CHAPTER O

PRELIMINARIES :

This chapter gives all necessary prerequisite for the following chapters:

<u>Definition 0.1</u> A series of the form:

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

is called a power series in x. And a series of the form

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots$$

is called a power series in x-a.

In general, a power series in n variables is given in the form : $\sum_{m_1,m_2,\ldots,m_n=0}^{\infty} C_{m_1 m_2 \ldots m_n} (x_1-a_1)^{m_1}(x_2-a_2)^{m_2} \ldots (x_n-a_n)^{m_n}$. In the next chapter we will consider a power series in x, y and write it in the form:

$$\sum_{m,n=0}^{\infty} C_{mn} x^m y^n = C_{00} + C_{10}x + C_{01}y + C_{20}x^2 + C_{11}xy + C_{02}y^2 + C_{30}x^3 + C_{21}x^2y + C_{12}xy^2 + C_{03}y^3 + \cdots$$

We also define the operations of series as in [6] page 111-113.

<u>Definition 0.2</u> A function of n variables, $f(x_1,x_2,\ldots,x_n)$ is said to be an <u>analytic function</u> at a point $(a_1,a_2,\ldots a_n)$ if it can be expanded into a power series which converges to the function in a neighborhood of $(a_1,a_2,\ldots a_n)$, i.e.

$$f(x_1,x_2,\ldots x_n)= \sum_{m_1,m_2,\ldots m_n=0}^{\infty} C_{m_1 m_2 \ldots m_n}(x_1-a_1)^{m_1}(x_2-a_2)^{m_2}\ldots \ldots\ldots(x_n-a_n)^{m_n}.$$

If f is of two variables x, y which is analytic at (0,0) then we can write f in the form:

$$f(x,y) = \sum_{m,n=0}^{\infty} C_{mn}x^m y^n = C_{00}+C_{10}x+C_{01}y+C_{20}x^2+C_{11}xy+C_{02}y^2+ C_{30}x^3+C_{21}x^2y+C_{12}xy^2+C_{03}y^3+\ldots\ldots$$

<u>Theorem 0.3</u> (Uniqueness Theorem). If a function f(x) is equal to the power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ in a neighborhood of $x = a$ and if f(x) is also equal to the power series $\sum_{n=0}^{\infty} b_n(x-a)^n$ in a neighborhood of $x = a$, then this two power series are identical, coefficient by coefficient, i.e.

$$C_n = b_n \quad \forall n = 0,1,2,\ldots.$$

<u>Proof</u>: See [1] page 246.

<u>Corollary 0.4</u> If a power series vanishs identically in a neighborhood of the origin then each coefficient is zero. That is,

$$\text{if } \sum_{n=0}^{\infty} C_n x^n = 0 \text{ then } C_n = 0 \quad \forall n = 0,1,2,3,\ldots\ldots .$$

<u>Proof</u>: See [4] page 144.

<u>Section 0.5</u> We consider an analytic function of two variables and write $f(x,y) = C_{00}+C_{10}x+C_{01}y+C_{20}x^2+C_{11}xy+C_{02}y^2+\ldots\ldots$.

If this series converges at a point (u,v) and if $u \neq 0$, $v \neq 0$ then it evidently converges for all points (x,y) such that

$|x| \leq |u|$ and $|y| \leq |v|$ [4] page 115.

A convergent power series can be differentiated term by term at any point interior to the region of convergence and the differentiated series will converge to the derivative of the power series. [6] page 124-126. Also, the derivative series will admit at least the same region of convergence as the original series. It follows that if

$$\begin{aligned} f(x,y) = {} & C_{00}+C_{10}x+C_{20}x^2+C_{30}x^3+ \cdots\cdots \\ & +C_{01}y+C_{11}xy+C_{21}x^2y+C_{31}x^3y+\cdots\cdots \\ & +C_{02}y^2+C_{12}xy^2+C_{22}x^2y^2+C_{32}x^3y^2+\cdots\cdots \\ & +C_{03}y^3+C_{13}xy^3+C_{23}x^2y^3+C_{33}x^3y^3+\cdots\cdots \\ & +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad (1)$$

and we differentiate (1) with respect to y, then we get

$$\begin{aligned} \frac{\partial f}{\partial y}(x,y) = {} & 0 + 0 + 0 + \cdots\cdots \\ & +C_{01}+C_{11}x+C_{21}x^2+C_{31}x^3+\cdots\cdots \\ & +2C_{02}y+2C_{12}xy+2C_{22}x^2y+2C_{32}x^3y+\cdots\cdots \\ & +3C_{03}y^2+3C_{13}xy^2+3C_{23}x^2y^2+3C_{33}x^3y^2+\cdots\cdots \\ & +\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \qquad (2)$$

