), it follows that $\sum_{k=1}^{\infty} C_{m_ok}x^k$ converges in a neighborhood of 0. Clearly, $\sum_{k=1}^{\infty} \dfrac{C_{m_ok}}{C_{m_o1}} x^k$ converges in the same neighborhood of 0 as $\sum_{k=1}^{\infty} C_{m_ok}x^k$.

Now, we see that

$$\frac{d}{dx}\eta(x) = \frac{C_{m_o 1}}{C_{m_o 1}} + 2\,\frac{C_{m_o 2}}{C_{m_o 1}}\,x + 3\,\frac{C_{m_o 3}}{C_{m_o 1}}\,x^2 + \cdots\cdots$$

Therefore $\frac{d}{dx}\eta(0) = 1 \neq 0.$

By Theorem 0.6, there exists a neighborhood of 0 such that η is one-to-one. Now, choose a neighborhood U of 0 such that η is one-to-one and η converges in this neighborhood and $U' = \eta(U)$. Therefore, we have $\eta : U \to U'$, U and U' are neighborhoods of 0 in R such that η is one-to-one and onto. We want to prove that $\eta(\psi(t,x)) = \phi(t,\eta(x))$; therefore, we first expand $\eta(\psi(t,x))$ and $\phi(t,\eta(x))$, respectively.

$$\begin{aligned}
\eta(\psi(t,x)) &= \eta(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\ldots) \\
&= d_1(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\ldots) \\
&\quad +d_2(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x^2+C_{31}t^3x+\ldots)^2 \qquad (2.7)\\
&\quad +d_3(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x+C_{31}t^3x+\ldots)^3 \\
&\quad +d_4(C_{11}tx+C_{12}tx^2+C_{21}t^2x+C_{13}tx^3+C_{22}t^2x+C_{31}t^3x+\ldots)^4 \\
&\quad +\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

$$\begin{aligned}
\phi(t,\eta(x)) &= \phi(t,d_1x+d_2x^2+d_3x^3+d_4x^4+\cdots\cdots) \\
&= t^{m_o}(d_1x+d_2x^2+d_3x^3+d_4x^4+\cdots\cdots) \qquad (2.8)\\
&= d_1t^{m_o}x+d_2t^{m_o}x^2+d_3t^{m_o}x^3+d_4t^{m_o}x^4+\cdots\cdots
\end{aligned}$$

We want to show that (2.7) and (2.8) are equal, that is, we want to show that the coefficients of the term $t^m x^n$ in (2.7) and (2.8) are equal.

Let m, n be any integers. We consider the coefficients of the term $t^m x^n$ in (2.7) and (2.8), respectively.

Case 2.1 We assume that $m \neq m_o$.

Subcase 2.1.1 $m \leq n$.

$m \neq m_o$ implies that the coefficient of the term $t^m x^n$ in (2.8) is zero. The coefficient of this term in (2.7) is

$$d_1 C_{mn}$$

$$+ d_2 \Big(\sum_{\substack{i_1+i_2=m. \\ j_1+j_2=n.}} C_{i_1 j_1} C_{i_2 j_2} \Big)$$

$$+ d_3 \Big(\sum_{\substack{i_1+i_2+i_3=m. \\ j_1+j_2+j_3=n.}} C_{i_1 j_1} C_{i_2 j_2} C_{i_3 j_3} \Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ d_m \Big(\sum_{\substack{i_1+i_2+\cdots+i_m=m. \\ j_1+j_2+\cdots+j_m=n.}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_m j_m} \Big).$$

We substitute $\frac{C_{m_o k}}{C_{m_o 1}}$ for d_k. We see that we must show

that

$$\frac{C_{m_o 1}}{C_{m_o 1}} C_{mn}$$

$$+ \frac{C_{m_o 2}}{C_{m_o 1}} \Big(\sum_{\substack{i_1+i_2=m. \\ j_1+j_2=n.}} C_{i_1 j_1} C_{i_2 j_2} \Big)$$

$$(2.9) \quad + \frac{C_{m_o 3}}{C_{m_o 1}} \Big(\sum_{\substack{i_1+i_2+i_3=m. \\ j_1+j_2+j_3=n.}} C_{i_1 j_1} C_{i_2 j_2} C_{i_3 j_3} \Big) = 0$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\frac{C_{m_o m}}{C_{m_o 1}}\Big(\sum_{\substack{i_1+i_2+\dots+i_m=m.\\ j_1+j_2+\dots+j_m=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_mj_m}\Big).$$

Now, use the fact that (2.3) and (2.4) are equal and consider the coefficient of the term $s^{m_o}t^mx^n$ ($m_o \neq m$, $m \leqslant n$). In (2.3), since $m_o \neq m$, we get that the coefficient of this term is zero. Since the coefficients of the term $s^{m_o}t^mx^n$ ($m \neq m_o, m \leqslant n$) in (2.3) and (2.4) are equal, we get that

$$C_{m_o1}C_{mn}$$

$$+\,C_{m_o2}\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

(2.10) $$+\,C_{m_o3}\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\,C_{m_om}\Big(\sum_{\substack{i_1+i_2+\dots+i_m=m.\\ j_1+j_2+\dots+j_m=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_mj_m}\Big) = 0.$$

Since $C_{m_o1} \neq 0$, we can divide both sides of (2.10) by C_{m_o1} and thus we prove (2.9). Therefore when $m_o \neq m$ and $m \leqslant n$ the coefficients of the term t^mx^n in (2.7) and (2.8) are equal.

