CHAPTER IV

HARMONIC FUNCTIONS IN $\mathbb{R}^{n-1} \times (a,\infty)$

In this chapter we further study the representations of harmonic functions in a half-space $\mathbb{R}^{n-1} \times (a,\infty)$ by using Kelvin transformation and the passage.

4.1 Kelvin Transformation

4.1.1 Definition Let $B(y,\rho)$ be a ball in $\mathbb{R}^n$. For $x \neq y$ in $\mathbb{R}^n$ choose $\hat{x}$ on the radial line joining y to x so that

$$\|\hat{x} - y\| \, \|x - y\| = \rho^2$$

We call $\hat{x}$ the inverse of x relative to $\partial B(y,\rho)$.

The mapping $x \mapsto \hat{x}$ defined by

$$\hat{x} = y + \frac{\rho^2}{\|x - y\|^2}(x-y) \qquad (x \neq y)$$

is known as an inversion relative to $\partial B(y,\rho)$.

4.1.2 Definition Let G be an open subset of $\mathbb{R}^n \setminus \{y\}$, $\hat{G}$ be the image of G under the inversion. If $\hat{\varphi}$ is a function defined on $\hat{G}$, then the equation

$$\psi(x) = \frac{\rho^{n-2}}{\|x - y\|^{n-2}} \hat{\varphi}(\hat{x})$$

defines a function ψ on G. The mapping $\hat{\psi} \mapsto \psi$ is called the Kelvin transformation.

4.1.3 Theorem The Kelvin transformation preserves positivity and harmonicity.

Proof That positivity is followed directly from the definition. By computing the Laplacian of φ,

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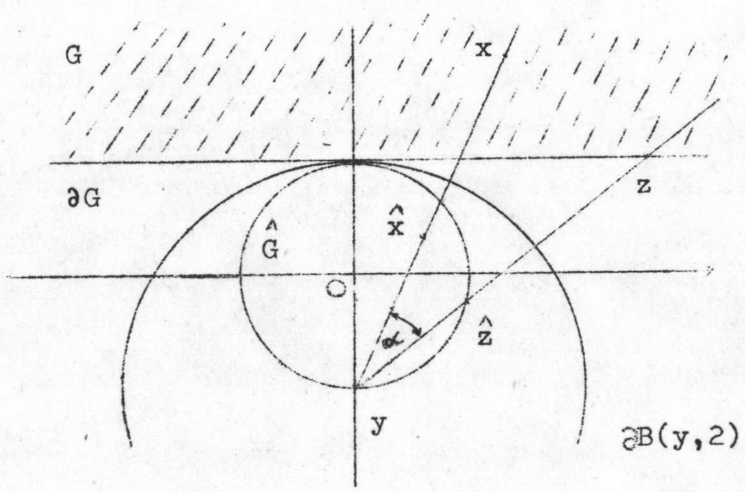
then the proof that ψ is harmonic in G whenever $\hat{\psi}$ is harmonic in $\hat{G}$ is accomplished. ([4], page 37)

4.2 Representation of a harmonic function in $\mathbb{R}^{n-1} \times (a,\infty)$

First consider the half-space $\mathbb{R}^{n-1} \times (1,\infty)$ and for any half-space $\mathbb{R}^{n-1} \times (a,\infty)$ will be taken at last.

Let $G = \mathbb{R}^{n-1} \times (1,\infty)$, $y = (0,\ldots,0,-1) \in \mathbb{R}^n$ and consider the inversion relative to $\partial B(y,2)$.



4.2.1 Lemma The inversion relative to $\partial B(y,2)$ maps G onto $B(0,1)$ and ∂G onto $\partial B \setminus \{y\}$. Moreover the inversion is 1-1.

Proof Let $x = (x_1,\ldots,x_n) \in G$, i.e. $x_n > 1$.

Since $$\hat{x} = y + \frac{\rho^2}{\|x - y\|^2}(x-y)$$

$$= (0,\ldots,0,-1) + \frac{4}{\|x - y\|^2}(x_1,\ldots,x_n+1)$$

$$= \left(\frac{4x_1}{\|x - y\|^2},\ldots,\frac{4x_{n-1}}{\|x - y\|^2}, \frac{4x_n + 4}{\|x - y\|^2} - 1\right),$$

we get $$\|\hat{x}\|^2 = \frac{8(1-x_n)}{\|x - y\|^2} + 1 ,$$

i.e. $$1 - \|\hat{x}\|^2 = \frac{8(x_n - 1)}{\|x - y\|^2} \qquad (1)$$

Since $x_n > 1$, $1 - \|\hat{x}\|^2 > 0$. Hence $\hat{x} \in B(0,1)$

If $z \in \partial G$, then note that $\hat{z} \in \partial B(0,1)$.