Suppose that the power series (1) converges for all points (x,y) belonging to the ball center at (0,0) and radius $R > 0$ which we shall denote by $D(0;R)$. In (2), set $y = 0$ then the series is reduced to:

$$C_{01}+C_{11}x+C_{21}x^2+C_{31}x^3+C_{41}x^4+\cdots\cdots\cdots\cdots\cdots$$

Therefore, $C_{01}+C_{11}x+C_{21}x^2+C_{31}x^3+C_{41}x^4+\cdots\cdots\cdots$ converges and the radius of convergence is at least R. Thus, this series converges in the interval $(-R,R)$.

Again, by differentiating (2) with respect to y and setting $y = 0$, we get that the series

$C_{02}+C_{12}x+C_{22}x^2+C_{32}x^3+C_{42}x^4+\ldots\ldots\ldots\ldots\ldots\ldots$ converges in the interval $(-R,R)$

On repeating the reasoning we get that for all n the series $\sum_{m=0}^{\infty} C_{mn}x^m$ converges in an interval containing 0 of radius at least R.

Now, we differentiate (1) with respect to x, we get that

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x,y) = {} & 0 + C_{10} + 2C_{20}x + 3C_{30}x^2 + \ldots\ldots \\
& + 0 + C_{11}y + 2C_{21}xy + 3C_{31}x^2y + \ldots\ldots \\
& + 0 + C_{12}y^2 + 2C_{22}xy^2 + 3C_{32}x^2y^2 + \ldots\ldots \qquad (3) \\
& + 0 + C_{13}y^3 + 2C_{23}xy^3 + 3C_{33}x^2y^3 + \ldots\ldots \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

In (3), set $x = 0$, then the series is reduced to:

$C_{10}+C_{11}y+C_{12}y^2+C_{13}y^3+C_{14}y^4+\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Therefore, $C_{10}+C_{11}y+C_{12}y^2+C_{13}y^3+C_{14}y^4+\ldots\ldots = \sum_{n=0}^{\infty} C_{1n}y^n$ converges in the interval $(-R,R)$. Again, by differentiating (3) with respect to x and setting $x = 0$, we get that the series $\sum_{n=0}^{\infty} C_{2n}y^n$ converges in $(-R,R)$. As before, we get that $\sum_{n=0}^{\infty} C_{mn}y^n$ converges for all m and that each series converges in the interval $(-R,R)$.

Hence if $\sum_{m,n=0}^{\infty} C_{mn}x^my^n$ converges in $D(0,R)$ then the series $\sum_{m=0}^{\infty} C_{mn}x^m$ and $\sum_{n=0}^{\infty} C_{mn}y^n$ converge in $(-R,R)$ $\forall m \; \forall n$.

<u>Theorem: 0.6.</u> If f is a continuous real-valued function on the interval J, and if $f'(x) > 0$ for all x in J except possibly the end points of J (if there are any), then f is strictly increasing on J (and hence f is one-to-one)

<u>Proof:</u> See [5] page 181.

<u>Definition 0.7</u> Let S be a set and let

$* : S \times S \to S$ be a map such that for all x,y,z belonging to S, $x * (y*z) = (x*y) * z$ then the pair $(S,*)$ is said to be <u>semigroup</u>.

If there exist a point 0 belonging to S such that for all x in S, $x * 0 = 0 * x = 0$, then we call the point 0 a <u>zero of S</u> and we call the semigroup S to which 0 belongs a <u>semigroup S with zero</u>. We write a semigroup with zero as $(S,*,0)$ [2].

<u>Remark:</u> If a zero exists then it is unique.

<u>Definition 0.8</u> A set S is said to be a <u>topological semigroup</u> if:

1) S is a semigroup,

2) S is a topological space,

3) The semigroup operation in S is continuous in the topological space S.

<u>Definition 0.9</u> Two topological semigroups with zero $(S,\cdot,0)$ and $(S,\cdot',0')$ are said to be <u>locally isomorphic</u> if there exists neighborhood U and U' of 0 and $0'$, respectively, and a bijection $f: U \to U'$ such that

1) if x,y and $x \cdot y$ belong to U then $f(x \cdot y) = f(x) \cdot' f(y)$

2) $f(0) = 0'$

We call f a <u>local isomorphism</u>.

<u>Definition 0.10</u> Let G be a group, X be a set and let $\psi : G \times X \to X$ be a map such that for any g, h belonging to G and x belonging to X:

$$\psi(gh,x) = \psi(g,\psi(h,x))$$

and $\psi(e,x) = x$ where e is the group identity.

