# CHAPTER I

# ANALYTIC SEMIGROUPS ON $\mathbb{R}$

The purpose of this chapter is to classify analytic semigroups with 0 on $\mathbb{R}$ up to local analytic isomorphism.

Let $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a semigroup multiplication which is analytic such that $0 \in \mathbb{R}$ is a zero of $\mu$. Therefore $\mu$ has the following properties:

(1.1) (1) $\mu(x,y) = \sum_{m,n=0}^{\infty} C_{mn} x^m y^n$ in some neighborhood of $(0,0)$

(2) $\mu(0,x) = \mu(x,0) = 0 \quad \forall\, x \in \mathbb{R}$

(3) $\mu(x,\mu(y,z)) = \mu(\mu(x,y),z) \quad \forall\, x,y,z \in \mathbb{R}$

Since $\mu$ has property (2), it follows that $0 = \mu(x,0) = C_{00}+C_{10}x+C_{20}x^2+C_{30}x^3+C_{40}x^4+\ldots\ldots\ldots\ldots$ . Therefore by corollary (0.4) we get that $C_{00}= C_{10}= C_{20}= C_{30}= \ldots\ldots = 0$. That is,

(1.2) $C_{n0} = 0 \quad \forall\, n = 0,1,2,3,\ldots\ldots$ .

Next, $0 = \mu(0,y) = C_{01}y+C_{02}y^2+C_{03}y^3+C_{04}y^4+\ldots\ldots\ldots$ . Therefore by corollary (0.4) we get that $C_{01}= C_{02}= C_{03}=\ldots\ldots = 0$. That is,

(1.3) $C_{0n} = 0 \quad \forall\, n = 1,2,3,4,\ldots\ldots$ .

We substitute (1.2) and (1.3) in equation (1.1), then (1.1) becomes:

$$\mu(x,y) = C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+C_{31}x^3y+ C_{14}xy^4+C_{23}x^2y^3+C_{32}x^3y^2+C_{41}x^4y+\ldots\ldots\ldots\ldots$$

$$= \sum_{m,n=1}^{\infty} C_{mn} x^m y^n \; .$$

Now, let's consider the implications of property (3).

$$\mu(x,\mu(y,z)) = \sum_{m,n=1}^{\infty} C_{mn}x^m(\mu(y,z))^n$$

$$= C_{11}x\mu(y,z)+C_{12}x(\mu(y,z))^2+C_{21}x^2\mu(y,z)+$$
$$C_{13}x(\mu(y,z))^3+C_{22}x^2(\mu(y,z))^2+C_{31}x^3\mu(y,z)+$$
$$C_{14}x(\mu(y,z))^4+C_{23}x^2(\mu(y,z))^3+C_{32}x^3(\mu(y,z))^2+$$
$$C_{41}x^4\mu(y,z)+\cdots\cdots$$
$$= C_{11}x(C_{11}yz+C_{12}yz^2+C_{21}y^2z+C_{13}yz^3+C_{22}y^2z^2+C_{31}y^3z+\ldots)+$$
$$C_{12}x(C_{11}yz+C_{12}yz^2+C_{21}y^2z+C_{13}yz^3+C_{22}y^2z^2+C_{31}y^3z+\ldots)^2+$$
$$C_{21}x^2(C_{11}yz+C_{12}yz^2+C_{21}y^2z+C_{13}yz^3+C_{22}y^2z^2+C_{31}y^3z+\ldots)+$$
$$(1.5)\qquad C_{13}x(C_{11}yz+C_{12}yz^2+C_{21}y^2z+C_{13}yz^3+C_{22}y^2z^2+C_{31}y^3z+\ldots)^3+$$
$$C_{22}x^2(C_{11}yz+C_{12}yz^2+C_{21}y^2z+C_{13}yz^3+C_{22}y^2z^2+C_{31}y^3z+\ldots)^2+$$
$$C_{31}x^3(C_{11}yz+C_{12}yz^2+C_{21}y^2z+C_{13}yz^3+C_{22}y^2z^2+C_{31}y^3z+\ldots)+$$
$$C_{14}x(C_{11}yz+C_{12}yz^2+C_{21}y^2z+C_{13}yz^3+C_{22}y^2z^2+C_{31}y^3z+\ldots)^4+$$
$$+\cdots\cdots$$
$$= C_{11}C_{11}xyz + C_{21}C_{11}x^2yz+C_{11}C_{12}xyz^2+C_{11}C_{21}xy^2z+$$
$$C_{31}C_{11}x^3yz+C_{21}C_{21}x^2y^2z+C_{21}C_{12}x^2yz^2+(C_{11}C_{22}+$$
$$C_{12}C_{11}^2)xy^2z^2+C_{11}C_{31}xy^3z+C_{11}C_{13}xyz^3+\cdots\cdots .$$

We see that the coefficient of the term $x^my^nz^k$ $(n \leq k)$ is

$$C_{m1}C_{nk}$$
$$+\ C_{m2}\Big(\sum_{\substack{i_1+i_2=n.\\ j_1+j_2=k.}} C_{i_1j_1}C_{i_2j_2}\Big)$$
$$+\ C_{m3}\Big(\sum_{\substack{i_1+i_2+i_3=n.\\ j_1+j_2+j_3=k.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{mn}\Big(\sum_{\substack{i_1+i_2+\cdots+i_n=n.\\ j_1+j_2+\cdots+j_n=k.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big).$$

Note that the terms involving $C_{m,n+1}$, $C_{m,n+2}$, ...... do not appear. To see this, suppose that we have

$$C_{m,n+1}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{n+1}=n.\\ j_1+j_2+\cdots+j_{n+1}=k.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{n+1}j_{n+1}}\Big).$$

But $i_\alpha \geqslant 1 \ \forall\alpha$ and $n = i_1+i_2+\cdots+i_{n+1}$ ($n+1$ terms) $\geqslant 1+1+\cdots+1 = n+1$. This implies that $n \geqslant n+1$, a contradiction. Using the same proof as above, the other terms do not appear.

