CHAPTER II

DEFINITIONS AND EXISTENCE THEOREM FOR RAMSEY NUMBERS

2.0 Introduction

In this Chapter we give a precise definition of Ramsey numbers. It will be proved in Theorem 2.5.3 that they always exist.

2.1 General Probelm

In order to generalize the problem mentioned in the previous chapter, we need some definitions and notations.

2.1.1 Definition We say that a set S is an n-set if it contains n elements. By an r-subset of a set S we shall mean any r-set which is a subset of S.

We may view a coloring of lines of any complete graph K_n in a new way as follows :

Let S denote the set of points of the complete graph K_n. Hence S is an n-set. Note that each line of the complete graph K_n joins precisely two points of S and any two points of S determine a unique line of K_n. Hence any line can be represented by a pair of points, i.e. any line PQ can be represented by $\{P, Q\}$. With this representation of K_n, the lines of K_n are represented by the 2-subsets of S. If we let $P_2(S)$ denote the family of all 2-subsets of S, then a red-blue coloring of the lines of K_n induces a partition of $P_2(S)$ into 2 classes, namely the class R of all 2-subsets that correspond to the red lines and the class B of all

2-subsets that correspond to the blue lines. Hence a red-blue coloring of lines of K_n corresponds to a partition of $P_2(S)$. We may restate the problem posed at the end of the previous chapter in our new terminologies as follows :

Find the smallest integer N such that if S is any n-set with $n \geq N$, then any partition of $P_2(S)$ into 2 classes C_1, C_2 there must exist a 3-subset of S with all its 2-subsets in C_1 or a 3-subset of S with all its 2-subsets in C_2.

Now, we are in the position to formulate our problem in a more general setting.

Find the smallest integer N such that if S is any n-set with $n \geq N$, then any partition of the class $P_r(S)$ of all r-subsets of S into ℓ class C_i, $i = 1, 2, \ldots, \ell$, there must exist at least one q_i-subset of S with all its r-subsets in C_i. Here r is any positive integer and q_i, $i = 1, 2, \ldots, \ell$ are any given positive integers such that $q_i \geq r$ for all i.

Note that if we let $r = 2$, $\ell = 2$, $q_1 = 3$, $q_2 = 3$, the above problem becomes our original problem. We shall show in Theorem 2.5.3 below that for any given $r, q_1, q_2, \ldots, q_\ell$ with $q_i \geq r$ for all i such integer N always exists, and depends only on $r, q_1, q_2, \ldots, q_\ell$. We shall denote such number by $N(q_1, q_2, \ldots, q_\ell ; r)$, and refer to it as a <u>Ramsey number</u>.

2.2 An Alternative Definition of Ramsey Numbers.

2.2.1 <u>Definition</u> Let S be a set and r be a positive integer. We say that $(C_1, C_2, \ldots, C_\ell)$ is a <u>partition</u> of $P_r(S)$ if

(i) C_i, $i = 1, 2, \ldots, \ell$ are disjoint subsets of $P_r(S)$,

(ii) $\bigcup_{i=1}^{\ell} C_i = P_r(s)$.

2.2.2 Definition Let S be a set. Let $r, q_1, q_2, \ldots, q_\ell$ be positive integers such that $r \geq 1$ and $q_i \geq r$, $i = 1, 2, \ldots, \ell$. We say that $(C_1, C_2, \ldots, C_\ell)$ is a $(q_1, q_2, \ldots, q_\ell\ ; r)$-partition of $P_r(S)$ if

(i) $(C_1, C_2, \ldots, C_\ell)$ is a partition of $P_r(S)$,

(ii) for each $i = 1, 2, \ldots, \ell$ no q_i-subset of S has all its r-subsets in C_i.

Example Let $S = \{P_1, P_2, P_3, P_4, P_5\}$.

Then $P_2(S) = \{\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_1, P_5\}, \{P_2, P_3\}, \{P_2, P_4\}, \{P_2, P_5\}, \{P_3, P_4\}, \{P_3, P_5\}, \{P_4, P_5\}\}$,

or briefly $P_2(S) = \{P_1P_2, P_1P_3, P_1P_4, P_1P_5, P_2P_3, P_2P_4, P_2P_5, P_3P_4, P_3P_5, P_4P_5\}$.