For $\hat{x} = (\xi_1,\ldots,\xi_n) \in B(0,1)$, i.e. $\sum_{i=1}^{n} \xi_i^2 < 1$,

we have $\hat{x} - y = (\xi_1,\ldots,\xi_n+1)$ and

$$\|\hat{x} - y\|^2 = \xi_1^2 + \ldots + \xi_{n-1}^2 + (\xi_n+1)^2 < 2 + 2\xi_n . \qquad (2)$$

Since $$x = y + \frac{4}{\|\hat{x} - y\|^2}(\hat{x} - y)$$

$$= \left(\frac{4\xi_1}{\|\hat{x} - y\|^2},\ldots,\frac{4\xi_{n-1}}{\|\hat{x} - y\|^2}, \frac{4\xi_n + 4 - \|\hat{x}-y\|^2}{\|\hat{x} - y\|^2}\right)$$

and the inequality $\frac{4\xi_n + 4 - \|\hat{x}-y\|^2}{\|\hat{x} - y\|^2} > 1$ which is deduced

from (2) we get $x \in G = \mathbb{R}^{n-1} \times (1,\infty)$.

If $\hat{x} \in \partial B \setminus \{y\}$, we can show that $x \in \partial G$.

This concludes that the inversion maps G onto B, ∂G onto $\partial B \setminus \{y\}$.

Finally we can check that the inversion is 1-1 #

Note that if T is the inversion which maps $\mathbb{R}^{n-1} \times (1,\infty)$ onto B, then T^{-1} maps B onto $\mathbb{R}^{n-1} \times (1,\infty)$.

4.2.2 <u>Theorem</u> Let h be positive and harmonic in $G = \mathbb{R}^{n-1} \times (1,\infty)$.

Then $$h(x) = c(x_n-1) + (x_n-1) \int_{\partial G} \frac{1}{\|x-z\|^n} \, d\mu(z) \qquad (x \in G)$$

where $x = (x_1,\ldots,x_n)$, c is a constant and μ is a measure on ∂G.

<u>Proof</u> Assume h is positive and harmonic in $G = \mathbb{R}^{n-1} \times (1,\infty)$.

The image of G under the inversion relative to $\partial B(y,2)$ is $B = B(0,1)$.

The Kelvin transformation $h \mapsto \hat{h}$ is defined by

$$\hat{h}(\hat{x}) = \frac{2^{n-2}}{\|y - \hat{x}\|} h\left(y + \frac{4}{\|\hat{x} - y\|^2}(\hat{x}-y)\right) \qquad (\hat{x} \in B).$$

By theorem 4.1.3, $\hat{h}$ is positive and harmonic in B.

Apply theorem 3.3.2 (Integral Representation), there is a positive Radon measure $\hat{\mu}$ on ∂B such that

$$\hat{h}(\hat{x}) = \int_{\partial B} \frac{1 - \|\hat{x}\|^2}{\|\hat{x} - \hat{z}\|^n} \, d\hat{\mu}(\hat{z})$$

$$= (1 - \|\hat{x}\|^2) \int_{\{y\}} \frac{1}{\|\hat{x} - \hat{z}\|^n} \, d\hat{\mu}(\hat{z})$$

$$+ \int_{\partial B \setminus \{y\}} \frac{1 - \|\hat{x}\|^2}{\|\hat{x} - \hat{z}\|^n} \, d\hat{\mu}(\hat{z}) \qquad (1)$$

Since $\hat{\mu}$ is a Radon measure, let $\hat{\mu}(\{y\}) = k$. (2)

From the proof of lemma 4.2.1, $1 - \|\hat{x}\|^2 = \dfrac{8(x_n - 1)}{\|x - y\|^2}$. (3)

Let α be the angle between the line segment joining $\hat{z}$ to y and the line segment joining $\hat{x}$ to y.