If $n > k$ the coefficient of the term $x^m y^n z^k$ equals:

$$C_{m1}C_{nk}$$

$$+ C_{m2}\Big(\sum_{\substack{i_1+i_2=n.\\ j_1+j_2=k.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ C_{m3}\Big(\sum_{\substack{i_1+i_2+i_3=n.\\ j_1+j_2+j_3=k.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{mk}\Big(\sum_{\substack{i_1+i_2+\cdots+i_k=n.\\ j_1+j_2+\cdots+j_k=k.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_kj_k}\Big)$$

Again, note that the terms involving $C_{m,k+1}$, $C_{m,k+2}$, ...... do not appear. To see this, suppose we have the term

$$C_{mk+1}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+1}=n. \\ j_1+j_2+\cdots+j_{k+1}=k.}} C_{i_1j_1}\,C_{i_2j_2}\cdots\cdots C_{i_{k+1}j_{k+1}}\Big)$$

Since $j_\beta \geqslant 1 \quad \forall\beta$ and $k = j_1+j_2+\cdots+j_{k+1}$ ($k+1$ terms), it follows that $k = j_1+j_2+\cdots+j_{k+1} \geqslant 1+1+\cdots+1 = k+1$, a contradiction. We can prove that the other terms do not appear by using the same proof as above.

$$\begin{aligned}
\mu(\mu(x,y),z) &= \sum_{m,n=1}^{\infty} C_{mn}(\mu(x,y))^m z^n \\
&= C_{11}\mu(x,y)z+C_{12}\mu(x,y)z^2+C_{21}(\mu(x,y))^2z+ \\
&\quad C_{13}\mu(x,y)z^3+C_{22}(\mu(x,y))^2z^2+C_{31}(\mu(x,y))^3z+ \\
&\quad C_{14}\mu(x,y)z^4+C_{23}(\mu(x,y))^2z^3+C_{32}(\mu(x,y))^3z^2+ \\
&\quad C_{41}(\mu(x,y))^4z+\cdots\cdots\cdots \\
&= C_{11}(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+C_{31}x^3y+\cdots)z+ \\
&\quad C_{12}(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+C_{31}x^3y+\cdots)z^2+ \\
&\quad C_{21}(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+C_{31}x^3y+\cdots)^2z+ \qquad (1.6) \\
&\quad C_{13}(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+C_{31}x^3y+\cdots)z^3+ \\
&\quad C_{22}(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+C_{31}x^3y+\cdots)^2z^2+ \\
&\quad C_{31}(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+C_{31}x^3y+\cdots)^3z+ \\
&\quad C_{14}(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+C_{31}x^3y+\cdots)z^4+ \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&= C_{11}C_{11}xyz+C_{11}C_{12}xy^2z+C_{11}C_{21}x^2yz+C_{11}C_{13}xy^3z \\
&\quad +C_{12}C_{11}xyz^2+C_{11}C_{31}x^3yz+C_{13}C_{11}xyz^3+C_{12}C_{12}xy^2z^2 \\
&\quad +(C_{11}C_{22}+C_{21}C_{11}^2)x^2y^2z+\cdots\cdots\cdots
\end{aligned}$$

If $m \leqslant n$, then the coefficient of the term $x^m y^n z^k$ in (1.6) equals:

$$C_{1k}C_{mn}$$

$$+ C_{2k}\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ C_{3k}\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{mk}\Big(\sum_{\substack{i_1+i_2+\cdots+i_m=m.\\ j_1+j_2+\cdots+j_m=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_mj_m}\Big).$$

Note that the terms involving $C_{m+1,k}, C_{m+2,k}, \ldots\ldots$ do not appear.

If $m > n$, the coefficient of the term $x^m y^n z^k$ in (1.6) equals:

$$C_{1k}C_{mn}$$

$$+ C_{2k}\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ C_{3k}\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{nk}\Big(\sum_{\substack{i_1+i_2+\cdots+i_n=m.\\ j_1+j_2+\cdots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big)$$

Note that the terms involving $C_{n+1,k}, C_{n+2,k}, \ldots\ldots$ do not appear.

Since (1.5) and (1.6) are equal, it follows that the coefficient of the term $x^m y^n z^k$ $(m,n,k \in \mathbb{N})$ in (1.5) and (1.6) are equal.

With this as background we are now ready to prove the following theorem:

<u>Theorem:</u> Let $\mu$ be an analytic semigroup on $\mathbb{R}$. Then $\mu$ is identically zero or $\mu$ is locally analytically isomorphic to the usual multiplication.

<u>Proof:</u> From (1.4) the coefficient of the lowest degree term is $C_{11}$. Regarding this coefficient we shall consider two cases:

<u>Case 1.</u> We assume that $C_{11} = 0$.

In this case we claim that $C_{mn} = 0 \quad \forall\, m = 1,2,3,\ldots \forall\, n = 1,2,3,\ldots$

$$
\begin{array}{llllllll}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & \cdots\cdots\cdots & & \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & \cdots\cdots\cdots & & \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & \cdots\cdots\cdots & & \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & \cdots\cdots\cdots & & \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & \cdots\cdots\cdots & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & C_{kk} & C_{k,k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & & C_{k+1k} & 
\end{array}
$$

To prove the claim, it is sufficient to prove that $C_{k+(\ell-1),k} = 0 = C_{k,k+(\ell-1)}$ $\forall k = 1,2,3,4,\ldots$ $\forall \ell = 1,2,3,4,\ldots$

Induction on k (on a diagonal line).

For k = 1, we have to show that $C_{\ell 1} = 0 = C_{1\ell}$ $\forall \ell = 1,2,3,\ldots$.