If all 2-subsets of S are partition according to the red-blue coloring of K_5 in the previous chapter, i.e. corresponding to the red lines $P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_1$ we form a set

$R = \{P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_1\}$

and corresponding to the blue lines $P_1P_3, P_1P_4, P_2P_4, P_2P_5, P_3P_5$ we form a set $B = \{P_1P_3, P_1P_4, P_2P_4, P_2P_5, P_3P_5\}$.

Then (R,B) is a (3, 3; 2)-partition of $P_2(S)$.

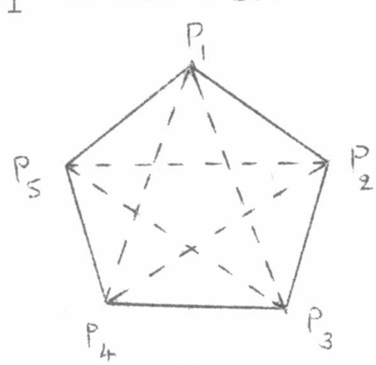

Fig. 2.1

The above example shows a (3, 3; 2)-partition of a 5-set S. It follows from what has been shown in the previous chapter that if S is an n-set with $n \geqslant 6$, then $P_2(S)$ does not have any (3, 3; 2)-partition. Hence the Ramsey number N(3, 3; 2) is the smallest integer N such that if $n \geqslant N$, then for any n-set S the family $P_2(S)$ has no (3, 3; 2)-partition. In general we have the following.

2.2.3 Remark The Ramsey number $N(q_1, q_2, \ldots, q_\ell ; r)$ is the smallest integer N such that if $n \geqslant N$, then for any n-set S the family $P_r(S)$ has no $(q_1, q_2, \ldots, q_\ell ; r)$-partition.

2.3 Induced Partitions

2.3.1 Proposition Let $(C_1, C_2, \ldots, C_\ell)$ be a partition of $P_r(S)$. For any $a_o \in S$ if C_i' is a set of (r - 1)-subsets of $S- \{a_o\}$ such that for any (r - 1)-subset of $S- \{a_o\}$ in C_i' together with a_o forms an r-subset of S in C_i where $i = 1, 2, \ldots, \ell$, then $(C_1', C_2', \ldots, C_\ell')$ is a partition of $P_{r-1}(S- \{a_o\})$.

Proof : We shall show that C_i', $i = 1, 2, \ldots, \ell$ are disjoint subsets of $P_{r-1}(S- \{a_o\})$, and $\bigcup_{i=1}^{\ell} C_i' = P_{r-1}(S- \{a_o\})$. Since $(C_1, C_2, \ldots, C_\ell)$ is a partition of $P_r(S)$, hence C_i, $i = 1,2,\ldots,\ell$ are disjoint subsets of $P_r(S)$. Therefore, C_i', $i = 1,2,\ldots, \ell$ are disjoint subsets of $P_{r-1}(S- \{a_o\})$. It is clear that $\bigcup_{i=1}^{\ell} C_i' \subseteq P_{r-1}(S- \{a_o\})$. Let A be any element of $P_{r-1}(S- \{a_o\})$. Then the set $A \cup \{a_o\}$ is in C_i for some i. Thus A is in C_i' for some i and hence $P_{r-1}(S- \{a_o\}) \subseteq \bigcup_{i=1}^{\ell} C_i'$. Therefore,

$\bigcup_{i=1}^{\ell} C_i' = P_{r-1}(S-\{a_o\})$. Hence $(C_1', C_2', \ldots, C_\ell')$ is a partition of $P_{r-1}(S-\{a_o\})$.

Q.E.D.

Note that $(C_1', C_2', \ldots, C_\ell')$ is called a partition of $P_{r-1}(S-\{a_o\})$ induced by $(C_1, C_2, \ldots, C_\ell)$ or the induced partition of $P_{r-1}(S-\{a_o\})$.