Subcase 2.1.2 $\quad m > n.$

Since $m \neq m_o$, we see that the coefficient of the term t^mx^n in (2.8) must be zero and the coefficient of t^mx^n ($m \neq m_o$, $m > n$). In (2.7) is :

$$d_1C_{mn}$$

$$+\,d_2\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+\ d_3\Big(\sum_{\substack{i_1+i_2+i_3=m. \\ j_1+j_2+j_3=n.}} C_{i_1j_2}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\ d_n\Big(\sum_{\substack{i_1+i_2+\cdots\cdots+i_n=m. \\ j_1+j_2+\cdots\cdots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big).$$

Again we substitute $\dfrac{C_{m_ok}}{C_{m_o1}}$ for d_k. Thus we need to show that

$$\frac{C_{m_o1}}{C_{m_o1}}\ C_{mn}$$

$$+\ \frac{C_{m_o2}}{C_{m_o1}}\Big(\sum_{\substack{i_1+i_2=m. \\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$\text{(2.11)}\quad +\ \frac{C_{m_o3}}{C_{m_o1}}\Big(\sum_{\substack{i_1+i_2+i_3=m. \\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\ \frac{C_{m_on}}{C_{m_o1}}\Big(\sum_{\substack{i_1+i_2+\cdots+i_n=m. \\ j_1+j_2+\cdots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big) = 0.$$

Now, using the fact that (2.3) and (2.4) are equal. We consider the coefficients of the term $s^{m_o}t^mx^n$ (when $m > n, m \neq m_o$). From (2.3), since $m \neq m_o$, we get that the coefficient of this term is zero. Since the coefficients of the term $s^{m_o}t^mx^n$ (when $m > n$, $m \neq m_o$) in (2.3) and (2.4) are equal, we get that

$$C_{m_o 1} C_{mn}$$

$$+ C_{m_o 2}\Big(\sum_{\substack{i_1+i_2=m. \\ j_1+j_2=n.}} C_{i_1 j_1} C_{i_2 j_2} \Big)$$

(2.12) $$+ C_{m_o 3}\Big(\sum_{\substack{i_1+i_2+i_3=m. \\ j_1+j_2+j_3=n.}} C_{i_1 j_1} C_{i_2 j_2} C_{i_3 j_3} \Big)$$

+ ....

+ ....

$$+ C_{m_o n}\Big(\sum_{\substack{i_1+i_2+\ldots+i_n=m. \\ j_1+j_2+\ldots+j_n=n.}} C_{i_1 j_1} C_{i_2 j_2} \ldots\ldots C_{i_n j_n} \Big) = 0.$$

Since $C_{m_o 1} \neq 0$, we can divide both sides of (2.12) by $C_{m_o 1}$ and thus we prove (2.11). Therefore when $m_o \neq m$ and $m > n$ the coefficients of the term $t^m x^n$ in (2.7) and (2.8) are equal. Hence when $m_o \neq m$ we get that the coefficients of the term $t^m x^n$ in (2.7) and (2.8) are equal.

<u>Case 2.2</u> We assume that $m = m_o$.

<u>Subcase 2.2.1</u> $m \leq n$, that is, $m_o \leq n$.

Since $m = m_o$, we see that the coefficient of the term $t^m x^n$ in (2.8) is equal $d_n = \dfrac{C_{m_o n}}{C_{m_o 1}}$. For $m_o \leq n$ the coefficient of this term in (2.7) is.

$$d_1 C_{m_o n}$$

$$+ d_2\Big(\sum_{\substack{i_1+i_2=m_o \\ j_1+j_2=n.}} C_{i_1 j_1} C_{i_2 j_2} \Big)$$

$$+ d_3\Big(\sum_{\substack{i_1+i_2+i_3=m_o. \\ j_1+j_2+j_3=n.}} C_{i_1 j_1} C_{i_2 j_2} C_{i_3 j_3} \Big)$$

$+ \cdots\cdots\cdots\cdots$

$+ \cdots\cdots\cdots\cdots$

$$+ d_{m_o}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{m_o}=m_o.\\ j_1+j_2+\cdots+j_{m_o}=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{m_o}j_{m_o}}\Big)$$

We substitute $\frac{C_{m_ok}}{C_{m_o1}}$ for d_k. Thus we need to show that:

$$\frac{C_{m_o1}}{C_{m_o1}} C_{m_on}$$

$$+ \frac{C_{m_o2}}{C_{m_o1}}\Big(\sum_{\substack{i_1+i_2=m_o.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$(2.13) \quad + \frac{C_{m_o3}}{C_{m_o1}}\Big(\sum_{\substack{i_1+i_2+i_3=m_o.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$+ \cdots\cdots\cdots\cdots$