To prove that $C_{\ell 1} = 0$, we use induction on $\ell$. For $\ell = 1$, $C_{11} = 0$ by assumption. Now assume it is true for all natural number less than $\ell$. To prove $C_{\ell 1} = 0$, consider the coefficient of the term $x^{\ell} y^{\ell} z$ in (1.5) and (1.6), respectively. The coefficient of this term in (1.5) is $C^2_{\ell,1}$ and in (1.6) is

$$
\begin{aligned}
& C_{11} C_{\ell\ell} \\
& + C_{21}\Big( \sum_{\substack{i_1+i_2=\ell \\ j_1+j_2=\ell}} C_{i_1 j_1} C_{i_2 j_2} \Big) \\
& + C_{31}\Big( \sum_{\substack{i_1+i_2+i_3=\ell \\ j_1+j_2+j_3=\ell}} C_{i_1 j_1} C_{i_2 j_2} C_{i_3 j_3} \Big) \qquad (1.7) \\
& + \cdots\cdots\cdots\cdots \\
& + \cdots\cdots\cdots\cdots \\
& + C_{\ell-1,1}\Big( \sum_{\substack{i_1+i_2+\cdots+i_{\ell-1}=\ell \\ j_1+j_2+\cdots+j_{\ell-1}=\ell}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_{\ell-1} j_{\ell-1}} \Big) \\
& + C_{\ell,1}\Big( \sum_{\substack{i_1+i_2+\cdots+i_{\ell}=\ell \\ j_1+j_2+\cdots+i_{\ell}=\ell}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_{\ell} j_{\ell}} \Big)
\end{aligned}
$$

Claim that the terms involving $C_{\ell+1,1}, C_{\ell+2,1}, \ldots$ do not appear. To prove this, suppose that we have

$$
C_{\ell+1,1}\Big( \sum_{\substack{i_1+i_2+\cdots+i_{\ell+1}=\ell \\ j_1+j_2+\cdots+j_{\ell+1}=\ell}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_{\ell+1} j_{\ell+1}} \Big)
$$

Now $i_\alpha \geqslant 1\ \forall\alpha$ and $\ell = i_1+i_2+\ldots+i_{\ell+1}$ ($\ell+1$ terms) $\geqslant 1+1+\ldots\ldots+1 = \ell+1$. This implies that $\ell \geqslant \ell+1$, a contradiction. Therefore (1.7) is reduced to $C_{\ell,1}\left(\sum_{\substack{i_1+i_2+\ldots+i_\ell=\ell \\ j_1+j_2+\ldots+j_\ell=\ell}} C_{i_1j_1}C_{i_2j_2}\ldots\ldots C_{i_\ell j_\ell}\right)$

because by the induction hypothesis $C_{11}, C_{21}, C_{31}, \ldots\ldots C_{\ell-1,1}$ are all zero. Since $i_\alpha \geqslant 1\ \forall\alpha$ and $i_1+i_2+\ldots+i_\ell = \ell$, therefore $i_\alpha = 1\ \forall\alpha$. Similarly $j_\beta = 1\ \forall\beta$. Hence $C_{i_\alpha j_\beta} = C_{11} = 0\ \ \forall\alpha\forall\beta$ implying that the term

$$C_{\ell,1}\left(\sum_{\substack{i_1+i_2+\ldots+i_\ell=\ell \\ j_1+j_2+\ldots+j_\ell=\ell}} C_{i_1j_1}C_{i_2j_2}\ldots\ldots C_{i_\ell j_\ell}\right) = 0.$$

Therefore $C^2_{\ell,1} = 0$, so $C_{\ell,1} = 0$.

Next we shall prove that $C_{1,\ell} = 0$. As before, we shall use induction on $\ell$. For $\ell = 1$, $C_{11} = 0$ by assumption. Assume it is true for all natural number less than $\ell$. To show that $C_{1\ell} = 0$, we consider the coefficient of the term $xy^\ell z^\ell$ in (1.5) and (1.6), respectively. The coefficient of this term in (1.6) is $C^2_{1\ell}$ and in (1.5) the coefficient is

$$C_{11}C_{\ell\ell}$$

$$+\ C_{12}\left(\sum_{\substack{i_1+i_2=\ell \\ j_1+j_2=\ell}} C_{i_1j_1}C_{i_2j_2}\right)$$

$$+\ C_{13}\left(\sum_{\substack{i_1+i_2+i_3=\ell \\ j_1+j_2+j_3=\ell}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\right)$$

(1.8) $\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$+ C_{1,\ell-1}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{\ell-1}=\ell. \\ j_1+j_2+\cdots+j_{\ell-1}=\ell.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{\ell-1}j_{\ell-1}}\Big)$$

$$+ C_{1\ell}\Big(\sum_{\substack{i_1+i_2+\cdots+i_\ell=\ell. \\ j_1+j_2+\cdots+j_\ell=\ell.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_\ell j_\ell}\Big).$$

The same proof as before shows that the term involving $C_{1,\ell+1}, C_{1,\ell+2},\ldots\ldots$ do not appear. Since $C_{11}=0$ and by the induction hypothesis $C_{12}, C_{13},\ldots\ldots C_{1\ell-1}$ are all zero, we see that (1.8) becomes:

$$0 + C_{1\ell}\Big(\sum_{\substack{i_1+i_2+\cdots+i_\ell=\ell. \\ j_1+j_2+\cdots+j_\ell=\ell.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_\ell j_\ell}\Big).$$

As before, the sum of the products above is zero. So, we get that

$$C_{1\ell}^2 = 0.$$

so, $\quad C_{1\ell} = 0.$

Hence $\quad C_{\ell,1} = 0 = C_{1,\ell}.$

Next, by induction we assume that it is true for all natural numbers less than k, i.e.

$$\forall\, t < k,\ C_{t+(\ell-1),t} = 0 = C_{t,t+(\ell-1)} \quad \forall \ell = 1,2,3,4,\ldots\ldots$$

We need to show that $C_{k+(\ell-1),k} = 0 = C_{k,k+(\ell-1)} \quad \forall\, \ell = 1,2,3,4,\ldots\ldots$

Again, we use induction on $\ell$. For $\ell = 1$ we need to show that $C_{kk} = 0$. To prove this, consider the coefficient of the term $x^k y^{k^2} z^{k^2}$ in (1.5) and (1.6), respectively. The coefficient of this term in (1.5) is

$$C_{k,1}C_{k^2k^2}$$

$$+ C_{k,2}\Big(\sum_{\substack{i_1+i_2=k^2\\ j_1+j_2=k^2}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ C_{k,3}\Big(\sum_{\substack{i_1+i_2+i_3=k^2.\\ j_1+j_2+j_3=k^2.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{k,k-1}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k-1}=k^2.\\ j_1+j_2+\cdots+j_{k-1}=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{k-1}j_{k-1}}\Big).$$