Then ψ is said to be a <u>left action of G on X</u>.

A map $\psi: X \times G \to X$ such that for any g, h belonging to G and x belonging to X:

$$\psi(x,gh) = \psi(\psi(x,g),h),$$

and $\psi(x,e) = x$ where e is the group identity is said to be <u>right action of G on X</u>.

In general when we talk about actions we use the word group action to mean a left action. We now extend this concept to the semigroup with zero case.

Let S be a semigroup with zero 0, X a set. A map $\psi: S \times X \to X$ such that for any s,t belonging to S and x belonging to X

$$\psi(st,x) = \psi(s,\psi(t,x))$$

and $\psi(0,x)$ is a constant map

is said to be an <u>action of the semigroup S with zero on X</u>. If X is open in $\mathbb{R}^n$ and S is open in $\mathbb{R}^m$ and ψ is analytic then ψ is called an <u>analytic semigroup action</u> with zero.

Definition 0.11 Let G be a group, X, X' be sets and let G act on X and X', respectively, that is, there exist group action $\Phi : G \times X \to X$ and $\psi : G \times X' \to X'$.

The group actions Φ and ψ are said to be isomorphic if there exists a bijection $\eta : X \to X'$ such that for any g belonging to G and x belonging to X:

$$\eta(\Phi(g,x)) = \psi(g,\eta(x)) .$$

Remark If $\eta : X \to X'$ satisfying $\eta(\Phi(g,x)) = \psi(g,\eta(x))$ then η is called a G-homomorphism.

We now extend this concept to the semigroup with zero case. If a semigroup with zero S acts on X and X', respectively, then the semigroup actions Φ and ψ are said to be isomorphic if there exists a bijection $\eta : X \to X'$ such that for any t belonging to S and x belonging to X,

$$\eta(\Phi(t,x)) = \psi(t,\eta(x)).$$

Remark If X and X' are open in $\mathbb{R}^n$ and S is open in $\mathbb{R}^m$ and Φ, ψ are analytic then we shall require η to be an analytic isomorphism.

Definition 0.12 Let G be a group, X be a set and $\psi : G \times X \to X$ a group action. A point x belonging to X is said to be an invariant of the group action ψ if for all g belonging to G,

$$\psi(g,x) = x.$$

Example $x = \mathbb{R}^n$, $G = GL(n,\mathbb{R}) = \{(a_{ij}) \;\; \begin{matrix} i = 1,2,..n \\ j = 1,2,..n \end{matrix} \mid \det a_{ij} \neq 0\}$.

The action is the ordinary linear action,

$$\left[\psi[(a_{ij}),(x_i)]\right]_i = \sum_{j=1}^{n} a_{ij}x_j .$$

Then 0 in $\mathbb{R}^n$ is an invariant of the group action.

<u>Definition 0.13</u> Let M,N be vector spaces over a field K and Hom_K (M,N) denote the set of linear transformations. The group of units in Hom_K (M,M) is called the <u>general linear group</u> and is denoted by GL(M). Therefore an element ψ of Hom_K (M,M) is in GL(M) if and only if ψ is invertible i.e. ψ^{-1} exists in Hom_K(M,M). [3]. Let us fix a basis of M, then to each $\psi \in \mathrm{Hom}_K$(M,M) there corresponds a matrix $\psi(m)$. The mapping $\psi \to \psi(m)$ maps GL(M) onto the group of invertible matrices which we denote by GL(n,K).

<u>Remark</u>. $\psi \in$ GL(M) if and only if det $\psi(m) \neq 0$.

<u>Definition 0.14</u> Let G be a group. A <u>matrix representation</u> of G of degree n is a homomorphism $\psi : G \to GL(n,K)$. Two matrix representations ψ and ψ' are said to be <u>equivalent</u> if there exists an invertible matrix S in GL(n,K) such that

$$\psi'(g) = S\psi(g)S^{-1}, \quad \forall g \in G.$$

We now extend to the semigroup with zero case. Let $(S,\cdot,0)$ be a semigroup with zero. Then we say that a <u>representation</u> of $(S,\cdot,0)$ is a homomorphism $\psi : S \to M(n,K)$ where M(n,K) denotes the semigroup of all $n \times n$ matrices, and two representations ψ and $\psi' : S \to M(n,K)$ are said to be <u>equivalent</u> if there exists a non-singular matrix A such that

$$\psi'(x) = A\psi(x)A^{-1} \qquad \forall x \in S. \qquad \#$$