CHAPTER II

DIGRAPHS

In this chapter we collect various definitions and theorems on digraphs.

2.1 Directed Graphs or Digraphs

Let V be a finite nonempty set and E be a subset of $V \times V$. Then an ordered pair (V, E) is called a directed graph or digraph. Elements of V and E are called vertices and arcs of (V, E) respectively. To represent a digraph (V, E) by a diagram, we represent each vertex v by a point v and each arc (u, v) by an arrow from the point u to the point v.

For example, let $V = \{u, v, w\}$,

and $E = \{(u, v), (u, w), (v, v), (w, u), (w, v)\}$. Then (V, E) is a digraph. This digraph can be represented by the following diagram.

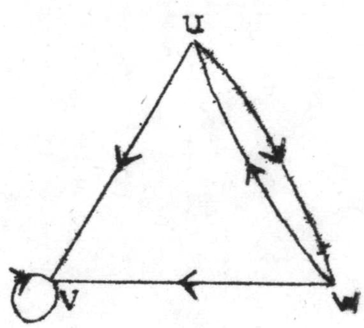

Fig. 2.1.1

2.2 In - degree, Out - degree and Degree

Let (V, E) be a digraph For any vertex v of (V, E), let

$$\delta(v) = \{ u \in V \mid (v, u) \in E \}$$

and $$\rho(v) = \{ w \in V \mid (w, v) \in E \}.$$

We shall call $|\delta(v)|$ and $|\rho(v)|$, where $|S|$ denotes the cardinal number of S, respectively the out - degree and the in - degree of v. If $|\delta(v)| = |\rho(v)|$, then we call $|\delta(v)|$ the degree of v.

2.3 Regular Digraphs, Normal Regular Digraphs and Degree of Digraphs

A digraph (V, E) is called a regular digraph if and only if for every two vertices u, v of (V, E)

$$|\delta(u)| = |\delta(v)| = |\rho(u)| = |\rho(v)|.$$

By the degree of any regular digraph (V, E) we mean the degree of any of its vertices.

A regular digraph (V, E) is called a normal regular digraph if and only if either $(v, v) \in E$ for each $v \in V$, or $(v, v) \notin E$ for each $v \in V$.

For example, consider the digraphs (V_1, E_1) and (V_2, E_2) represented by Fig. 2.3.1 and Fig. 2.3.2 respectively.

(V_1, E_1)

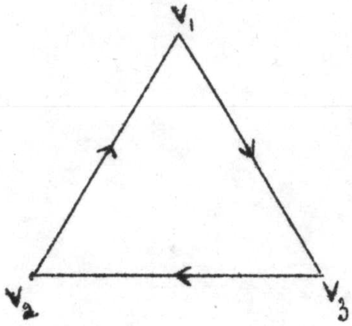

Fig. 2.3.1

(V_2, E_2)

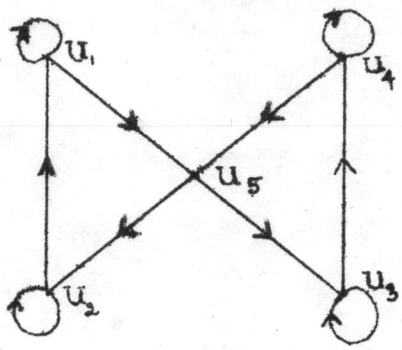

Fig. 2.3.2

It is clear that (V_1, E_1) and (V_2, E_2) are regular digraphs of degree 1 and 2 respectively. Since $(v, v) \notin E_1$ for any vertex v in (V_1, E_1). Hence it is normal regular. In Fig. 2.3.2, we see that $(u_5, u_5) \notin E_2$ which $(u, u) \in E_2$ for all other vertices u. Hence (V_2, E_2) is not a normal regular digraph.

2.3.1 __Theorem__ Let (V, E) be a digraph of n vertices. If $E = V \times V$, then (V, E) is a regular digraph of degree n.

__Proof__ : Let (V, E) be a digraph of n vertices and $E = V \times V$. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$. For each $v_i \in V$, we have

$$\delta(v_i) = \{u \in V \mid (v_i, u) \in E\} = \{v_1, v_2, v_3, \ldots, v_n\}$$

and $\rho(v_i) = \{w \in V \mid (w, v_i) \in E\} = \{v_1, v_2, v_3, \ldots, v_n\}$.

Thus $|\delta(v_i)| = |\rho(v_i)| = n$.

Hence $(V, V \times V)$ is a regular digraph of degree n.

__Q.E.D.__

2.3.2 <u>Theorem</u> Let (V, E_1) and (V, E_2) be regular digraphs of degree m and degree k respectively. If $E_2 \subseteq E_1$ and $E_3 = E_1 - E_2$, then (V, E_3) is a regular digraph of degree $m - k$.

<u>Proof</u> : Let v be an arbitrary element of V. For each $i = 1,2,3$, let

$$\sigma_i(v) = \left\{ u \in V \;\middle|\; (v, u) \in E_i \right\}$$

and $$\rho_i(v) = \left\{ w \in V \;\middle|\; (w, v) \in E_i \right\}.$$

Since (V, E_1) and (V, E_2) are regular digraphs of degree m and k respectively. Hence

$$|\sigma_1(v)| = |\rho_1(v)| = m \quad \text{and} \quad |\sigma_2(v)| = |\rho_2(v)| = k .$$

Claim that $|\sigma_3(v)| = |\rho_3(v)| = m-k$.