(1.9)

$$+ C_{k,k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_k=k^2.\\ j_1+j_2+\cdots+j_k=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_kj_k}\Big)$$

$$+ C_{k,k+1}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+1}=k^2.\\ j_1+j_2+\cdots+j_{k+1}=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{k+1}j_{k+1}}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{k,k^2}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k^2}=k^2.\\ j_1+j_2+\cdots+j_{k^2}=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{k^2}j_{k^2}}\Big).$$

By assumption the terms involving $C_{k,1}, C_{k,2}, \ldots. C_{k,k-1}$ are all zero and we also have that $C_{kp}$ does not appear if $p > k^2$. Now, let's consider the term:

$$C_{k,k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_k=k^2.\\ j_1+j_2+\cdots+j_k=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_kj_k}\Big).$$ Consider the product $C_{i_1j_1}C_{i_2j_2}\cdots C_{i_kj_k}$.

If there exists $\alpha_o$ such that $i_{\alpha_o} \neq k$ then we have two possibilities: either $i_{\alpha_o} > k$ or $i_{\alpha_o} < k$.

If $i_{\alpha_o} < k$ then $C_{i_{\alpha_o} j_\beta} = 0 \ \forall \beta$. Therefore $C_{i_1 j_1} C_{i_2 j_2} \cdots$ $\cdots C_{i_k j_k} = 0$.

If $i_{\alpha_o} > k$ then claim that there exists an $\alpha'$ such that $i_{\alpha'} < k$. To prove this, suppose not, then for any $\alpha$ (except $\alpha_o$) $i_\alpha \geqslant k$. Therefore $k^2 = i_1+i_2+\cdots+i_k \geqslant k+k+\cdots+k$ (k-1 terms)$+ i_{\alpha_o}$ $> k+k+\cdots+k$ (k terms) $= k^2$. This implies that $k^2 > k^2$ which is a contradiction. Hence there exist an $\alpha'$ such that $i_{\alpha'} < k$. Therefore $C_{i_{\alpha'} j_\beta} = 0$ hence $C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_k j_k} = 0$. This implies that the sum of the products is zero. If $i_\alpha = k \ \forall \alpha$ then we consider $j_\beta$ as follows:

If there exists $\beta_o$ such that $j_{\beta_o} < k$ then $C_{i_\alpha j_{\beta_o}} = 0$. Therefore $C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_k j_k} = 0$.

If there exists $\beta_o$ such that $j_{\beta_o} > k$ then we can prove that there exists $\beta'$ such that $j_{\beta'} < k$. Therefore $C_{i_\alpha j_{\beta'}} = 0$ implying that $C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_k j_k} = 0$ and hence the sum of the products is zero.

If $j_\beta = k \ \forall \beta$ then the sum of the products is $C_{k,k}^k$. Hence the term

$$C_{k,k}\Big( \sum_{\substack{i_1+i_2+\cdots+i_k=k^2. \\ j_1+j_2+\cdots+j_k=k^2.}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_k j_k} \Big) \text{ is reduced}$$

to $C_{k,k}^{k+1}$.

Next, we consider the term

$$C_{k,k+1}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+1}=k^2. \\ j_1+j_2+\cdots+j_{k+1}=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{k+1}j_{k+1}}\Big).$$

If $j_\beta \geqslant k \quad \forall\beta$ then $k^2 = j_1+j_2+\cdots+j_{k+1} \geqslant k+k+\cdots+k$ ($k+1$ terms) $\geqslant k(k+1) = k^2+k$. Therefore $k^2 \geqslant k^2+k$ which is a contradiction. Therefore, there exists $\beta_o$ such that $j_{\beta_o} < k$ and therefore $C_{i_\alpha j_{\beta_o}} = 0 \;\forall\alpha$. Therefore as before the sum of the product is zero. Hence this term is zero.

In this same way, we can show that the sum of the products are all zero for the other terms. Therefore the coefficient of the term $x^k y^{k^2} z^{k^2}$ in (1.5) is reduced to $C_{k,k}^{k+1}$.

The coefficient of the term $x^k y^{k^2} z^{k^2}$ in (1.6) is

$$\begin{aligned}
& C_{1,k^2}\, C_{k,k^2} \\
& + C_{2,k^2}\Big(\sum_{\substack{i_1+i_2=k. \\ j_1+j_2=k^2.}} C_{i_1j_1}C_{i_2j_2}\Big) \\
& + C_{3,k^2}\Big(\sum_{\substack{i_1+i_2+i_3=k. \\ j_1+j_2+j_3=k^2.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + C_{k-1,k^2}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k-1}=k. \\ j_1+j_2+\cdots+j_{k-1}=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_{k-1}j_{k-1}}\Big)
\end{aligned} \tag{1.10}$$

$$+ C_{k,k^2}\left(\sum_{\substack{i_1+i_2+\cdots+i_k=k.\\ j_1+j_2+\cdots+j_k=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_kj_k}\right)$$

Now by assumption, $C_{1,k^2}, C_{2,k^2}, \ldots\ldots C_{k-1,k^2}$ are all zero.

We consider the term $C_{k,k^2}\left(\sum_{\substack{i_1+i_2+\cdots+i_k=k.\\ j_1+j_2+\cdots+j_k=k^2.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_kj_k}\right)$.

We see that $i_\alpha = 1 \;\forall\, \alpha$ . Therefore $C_{i_\alpha j_\beta} = C_{1j_\beta} = 0$. Hence as before the sum of the products is zero. Since (1.9) and (1.10) are equal, it follows that

$$C^{k+1}_{k,k} = 0.$$

$$\text{so,} \quad C_{k,k} = 0 .$$

Next, we assume that it is true for all natural numbers less than $\ell$ . We must show that $C_{k+(\ell-1),k} = 0 = C_{k,k+(\ell-1)}$.