$+ \cdots\cdots\cdots\cdots$

$$+ \frac{C_{m_om_o}}{C_{m_o1}}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{m_o}=m_o\\ j_1+j_2+\cdots+j_{m_o}=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{m_o}j_{m_o}}\Big) = \frac{C_{m_on}}{C_{m_o1}}\ .$$

Using the fact that (2.3) and (2.4) are equal, we consider the coefficient of the term $s^{m_o}t^{m_o}x^n$ $(m_o \leq n)$. For $m_o \leq n$, the coefficient of this term in (2.3) is C_{m_on}. Since the coefficients of the term $s^{m_o}t^{m_o}x^n$ $(m_o \leq n)$ in (2.3) and (2.4) are equal, we get that :

$$C_{m_o1}C_{m_on}$$

$$+ C_{m_o2}\Big(\sum_{\substack{i_1+i_2=m_o.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ C_{m_o 3}\Big(\sum_{\substack{i_1+i_2+i_3=m_o\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

(2.14) $$+ \cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots$$

$$+ C_{m_o m_o}\Big(\sum_{\substack{i_1+i_2+\dots+i_{m_o} = m_o\\ j_1+j_2+\dots+j_{m_o} = n,}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{m_o}j_{m_o}}\Big) = C_{m_o n}\,.$$

Next, we can divide both sides of (2.14) by $C_{m_o 1}$($C_{m_o 1} \neq 0$) and thus we prove (2.13). Therefore when $m = m_o$, $m_o \leq n$ the coefficients of the term $t^m x^n$ ($t^{m_o} x^n$) in (2.7) and (2.8) are equal.

<u>Subcase 2.2.2</u> $m > n$, that is $m_o > n$.

Since $m = m_o$, the coefficient of the term $t^{m_o} x^n$ in (2.8) is equal to $d_n = \dfrac{C_{m_o n}}{C_{m_o 1}}$. But the coefficient of the term $t^m x^n$ in (2.7) is

$$d_1 C_{m_o n}$$

$$+ d_2\Big(\sum_{\substack{i_1 i_2=m_o\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ d_3\Big(\sum_{\substack{i_1+i_2+i_3=m_o\\ j_1+j_2+j_3=n}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots$$

$$+ d_n\Big(\sum_{\substack{i_1+i_2+\dots+i_n=m_o.\\ j_1+j_2+\dots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big)$$

We substitute $\frac{C_{m_o k}}{C_{m_o 1}}$ for d_k, therefore we see that we have to show that

$$\frac{C_{m_o 1}}{C_{m_o 1}} C_{m_o n}$$

$$+ \frac{C_{m_o 2}}{C_{m_o 1}} \Big(\sum_{\substack{i_1+i_2=m_o \\ j_1+j_2=n.}} C_{i_1 j_1} C_{i_2 j_2} \Big)$$

(2.15) $$+ \frac{C_{m_o 3}}{C_{m_o 1}} \Big(\sum_{\substack{i_1+i_2+i_3=m_o. \\ j_1+j_2+j_3=n.}} C_{i_1 j_1} C_{i_2 j_2} C_{i_3 j_3} \Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \frac{C_{m_o n}}{C_{m_o 1}} \Big(\sum_{\substack{i_1+i_2+\cdots+i_n=m_o. \\ j_1+j_2+\cdots+j_n=n.}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_n j_n} \Big) = \frac{C_{m_o n}}{C_{m_o 1}} .$$

Using the fact that (2.3) and (2.4) are equal, we consider the term $s^{m_o} t^{m_o} x^n$ $(m_o > n)$. For $m_o > n$, the coefficient in (2.4) is $C_{m_o n}$. Since the coefficients of this term in (2.3) and (2.4) are equal, we get that

$$C_{m_o 1} C_{m_o n}$$

$$+ C_{m_o 2} \Big(\sum_{\substack{i_1+i_2=m_o. \\ j_1+j_2=n.}} C_{i_1 j_1} C_{i_2 j_2} \Big)$$

(2.16) $$+ C_{m_o 3} \Big(\sum_{\substack{i_1+i_2+i_3=m_o \\ j_1+j_2+j_3=n.}} C_{i_1 j_1} C_{i_2 j_2} C_{i_3 j_3} \Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{m_o n} \left(\sum_{\substack{i_1+i_2+\cdots+i_n=m_o \\ j_1+j_2+\cdots+j_n=n}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_n j_n} \right) = C_{m_o n} \cdot$$

Since $C_{m_o 1} \neq 0$, we can divide both sides of (2.16) by $C_{m_o 1}$ and thus we prove (2.15). So we see that when $m_o > n$ the coefficients of the term $t^m x^n$ in (2.3) and (2.4) are equal. So, we see that in all cases the coefficients of the term $t^m x^n$ in (2.3) and (2.4) are equal. Hence this proves that $\eta(\psi(t,x)) = \Phi(t,\eta(x))$.