Example Let $S = \{a_o, a_1, a_2, a_3, a_4,\}$ and

$$C_1 = \{\{a_o,a_1,a_2\}, \{a_o,a_2,a_3\}, \{a_1,a_2,a_3\}, \{a_o,a_3,a_4\}, \{a_2,a_3,a_4\}\},$$

$$C_2 = \{\{a_o,a_1,a_3\}, \{a_o,a_1,a_4\}, \{a_o,a_2,a_4\}, \{a_1,a_2,a_4\}, \{a_1,a_3,a_4\}\}.$$

Clearly, (C_1,C_2) is a partition of $P_3(S)$, The induced partition of $P_2(S-\{a_o\})$ is (C_1', C_2') where

$$C_1' = \{\{a_1,a_2\}, \{a_2,a_3\}, \{a_3,a_4\}\},$$

$$C_2' = \{\{a_1,a_3\}, \{a_1,a_4\}, \{a_2,a_4\}\}.$$

2.4 Some Properties of Ramsey Numbers.

In this section, the properties concerning with the Ramsey numbers $N(q_1,q_2,\ldots, q_\ell ; r)$ will be discussed.

2.4.1 Theorem For any integers $q_1,q_2,\ldots,q_\ell, r$ such that $q_i \geqslant r \geqslant 1$, $i = 1,2,\ldots, \ell$. if $N(q_1,q_2,\ldots,q_\ell ; r)$ exists and $q_1',q_2',\ldots, q_\ell'$ is any permutation of $q_1,q_2,\ldots, q_\ell$, then $N(q_1',q_2',\ldots, q_\ell' ; r) = N(q_1,q_2,\ldots, q_\ell ; r)$.

Proof : Let S be any n-set with $n \geq N(q_1,q_2,\ldots,q_\ell ; r)$. Then $P_r(S)$ has no $(q_1,q_2,\ldots,q_\ell ; r)$-partition. Let $(C_1,C_2,\ldots,C_\ell)$ be any partition of $P_r(S)$. Thus S contains either a q_1-subset with all its r-subsets in C_1 or a q_2-subset with all its r-subsets in C_2, or...., or a q_ℓ -subset with all its r-subsets in C_ℓ . Since $q_1', q_2', \ldots, q_\ell'$ is a permutation of $q_1,q_2,\ldots, q_\ell$, hence there exists a permutation σ on $1,2,\ldots,\ell$ such that $q_i' = q_{\sigma(i)}$ for $i = 1,2,\ldots,\ell$. Observe that $(C_{\sigma(1)}, C_{\sigma(2)},\ldots, C_{\sigma(\ell)})$ is also a partition of $P_r(S)$. Thus we can see that S contains either a q_1'-subset with all its r-subsets in $C_{\sigma(1)}$, or a q_2'-subset with all its r-subsets in $C_{\sigma(2)}$, or..., or a q_ℓ' -subset with all its r-subsets in $C_{\sigma(\ell)}$. This shows that $P_r(S)$ has no $(q_1',q_2',\ldots,q_\ell' ; r)$-partition. Hence $N(q_1',q_2',\ldots, q_\ell' ;r)$ exists and $N(q_1',q_2',\ldots, q_\ell' ; r) \leq N(q_1,q_2,\ldots,q_\ell ; r)$.

To show equality let S be any n-set with $n = N(q_1,q_2,\ldots, q_\ell ; r)-1$. Thus $P_r(S)$ has a $(q_1,q_2,\ldots,q_\ell ; r)$-partition. Let $(C_1,C_2,\ldots, C_\ell)$ be a $(q_1,q_2,\ldots, q_\ell ; r)$-partition of $P_r(S)$. Since $q_1', q_2',\ldots, q_\ell'$ is a permutation of $q_1,q_2,\ldots,q_\ell$, so that there exists a permutation σ on $1,2,\ldots,\ell$ such that $q_i' = q_{\sigma(i)}$. Observe that $(C_{\sigma(1)}, C_{\sigma(2)},\ldots, C_{\sigma(\ell)})$ is a $(q_1', q_2',\ldots, q_\ell' ; r)$-partition of $P_r(S)$. Hence $N(q_1',q_2',\ldots, q_\ell' ; r) > N(q_1,q_2,\ldots, q_\ell ; r)-1$. Therefore, we have $N(q_1',q_2',\ldots, q_\ell' ; r) = N(q_1,q_2,\ldots, q_\ell ; r)$.