To prove that $C_{k+(\ell-1),k} = 0$, we consider the coefficient of the term $x^{[k+(\ell-1)]^2}y^{k[k+(\ell-1)]}z^k$ in (1.5) and (1.6), respectively. The coefficient of this term in (1.5) is

$$C_{[k+(\ell-1)]^2,1}\,C_{k[k+(\ell-1)],k}$$

$$+ C_{[k+(\ell-1)]^2,2}\left(\sum_{\substack{i_1+i_2=k[k+(\ell-1)].\\ j_1+j_2=k.}} C_{i_1j_1}C_{i_2j_2}\right)$$

$$+ C_{[k+(\ell-1)]^2,3}\left(\sum_{\substack{i_1+i_2+i_3=k[k+(\ell-1)].\\ j_1+j_2+j_3=k.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\right)$$

$$(1.11)\quad + \cdots\cdots\cdots$$
$$+ \cdots\cdots\cdots$$
$$+ C_{[k+(\ell-1)]^2,k-1}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k-1}=k[k+(\ell-1)]\\ j_1+j_2+\cdots+j_{k-1}=k.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k-1}j_{k-1}}\Big)$$
$$+ C_{[k+(\ell-1)]^2,k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_k=k[k+(\ell-1)]\\ j_1+j_2+\cdots+j_k=k.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_kj_k}\Big).$$

Note that the terms involving $C_{[k+(\ell-1)]^2,k+1}, C_{[k+(\ell-1)]^2,k+2},\ldots$ do not appear. Now by assumption $C_{[k+(\ell-1)]^2,1}, C_{k[k+(\ell-1)]^2,2},\ldots, C_{k[k+(\ell-1)]^2,k-1}$ are all zero.

Consider the term

$$C_{[k+(\ell-1)]^2,k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_k=k[k+(\ell-1)]\\ j_1+j_2+\cdots+j_k=k.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_kj_k}\Big).$$

Since $j_\beta \geqslant 1 \ \forall \beta$ and $j_1+j_2+\cdots+j_k=k$, it implies that $j_\beta = 1 \ \forall \beta$. Thus, we have $C_{i_\alpha,1} = 0 \ \forall \alpha$ and hence the sum of the products above equals zero. Therefore (1.11) is reduced to 0. That is, the coefficient of the term $x^{[k+(\ell-1)]^2}y^{k[k+(\ell-1)]}z^k$ in (1.5) is zero.

Next, the coefficient of this term in (1.6) is:

$$C_{1,k}\,C_{[k+(\ell-1)]^2,k[k+(\ell-1)]}$$
$$+ C_{2,k}\Big(\sum_{\substack{i_1+i_2=[k+(\ell-1)]^2\\ j_1+j_2=k[k+(\ell-1)].}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ C_{3,k}\Big(\sum_{\substack{i_1+i_2+i_3=[k+(\ell-1)]^2 \\ j_1+j_2+j_3=k[k+(\ell-1)]}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{[k+(\ell-2)],k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+(\ell-2)}=[k+(\ell-1)]^2 \\ j_1+j_2+\cdots+j_{k+(\ell-2)}=k[k+(\ell-1)]}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k+(\ell-2)}j_{k+(\ell-2)}}\Big) \tag{1.12}$$

$$+ C_{[k+(\ell-1)],k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+(\ell-1)}=[k+(\ell-1)]^2 \\ j_1+j_2+\cdots+j_{k+(\ell-1)}=k[k+(\ell-1)]}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k+(\ell-1)}j_{k+(\ell-1)}}\Big)$$

$$+ C_{[k+\ell],k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+\ell}=[k+(\ell-1)]^2 \\ j_1+j_2+\cdots+j_{k+\ell}=k[k+(\ell-1)]}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k+\ell}j_{k+\ell}}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{k[k+(\ell-1)],k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k[k+(\ell-1)]}=[k+(\ell-1)]^2 \\ j_1+j_2+\cdots+j_{k[k+(\ell-1)]}=k[k+(\ell-1)]}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k[k+(\ell-1)]}j_{k[k+(\ell-1)]}}\Big).$$

Since $j_\beta \geqslant 1 \ \forall \beta$, we see that the terms whose first index is $> k[k+(\ell-1)]$ do not appear. By assumption, $C_{1,k}, C_{2,k}, \ldots\ldots C_{k+(\ell-2),k}$ are all zero. We now consider the term

$$C_{[k+(\ell-1)],k}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+(\ell-1)}=[k+(\ell-1)]^2 \\ j_1+j_2+\cdots+j_{k+(\ell-1)}=k[k+(\ell-1)]}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k+(\ell-1)}j_{k+(\ell-1)}}\Big).$$

If there exists a $\beta_o$ such that $j_{\beta_o} \neq k$, then we have two possibilities: to consider; either $j_{\beta_o} > k$ or $j_{\beta_o} < k$.

If $j_{\beta_o} < k$ then $C_{i_\alpha j_{\beta_o}} = 0 \ \forall \alpha$. Therefore the product

$C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_{k+(\ell-1)} j_{k+(\ell-1)}} = 0.$

If $j_{\beta_o} > k$ then claim that there exists a $\beta'$ such that $j_{\beta'} < k$. To prove this, suppose not, then for any $\beta$ (except $\beta_o$) $j_\beta \geq k$. Therefore, $k[k+(\ell-1)] = j_1+j_2+\cdots+j_{\beta_o}+\cdots+j_{k+(\ell-1)} \geq k+k+\cdots+k$ $(k+\ell-2$ terms$) + j_{\beta_o} > k+k+\cdots+k$ $(k+(\ell-1)$ terms$) = k[k+(\ell-1)]$. This implies that $k[k+(\ell-1)] > k[k+(\ell-1)]$ which is a contradiction. Hence there exists a $\beta'$ such that $j_{\beta'} < k$. Therefore $C_{i_\alpha j_{\beta'}} = 0$ an hence $C_{i_1 j_1} C_{i_2 j_2} \cdots C_{i_{k+(\ell-1)} j_{k+(\ell-1)}} = 0$. This implies that the sum of the products is zero.

If $j_\beta = k \ \forall \beta$ then we consider $i_\alpha$ as follows:

If there exists $\alpha_o$ such that $i_{\alpha_o} < k+(\ell-1)$ then $C_{i_{\alpha_o} j_\beta} = 0$. Therefore as before the sum of the product is zero.