By using cosine's law

$$\|\hat{z} - \hat{x}\|^2 = \|\hat{z} - y\|^2 + \|\hat{x} - y\|^2 - 2\|\hat{z} - y\| \, \|\hat{x} - y\| \cos\alpha,$$

and the fact that $\|\hat{x} - y\| \, \|x - y\| = \rho^2 = 4$ we have

$$\|\hat{z} - \hat{x}\|^2 = \frac{16}{\|z-y\|^2} + \frac{16}{\|x-y\|^2} - 2 \cdot \frac{4}{\|z-y\|} \cdot \frac{4}{\|x-y\|} \cos\alpha$$

$$= \frac{16(\|x-y\|^2 + \|z-y\|^2 - 2\|x-y\| \, \|z-y\| \cos\alpha)}{\|z - y\|^2 \|x - y\|^2}$$

$$= \frac{16\|z - x\|^2}{\|z - y\|^2 \|x - y\|^2} \qquad (4)$$

From (2), (3), (4) and the fact $\|\hat{x} - y\| \, \|x - y\| = 4$, the equation (1) becomes

$$\hat{h}(\hat{x}) = \frac{8(x_n-1)k}{\|x-y\|^2\|\hat{x}-y\|^n} + \int_{\partial B\setminus\{y\}} \frac{8(x_n-1)}{\|x-y\|^2} \frac{\|z-y\|^n\|x-y\|^n}{4^n\|z-x\|^n} d\hat{\mu}(\hat{z})$$

$$= \frac{8(x_n-1)\|x-y\|^{n-2}}{4^n} + \frac{8(x_n-1)\|x-y\|^{n-2}}{4^n} \int_{\partial B\setminus\{y\}} \frac{\|z-y\|^n}{\|z-x\|^n} d\hat{\mu}(\hat{z}).$$

Denote the mapping $z \mapsto \hat{z}$ by T which maps ∂G onto $\partial B \setminus \{y\}$. The measure $\hat{\mu}$ induces a new measure μ_1 on ∂G defined by $\mu_1 = \hat{\mu}\, T$. Then we have

$$\frac{2^{n-2}}{\|x-y\|^n} \hat{h}(\hat{x}) = \frac{(x_n-1)k}{2^{n-1}} + \frac{(x_n-1)}{2^{n-1}} \int_{\partial G} \frac{\|z-y\|^n}{\|z-x\|^n} d\mu_1(z).$$

The left hand side of the last equation is just h(x).

Let $c = \frac{k}{2^{n-1}}$ and define a measure μ on ∂G by

$$\mu(F) = \int_F \frac{\|z-y\|^n}{2^{n-1}} d\mu_1(z) \quad \text{for any Borel set } F \text{ of } \partial G.$$

Then the last equation can be written as

$$h(x) = c(x_n-1) + (x_n-1) \int_{\partial G} \frac{1}{\|z-x\|^n} d\mu(z)$$

which is the desired result #

We end this thesis with the following theorem that comes directly from the preceeding theorem.

4.2.3 <u>Theorem</u> If h is positive and harmonic in $W = \mathbb{R}^{n-1} \times (a,\infty)$ where a is any real number, then there exist a constant c and a measure μ on ∂W such that

$$h(x) = c(x_n - a) + (x_n - a) \int_{\partial W} \frac{1}{\|x-z\|^n} \, d\mu(z) .$$

<u>Proof</u> Consider the mapping ψ defined on $W = \mathbb{R}^{n-1} \times (a,\infty)$ by $\psi(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-1}, x_n - a + 1)$.

It is clear that ψ is a 1-1 mapping from W onto $G = \mathbb{R}^{n-1} \times (1,\infty)$.

Assume h is positive and harmonic in W.

Define the function h_1 on G by

$$h_1(y_1,\ldots,y_n) = h(y_1,\ldots,y_{n-1}, y_n + a - 1) \qquad ((y_1,\ldots,y_n) \in G).$$

It follows that h_1 is positive and harmonic in G.

For $x = (x_1,\ldots,x_n) \in W$. $\psi(x) = (x_1,\ldots,x_{n-1}, x_n - a + 1) \in G$.

Hence by the preceeding theorem, there exist $c > 0$ and a measure μ_1 such that

$$h_1(x_1,\ldots,x_{n-1}, x_n - a + 1) = c(x_n - a) + (x_n - a) \int_{\partial G} \frac{1}{\|\psi(x) - z\|^n} \, d\mu_1(z).$$

Let μ be a measure on ∂W defined by $\mu = \mu_1 \psi$.

Since $\|\psi(x) - z\| = \|x - \psi^{-1}(z)\|$, then the last equation becomes

$$h(x) = c(x_n - a) + (x_n - a) \int_{\partial W} \frac{1}{\|x-N\|^n} \, d\mu(N)$$

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