Q.E.D.

2.4.2 Theorem For any $q_1,q_2,\ldots, q_\ell$, r such that $q_i \geq r \geq 1$, $i = 1,2,\ldots, \ell-1$ if $N(q_1,q_2,\ldots, q_{\ell-1}; r)$ exists, then $N(q_1,q_2,\ldots, q_{\ell-1}, r ; r)$ exists and

$N(q_1, q_2, \ldots, q_{\ell-1}, r ; r) = N(q_1, q_2, \ldots, q_{\ell-1}; r)$.

Proof : Let S be any n-set with $n \geq N(q_1, q_2, \ldots, q_{\ell-1}; r)$. We shall show that $P_r(S)$ has no $(q_1, q_2, \ldots, q_{\ell-1}, r; r)$-partition. Let $(C_1, C_2, \ldots, C_\ell)$ be any partition of $P_r(S)$. Suppose that there exists no r-subset of S in C_ℓ, then all r-subsets of S are in $\bigcup_{i=1}^{\ell-1} C_i$. Hence $\bigcup_{i=1}^{\ell-1} C_i = P_r(S)$. Therefore, $(C_1, C_2, \ldots, C_{\ell-1})$ is a partition of $P_r(S)$. Since $n \geq N(q_1, q_2, \ldots, q_{\ell-1}; r)$, thus there exists either a q_1-subset of S with all its r-subsets in C_1, or a q_2-subset of S with all its r-subsets in C_2, or ...., or a $q_{\ell-1}$-subset of S with all its r-subsets in $C_{\ell-1}$. This shows that $P_r(S)$ has no $(q_1, q_2, \ldots, q_{\ell-1}, r ; r)$-partition. Hence $N(q_1, q_2, \ldots, q_{\ell-1}, r ; r)$ exists and

$N(q_1, q_2, \ldots, q_{\ell-1}, r ; r) \leq N(q_1, q_2, \ldots, q_{\ell-1}; r)$.

To show equality we need to construct $(q_1, q_2, \ldots, q_{\ell-1}, r ; r)$-partition of $P_r(S)$ where S is an n-set with $n = N(q_1, q_2, \ldots, q_{\ell-1}; r)-1$. Thus $P_r(S)$ has a $(q_1, q_2, \ldots, q_{\ell-1}; r)$-partition. Let $(C_1, C_2, \ldots, C_{\ell-1})$ be a $(q_1, q_2, \ldots, q_{\ell-1}; r)$-partition of $P_r(S)$. Then $(C_1, C_2, \ldots, C_\ell)$, where C_ℓ is empty, is a $(q_1, q_2, \ldots, q_{\ell-1}, r ; r)$-partition of $P_r(S)$. Hence $N(q_1, q_2, \ldots, q_{\ell-1}, r : r) > N(q_1, q_2, \ldots, q_{\ell-1}; r)-1$. Therefore, $N(q_1, q_2, \ldots, q_{\ell-1}, r ; r) = N(q_1, q_2, \ldots, q_{\ell-1}; r)$.

Q.E.D.

2.5 The Existence of the Ramsey Numbers $N(q_1, q_2, \ldots, q_\ell ; r)$

In this section we shall show that the Ramsey numbers $N(q_1, q_2, \ldots, q_\ell ; r)$ exist for all $q_1, q_2, \ldots, q_\ell$, r such that

$q_i \geq r \geq 1$, $i = 1,2,\ldots, \ell$. The proof will be by induction on $q_1, q_2, \ldots, q_\ell$, r. Theorem A - 2 of the appendix justifies our inductive proof.

2.5.1 Lemma For any $q_1, q_2, \ldots, q_\ell$ such that $q_i \geq 1$, $i = 1,2,\ldots, \ell$ $N(q_1, q_2, \ldots, q_\ell ; 1)$ exists and is given by $N(q_1, q_2, \ldots, q_\ell ; 1) = q_1 + q_2 + \cdots + q_\ell - \ell + 1$.