If there exists $\alpha_o$ such that $i_{\alpha_o} > k+(\ell-1)$ then claim that there exists $\alpha'$ such that $i_{\alpha'} < k+(\ell-1)$. To prove this, suppose it is not true. Therefore for any $\alpha$ (except $\alpha_o$) $i_\alpha \geq k+(\ell-1)$. Therefore $[k+(\ell-1)]^2 = i_1+i_2+\cdots+i_{\alpha_o}+\cdots+i_{k+(\ell-1)}$ $(k+(\ell-1)$ terms$) \geq$ $k+(\ell-1)+k+(\ell-1)+\cdots+k+(\ell-1)$ $(k+(\ell-2)$ terms$) + i_{\alpha_o} > k+(\ell-1)+k+(\ell-1)+\cdots$ $+k+(\ell-1)$ $(k+(\ell-1)$ terms$) = [k+(\ell-1)]^2$. This implies that $[k+(\ell-1)]^2 > [k+(\ell-1)]^2$ which is a contradiction. Hence there exists $\alpha'$ such that $i_{\alpha'} < k+(\ell-1)$. Therefore $C_{i_{\alpha'} j_\beta} = 0$, so, the sum of the products is zero.

If $i_\alpha = k+(\ell-1)\ \forall\ \alpha$ then the sum of the products is

$C^{k+(\ell-1)}_{k+(\ell-1),k}$.

Therefore the above term is reduced to:

$$C_{[k+(\ell-1)],k}(C^{k+(\ell-1)}_{k+(\ell-1),k}) = C^{k+\ell}_{k+(\ell-1),k}.$$

Now, consider the next term:

The next term is $C_{k+\ell,k}\Big(\sum_{\substack{i_1+i_2+\dots+i_{k+\ell}=[k+(\ell-1)]^2 \\ j_1+j_2+\dots+j_{k+\ell}=k[k+(\ell-1)]}} C_{i_1j_1}C_{i_2j_2}\dots C_{i_{k+\ell}j_{k+\ell}}\Big)$.

If $j_\beta \geqslant k$ for all $\beta$ then we get that $k[k+(\ell-1)] = j_1+j_2+\dots+j_{k+\ell}$ ($k+\ell$ terms) $\geqslant k+k+\dots+k$ ($k+\ell$ terms) $= k[k+\ell]$ which is a contradiction. Thus, there exists a $\beta_o$ such that $j_{\beta_o} < k$. Therefore $C_{i_\alpha j_{\beta_o}} = 0$, impling that the sum of the products is zero. As above the other terms are all zero.

Thus, the coefficient of the term $x^{[k+(\ell-1)]^2}y^{k[k+(\ell-1)]}z^k$ in (1.6) is equal to $C^{k+\ell}_{k+(\ell-1),k}$. Since (1.11) and (1.12) are equal, this gives us that

$$C^{k+\ell}_{k+(\ell-1),k} = 0.$$

So, $\quad C_{k+(\ell-1),k} = 0.$

To prove $C_{k,[k+(\ell-1)]} = 0$. We consider the term $x^k y^{k[k+(\ell-1)]} z^{[k+(\ell-1)]^2}$ in (1.5) and (1.6), respectively. The coefficient of this term in (1.5) is

$$C_{k,1}C_{k[k+(\ell-1)],\,[k+(\ell-1)]^2}.$$

$$+\ C_{k,2}\Big(\sum_{\substack{i_1+i_2=k[k+(\ell-1)].\\ j_1+j_2=[k+(\ell-1)]^2.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+\ C_{k,3}\Big(\sum_{\substack{i_1+i_2+i_3=k[k+(\ell-1)].\\ j_1+j_2+j_3=[k+(\ell-1)]^2.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

+ ..............................

+ ..............................

(1.13) $$+\ C_{k,k+(\ell-2)}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+(\ell-2)}=k[k+(\ell-1)].\\ j_1+j_2+\cdots+j_{k+(\ell-2)}=[k+(\ell-1)]^2.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k+(\ell-2)}j_{k+(\ell-2)}}\Big)$$

$$+\ C_{k,k+(\ell-1)}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+(\ell-1)}=k[k+(\ell-1)].\\ j_1+j_2+\cdots+j_{k+(\ell-1)}=[k+(\ell-1)]^2.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k+(\ell-1)}j_{k+(\ell-1)}}\Big)$$

$$+\ C_{k,k+\ell}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+\ell}=k[k+(\ell-1)].\\ j_1+j_2+\cdots+j_{k+\ell}=[k+(\ell-1)]^2.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k+\ell}j_{k+\ell}}\Big)$$

+ ..............................

+ ..............................

$$+\ C_{k,k[k+(\ell-1)]}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k[k+(\ell-1)]}=k[k+(\ell-1)].\\ j_1+j_2+\cdots+j_{k[k+(\ell-1)]}=[k+(\ell-1)]^2.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k[k+(\ell-1)]}j_{k[k+(\ell-1)]}}\Big).$$

As before, the terms involving $C_{k,1}, C_{k,2}\cdots\cdots C_{k,k+(\ell-2)}$ are all zero. The term $C_{k,k+(\ell-1)}\Big(\sum_{\substack{i_1+i_2+\cdots+i_{k+(\ell-1)}=k[k+(\ell-1)].\\ j_1+j_2+\cdots+j_{k+(\ell-1)}=[k+(\ell-1)]^2.}} C_{i_1j_1}C_{i_2j_2}\cdots C_{i_{k+(\ell-1)}j_{k+(\ell-1)}}\Big)$

is reduced to $C^{k+\ell}_{k,k+(\ell-1)}$ ; and the other terms are all zero.

Therefore we get that (1.13) equals $C^{k+\ell}_{k,k+(\ell-1)}$.