Let $u \in \sigma_2(v)$, hence $(v, u) \in E_2$. Since $E_2 \subseteq E_1$, hence $(v, u) \in E_1$. That is $u \in \sigma_1(v)$. Therefore $\sigma_2(v) \subseteq \sigma_1(v)$.

Similarly we can prove that $\rho_2(v) \subseteq \rho_1(v)$.

Let w be any element of $\sigma_3(v)$. Since $E_3 = E_1 - E_2$, hence

$$\begin{aligned}
w \in \sigma_3(v) &\iff (v,w) \in E_3 = E_1 - E_2 \\
&\iff (v, w) \in E_1 \text{ and } (v, w) \notin E_2 \\
&\iff w \in \sigma_1(v) \text{ and } w \notin \sigma_2(v) \\
&\iff w \in \sigma_1(v) - \sigma_2(v) .
\end{aligned}$$

Hence $\delta_3(v) = \delta_1(v) - \delta_2(v)$.

Similarly we can prove that $\rho_3(v) = \rho_1(v) - \rho_2(v)$.

Hence $|\delta_3(v)| = m - k$ and $|\rho_3(v)| = m - k$.

Therefore (V, E_3) is a regular digraph of degree $m - k$.

Q.E.D.

2.4 Digraph Isomorphisms, Digraph Automorphisms and Isomorphic Digraphs

Let (V_1, E_1) and (V_2, E_2) be digraphs. A one - to- one mapping φ from V_1 onto V_2 is called a digraph isomorphism from (V_1, E_1) onto (V_2, E_2) iff for each $u, v \in V_1$

$$(u,v) \in E_1 \iff (u\varphi, v\varphi) \in E_2 .$$

If there is a digraph isomorphism from (V_1, E_1) onto (V_2, E_2), then we say (V_1, E_1) and (V_2, E_2) are isomorphic or (V_1, E_1) is isomorphic to (V_2, E_2) and write $(V_1, E_1) \cong (V_2, E_2)$.

For an example, consider the digraphs (V_1, E_1) and (V_2, E_2) in Fig. 2.4.1 and Fig. 2.4.2 respectively.

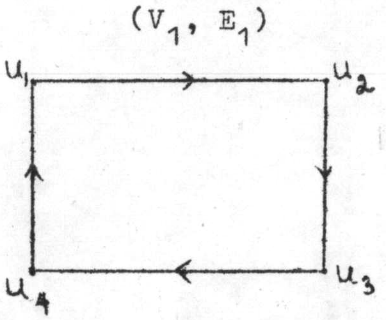

Fig. 2.4.1

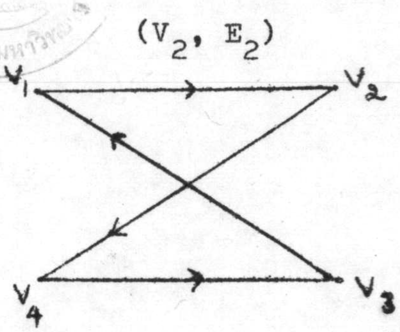

Fig. 2.4.2

Let $\varphi : V_1 \longrightarrow V_2$ be a mapping such that $u_1\varphi = v_1$, $u_2\varphi = v_2$, $u_3\varphi = v_4$ and $u_4\varphi = v_3$. It can be verified that φ is a digraph isomorphism from (V_1, E_1) onto (V_2, E_2). Hence $(V_1, E_1) \cong (V_2, E_2)$.

2.4.1 Remark If $(V_1, E_1) \cong (V_2, E_2)$, then it is clear that $|V_1| = |V_2|$ and $|E_1| = |E_2|$.

2.4.2 Remarks Observe that for any **digraph** (V, E), the identity mapping $1 : V \longmapsto V$ is a digraph isomorphism from (V, E) onto (V, E).

If φ is a digraph isomorphism from (V_1, E_1) onto (V_2, E_2), then it can be verified that φ^{-1} is a digraph isomorphism from (V_2, E_2) onto (V_1, E_1).

If $\varphi_1 : V_1 \longrightarrow V_2$ is a digraph isomorphism from (V_1, E_1) onto (V_2, E_2) and $\varphi_2 : V_2 \longrightarrow V_3$ is a digraph isomorphism from

(V_2, E_2) onto (V_3, E_3). Then it can be verified that $\varphi_1 \circ \varphi_2$, the composition of φ_1 with φ_2 is a digraph isomorphism from (V_1, E_1) onto (V_3, E_3).

From the above observation we see that

(1) $(V, E) \cong (V, E)$;

(2) if $(V_1, E_1) \cong (V_2, E_2)$, then $(V_2, E_2) \cong (V_1, E_1)$;

(3) if $(V_1, E_1) \cong (V_2, E_2)$ and $(V_2, E_2) \cong (V_3, E_3)$ then $(V_1, E_1) \cong (V_3, E_3)$.

If φ is a digraph isomorphism from (V, E) onto itself, then φ is called a <u>digraph automorphism</u> of (V, E).

2.4.3 <u>Remark.</u> The above remarks show that the set of all digraph automorphisms of a digraph (V, E) forms a group under composition. This group is known as the digraph automorphism group of (V,E). It will be denoted by $\Gamma(V, E)$.