Proof. Let S be any n-set with $n \geq q_1 + q_2 + \cdots + q_\ell - \ell + 1$. We shall show that $P_1(S)$ has no $(q_1, q_2, \ldots, q_\ell ; 1)$-partition. Let $(C_1, C_2, \ldots, C_\ell)$ be any partition of $P_1(S)$. Suppose that no q_i-subset of S has all its points, i.e. 1-subsets, in C_i for $i = 1,2,\ldots, \ell - 1$, then there are at most $(q_i - 1)$ points in C_i, $i = 1,2,\ldots, \ell - 1$. Thus there are at least $n - [(q_1 - 1) + (q_2 - 1) + \cdots + (q_{\ell-1} - 1)]$ points in C_ℓ. Since $n \geq q_1 + q_2 + \cdots + q_\ell - \ell + 1$, hence $n - [(q_1 - 1) + (q_2 - 1) + \cdots + (q_{\ell-1} - 1)] \geq q_\ell$. Therefore, there exists a q_ℓ-subset of S with all its points in C_ℓ. This shows that $P_1(S)$ has no $(q_1, q_2, \ldots, q_\ell ; 1)$-partition. Hence $N(q_1, q_2, \ldots, q_\ell ; 1)$ exists and $N(q_1, q_2, \ldots, q_\ell ; 1) \leq q_1 + q_2 + \cdots + q_\ell - \ell + 1$.

To show equality we have to construct a $(q_1, q_2, \ldots, q_\ell ; 1)$-partition of $P_1(S)$ where S is an n-set with $n = q_1 + q_2 + \cdots + q_\ell - \ell$. Let $(C_1, C_2, \ldots, C_\ell)$ be a partition of $P_1(S)$ such C_i contains $(q_i - 1)$ points of S for all $i = 1,2,\ldots, \ell$. Thus no q_i-subset of S has all its points in C_i, $i = 1,2,\ldots, \ell$. This shows that $(C_1, C_2, \ldots, C_\ell)$ is a $(q_1, q_2, \ldots, q_\ell ; 1)$-partition of $P_1(S)$. Therefore, $N(q_1, q_2, \ldots, q_\ell ; 1) > q_1 + q_2 + \cdots + q_\ell - \ell$. Hence we obtain $N(q_1, q_2, \ldots, q_\ell ; 1) = q_1 + q_2 + \cdots q_\ell - \ell + 1$. Q.E.D.

2.5.2 Lemma For any positive integers q_1, r such that $q_1 \geq r$ $N(q_1; r)$ exists and $N(q_1; r) = q_1$.

Proof : This is clear from the definition of $N(q_1; r)$

Q.E.D.

2.5.3 Theorem Let $q_1, q_2, \ldots q_\ell$, r be integers such that $q_i \geq r \geq 1$, $i = 1,2,\ldots,\ell$. Then $N(q_1,q_2,\ldots,q_\ell ; r)$ exists.

Proof : We shall prove this theorem by induction on ℓ. Let $P(\ell)$ be the statement "if $q_1,q_2,\ldots,q_\ell$, r are positive integers such that $q_i \geq r \geq 1$, $i = 1,2,\ldots,\ell$, then $N(q_1,q_2,\ldots,q_\ell ; r)$ exists." By Lemma 2.5.2, we see that P(1) holds. Now we assume that P(k) holds. To show that P(k + 1) holds, we must verify that for any integers $q_1,q_2,\ldots,q_{k+1}$, r such that $q_i \geq r \geq 1$, $i = 1,2,\ldots, k + 1$, $N(q_1,q_2,\ldots,q_{k+1}; r)$ exists. Let

$$S = \left\{ (q_1,q_2,\ldots,q_{k+1}; r) / \; q_i \geq r \geq 1, \; i = 1,2,\ldots,k + 1 \right\}$$

and $T = \left\{ (q_1,q_2,\ldots,q_{k+1}; r) / N(q_1,q_2,\ldots,q_{k+1}; r) \text{ exists} \right\}$. Thus $T \subseteq S$. We shall apply Theorem A-2 of the appendix to show that T = S. By Lemma 2.5.1, we see that

(1) $(q_1,q_2,\ldots,q_{k+1}; 1)$ belongs to T for all $q_i \geq 1$, $i = 1,2,\ldots,k + 1$. By Theorems 2.4.1, 2.4.2 and the assumption P(k), we see that

(2) if $q_i \geq r$ for all $i = 1,2,\ldots, k + 1$ and $q_i = r$ for some i, then $(q_1,q_2,\ldots,q_{k+1}; r)$ belongs to T.