Next we consider the coefficient of the term $x^k y^{k[k+(\ell-1)]}$ $z^{[k+(\ell-1)]^2}$ in (1.6), this is

$$C_{1,[k+(\ell-1)]^2} C_{k,k[k+(\ell-1)]}$$

$$+ C_{2,[k+(\ell-1)]^2} \Big( \sum_{\substack{i_1+i_2=k. \\ j_1+j_2=k[k+(\ell-1)].}} C_{i_1 j_1} C_{i_2 j_2} \Big)$$

$$+ C_{3,[k+(\ell-1)]^2} \Big( \sum_{\substack{i_1+i_2+i_3=k. \\ j_1+j_2+j_3=k[k+(\ell-1)].}} C_{i_1 j_1} C_{i_2 j_2} C_{i_3 j_3} \Big)$$

(1.14) $+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$$C_{k-1,[k+(\ell-1)]^2} \Big( \sum_{\substack{i_1+i_2+\cdots+i_{k-1}=k. \\ j_1+j_2+\cdots+j_{k-1}=k[k+(\ell-1)].}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_{k-1} j_{k-1}} \Big)$$

$$+ C_{k,[k+(\ell-1)]^2} \Big( \sum_{\substack{i_1+i_2+\cdots+j_k=k. \\ j_1+j_2+\cdots+j_k=k[k+(\ell-1)].}} C_{i_1 j_1} C_{i_2 j_2} \cdots\cdots C_{i_k j_k} \Big)$$

We see that $C_{1,[k+(\ell-1)]^2}, C_{2,[k+(\ell-1)]^2}, \ldots\ldots C_{k-1,[k+(\ell-1)]^2}$ are all zero by assumption. As before, the last term is zero. Therefore (1.14) is zero. Since (1.13) and (1.14) are equal, we obtain the following:

$$C^{k+\ell}_{k,[k+(\ell-1)]} = 0$$

so, $$C_{k,k+(\ell-1)} = 0 \quad .$$ #

Hence, this proves that $C_{mn} = 0 \quad \forall m \, \forall n$.

Case 2. In this case, we assume that $C_{11} \neq 0$.

In this case, we claim that $\mu$ is locally analytically isomorphic to the usual multiplication on $\mathbb{R}$. To prove this, we first prove the lemma.

Lemma: Let $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the usual multiplication on $\mathbb{R}$.

Define $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$x * y = C_{11} xy \qquad x, y \in \mathbb{R}.$$

Then $(\mathbb{R}, *)$ is a semigroup with zero and is analytically isomorphic to $(\mathbb{R}, \circ)$. Note that $0 \in \mathbb{R}$ is a zero of $(\mathbb{R}, *)$.

Proof: Let $x \in \mathbb{R}$. Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(x) = \frac{1}{C_{11}} x$$

$\psi$ is one-to-one, onto function. We must show that $\psi$ preserve the operations, that is $\psi(x \circ y) = \psi(x) * \psi(y)$. Let $x, y \in \mathbb{R}$ then $\psi(x \circ y) = \frac{1}{C_{11}} xy = C_{11} \frac{1}{C_{11}} x \cdot \frac{1}{C_{11}} y = C_{11} \psi(x) \circ \psi(y) = \psi(x) * \psi(y)$.

Therefore $\psi(x) * \psi(y) = \psi(x \circ y)$.

This proves that $(\mathbb{R}, *)$ is analytically isomorphic to $(\mathbb{R}, \circ)$.

To show that the analytic semigroup $\mu$ is locally analytically isomorphic to the usual multiplication $\circ$ on $\mathbb{R}$, it is sufficient to show that $\mu$ is locally isomorphic to $*$. To prove this we need to find a neighborhood $U$ and $U'$ of $0$ in $\mathbb{R}$ and a bijection map $\psi : U \to U'$ such that $\forall\, x, y \in U$, $\psi(\mu(x,y)) = \psi(x) * \psi(y)$.

Define $\psi$ by $\psi(x) = \sum_{n=1}^{\infty} b_n x^n$ , where $b_n = \frac{C_{1n}}{C_{11}}$ .

We see that $\psi(0) = 0$. Since $\mu(x,y)$ converges, this implies that there exists a neighborhood of $(0,0)$ such that $\mu(x,y)$ converges for all x, y belonging to this neighborhood. Now by section (0.5) $\sum_{n=1}^{\infty} C_{1n}x^n$ converges in a neighborhood of 0.

Clearly that $\sum_{n=1}^{\infty} \frac{C_{1n}}{C_{11}} x^n$ converges in the same neighborhood of 0 as $\sum_{n=1}^{\infty} C_{1n}x^n$.

Now, we see that $\psi(x) = \sum_{n=1}^{\infty} b_n x^n = \sum_{n=1}^{\infty} \frac{C_{1n}}{C_{11}} x^n$.

$$= \frac{C_{11}}{C_{11}} x + \frac{C_{12}}{C_{11}} x^2 + \frac{C_{13}}{C_{11}} x^3 + \cdots\cdots\cdots\cdots$$

Therefore $\frac{d}{dx}\psi(x) = 1 + 2\frac{C_{12}}{C_{11}} x + 3\frac{C_{13}}{C_{11}} x^2 + \cdots\cdots\cdots$ Thus $\frac{d}{dx}\psi(0)=1\neq 0$.

Then by Theorem 0.6, there exist a neighborhood of 0 such that $\psi$ is one-to-one.

Choose U be a neighborhood of 0 such that $\psi$ is one-to-one and converges in U and let $U' = \psi(U)$. Therefore

$\psi : U \to U'$ and $\psi$ converges in U. Thus $\psi$ is well-defined, one to one and onto. Next, we must show that $\psi$ preserves the operation, that is

$\psi(\mu(x,y)) = \psi(x) * \psi(y)$. Therefore we first expand $\psi(\mu(x,y))$ and $\psi(x) * \psi(y)$, respectively.

$$\psi(\mu(x,y)) = b_1(\mu(x,y))+b_2(\mu(x,y))^2+b_3(\mu(x,y))^3+\cdots$$
$$= b_1(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+\cdots)$$

$$+ b_2(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+\dots)^2$$

(1.15) $$+ b_3(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+\dots)^3$$

$$+ b_4(C_{11}xy+C_{12}xy^2+C_{21}x^2y+C_{13}xy^3+C_{22}x^2y^2+\dots)^4$$

$$+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\psi(x)*\psi(y) = C_{11}\psi(x)\cdot\psi(y)$$

(1.16) $$= C_{11}(b_1x+b_2x^2+b_3x^3+b_4x^4+\dots)(b_1y+b_2y^2+b_3y^3+\dots)$$

$$= C_{11}b_1b_1xy+C_{11}b_1b_2xy^2+C_{11}b_2b_1x^2y+C_{11}b_2b_2x^2y^2$$

$$+ C_{11}b_1b_3xy^3+C_{11}b_3b_1x^3y+ \dots\dots\dots\dots\dots$$

To prove that $\psi(\mu(x,y)) = \psi(x) * \psi(y)$, it suffices to show that the coefficient of the term $x^my^n$ ($\forall m, \forall n$) in (1.15) and (1.16) are equal. We have two possibilities; either $m \geq n$ or $m < n$.

<u>Case 2.1.</u> $m \geq n$.

Consider the coefficient of the term $x^my^n$ in (1.15) and (1.16), respectively. The coefficient of the term $x^my^n$ in (1.16) is

$$C_{11}b_mb_n$$

$$= C_{11}\frac{C_{1m}}{C_{11}}\frac{C_{1n}}{C_{11}}$$

(1.17) $$= \frac{C_{1m}C_{1n}}{C_{11}}$$

The coefficient of this term in (1.15) is

$$b_1C_{mn}$$

$$+ b_2\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ b_3\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ b_n\Big(\sum_{\substack{i_1+i_2+\cdots+i_n=m.\\ j_1+j_2+\cdots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big).$$

We substitute $\frac{C_{1i}}{C_{11}}$ for $b_i$, $i = 1,2,3,\ldots n.$ Then we get that the coefficient of the term $x^my^n$ in (1.15) is

(1.18)
$$\frac{C_{11}}{C_{11}} C_{mn}$$
$$+ \frac{C_{12}}{C_{11}}\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$
$$+ \frac{C_{13}}{C_{11}}\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{C_{1n}}{C_{11}}\Big(\sum_{\substack{i_1+i_2+\cdots+i_n=m.\\ j_1+j_2+\cdots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_nj_n}\Big)$$

Therefore when $m \geq n$ we want to prove that (1.17) and (1.18) are equal. To prove this, consider the coefficient of the term $x\, y^mz^n$ (when $m \geq n$) in (1.5) and (1.6), respectively. The coefficient of this term in (1.5) is

$$
\begin{aligned}
& C_{11}C_{mn} \\
& + C_{12}\Big(\sum_{\substack{i_1+i_2=m. \\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big) \\
& + C_{13}\Big(\sum_{\substack{i_1+i_2+i_3=m. \\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big) \qquad (1.19) \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \\
& + \cdots\cdots\cdots\cdots\cdots\cdots \\
& + C_{1n}\Big(\sum_{\substack{i_1+i_2+\ldots+i_n=m. \\ j_1+j_2+\ldots+j_n=n.}} C_{i_1j_1}C_{i_2j_2}\ldots\ldots C_{i_nj_n}\Big)
\end{aligned}
$$

and the coefficient in (1.6) is

$$C_{1m}C_{1n} \qquad (1.20)$$

Since (1.19) and (1.20) are equal; multiply (1.19) and (1.20) by $\frac{1}{C_{11}}$,it follows that (1.17) = (1.18). Therefore the coefficient of the term $x^my^n$ (when $m \geq n$) in (1.15) and (1.16) are equal.

<u>Case 2.2</u> $m < n$.

Similarly, consider the coefficient of the term $x^my^n$ in (1.15) and (1.16); respectively. The coefficient of this term in (1.15) is

$$
\begin{aligned}
& b_1C_{mn} \\
& + b_2\Big(\sum_{\substack{i_1+i_2=m. \\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)
\end{aligned}
$$

$$+ b_3\Big( \sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ b_m\Big( \sum_{\substack{i_1+i_2+\cdots+i_m=m.\\ j_1+j_2+\cdots+j_m=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_mj_m}\Big)$$

and the coefficient in (1.16)is

$$C_{11}b_m b_n$$

We substitute $\frac{C_{1i}}{C_{11}}$ for $b_i$, $i = 1,2,3,\ldots m$. Then we need to show that

$$\frac{C_{11}}{C_{11}} C_{mn}$$

$$+ \frac{C_{12}}{C_{11}} \Big( \sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

$$+ \frac{C_{13}}{C_{11}} \Big( \sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big) = C_{11}\frac{C_{1m}}{C_{11}}\frac{C_{1n}}{C_{11}} \tag{1.21}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{C_{1m}}{C_{11}}\Big( \sum_{\substack{i_1+i_2+\cdots+i_m=m.\\ j_1+j_2+\cdots+j_m=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_mj_m}\Big)$$

To prove (1.21), we consider the coefficient of the term $xy^mz^n$ (when $m < n$) in (1.5) and (1.6), respectively. We see that the coefficient of this term in (1.6) is

(1.22) $$C_{1m}C_{1n}$$

and the coefficient of this term in (1.5) is:

$$C_{11}C_{mn}$$

$$+ C_{12}\Big(\sum_{\substack{i_1+i_2=m.\\ j_1+j_2=n.}} C_{i_1j_1}C_{i_2j_2}\Big)$$

(1.23) $$+ C_{13}\Big(\sum_{\substack{i_1+i_2+i_3=m.\\ j_1+j_2+j_3=n.}} C_{i_1j_1}C_{i_2j_2}C_{i_3j_3}\Big)$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{1m}\Big(\sum_{\substack{i_1+i_2+\cdots+i_m=m.\\ j_1+j_2+\cdots+j_m=n.}} C_{i_1j_1}C_{i_2j_2}\cdots\cdots C_{i_mj_m}\Big) .$$

Since the coefficient of the term $xy^mz^n$(when $m < n$) in (1.5) and (1.6) are equal, we have that (1.22) = (1.23). Therefore we can divide (1.22) and (1.23) by $C_{11}$ and get (1.21). Hence, in all cases it follows that the coefficients of the term $x^my^n$ of $\psi(\mu(x,y))$ and $\psi(x) * \psi(y)$ are equal. Thus, $\psi(\mu(x,y)) = \psi(x) * \psi(y)$ and it follows that $\mu$ is locally isomorphic to the usual multiplication on $\mathbb{R}$.