To verify that T has the property (3) of the hypothesis of Theorem A-2, we assume that for all $r \geq 2$ and all $q_i \geq r$, $i = 1,2,\ldots,k + 1$ $(q_1^*,q_2^*,\ldots,q_{k+1}^* ; r - 1)$ belongs to T for all $q_i^* \geq r - 1$, $i = 1,2,\ldots, k + 1$, and $(q_1-1, q_2,\ldots,q_{k+1}; r)$, $(q_1,q_2-1,q_3,\ldots,q_{k+1}; r),\ldots,$ $(q_1,\ldots,q_k, q_{k+1}-1; r)$ belong to T. For convenience, we write

$$q_1' = N(q_1-1, q_2,\ldots, q_{k+1}; r),$$

$$q_2' = N(q_1,q_2-1,q_3,\ldots,q_{k+1}; r),$$

$$\vdots$$

$$q_{k+1}' = N(q_1,q_2,\ldots,q_k,q_{k+1}-1; r).$$

Let $N = N(q_1',q_2',\ldots,q_{k+1}' ; r - 1) + 1$.

Let S be any n-set with $n \geq N$. We shall show that $P_r(S)$ has no $(q_1,q_2,\ldots,q_{k+1}; r)$-partition. Let $(C_1,C_2,\ldots,C_{k+1})$ be any partition of $P_r(S)$. Let a_o be an element of S and $(C_1',C_2',\ldots,C_{k+1}')$ be the partition of $P_{r-1}(S - \{a_o\})$ induced by $(C_1,C_2,\ldots,C_{k+1})$. Since $n \geq N(q_1',q_2',\ldots,q_{k+1}'; r - 1) + 1$, then $(n - 1) \geq N(q_1',q_2',\ldots,q_{k+1}'; r - 1)$. Hence $S - \{a_o\}$ is an $(n - 1)$-set with $(n - 1) \geq N(q_1',q_2',\ldots,q_{k+1}'; r - 1)$. By definition of the Ramsey number $N(q_1',q_2',\ldots,q_{k+1}'; r - 1)$, thus $S - \{a_o\}$ contains some q_i'-subset S_i with all its $(r - 1)$-subsets in C_i'. By definition of q_i', S_i contains

(1) (q_i-1)-subset with all its r-subsets in C_i,

or (2) some q_j-subset, $j \neq i$, with all its r-subsets in C_j.

If (1) holds, then a $(q_i- 1)$-subset of $S - \{a_o\}$ has all its r-subsets in C_i, and this $(q_i- 1)$-subset together with a_o is a q_i-subset of S with all its r-subsets in C_i.

If (2) holds, there exists a q_j-subset of S with all its r-subsets in C_j. Hence we can find an integer N such that for any n-set S with $n \geq N$ $P_r(S)$ has no $(q_1, q_2, \ldots, q_{k+1}; r)$-partition. By well-ordering principle, smallest such N exists. Hence $N(q_1, q_2, \ldots, q_{k+1}; r)$ exists. Therefore, $(q_1, q_2, \ldots, q_{k+1}; r)$ belongs to T. Thus T = S. Hence $N(q_1, q_2, \ldots, q_{k+1}; r)$ exists for all $q_i \geq r \geq 1$, $i = 1, 2, \ldots, k+1$. Therefore, $P(k+1)$ holds.

Q.E.D.

2.5.4 Remark From the proof of Theorem 2.5.3, we obtain $N(q_1, q_2, \ldots, q_{k+1}; r) \leq N(q_1', q_2', \ldots, q_{k+1}'; r-1) + 1$ with

$$q_1' = N(q_1 - 1, q_2, \ldots, q_{k+1}; r),$$

$$q_2' = N(q_1, q_2 - 1, q_3, \ldots, q_{k+1}; r),$$

$$\vdots$$

$$q_{k+1}' = N(q_1, \ldots, q_k, q_{k+1} - 1; r).$$