CHAPTER III

ON THE SUPERHARMONICITY OF u^*

3.1 Notation

Recall that ω denotes an arbitrary open subset of the unit ball B in $\mathbb{R}^n$ and $\Omega = Q^{-1}[\omega]$ where Q is the function as in lemma 2.2.1.

Let Ω_1 be an open set such that

$$\Omega \subset \Omega_1 \subset \Omega \cup (\partial\Omega \cap E)$$

where $\partial\Omega$ is the boundary of Ω in $\mathbb{R}^{n+2}$ and let

$$\omega' = \{(\xi_1,\ldots,\xi_n) \varepsilon\ \partial B / (\xi_1,\ldots,\xi_n,0,0)\ \varepsilon\ \Omega_1\}.$$

Since the extension of superharmonic function to Ω_1 is interested, we drop the case $\Omega_1 = \Omega$. Note that $\Omega_1 \setminus \Omega \subset (\Omega \cup E) \setminus \Omega = E$ and if $\omega = B$, then we have $\Omega = \mathbb{R}^{n+2} \setminus E$, $\Omega_1 = \mathbb{R}^{n+2}$ and $\omega' = \partial B$.

3.2 Superharmonic Extension of u^*

In this section we take up the problem of continuation of superharmonic u^* across a polar set.

Since the function g defined by $g(x) = \int_E \frac{1}{\|x-z\|^n}\, d\sigma(z)$ is superharmonic in $\mathbb{R}^{n+2}$ and takes the value ∞ on E, then E is a polar set. Hence $E \cap \Omega_1$ is a polar set by 1.4.3. (3.2.1)

3.2.2 Theorem Let u be superharmonic in ω and

$$u^*(x) = f(x)\, u(Q(x)) \qquad (x \in \Omega).$$

Then the superharmonic function u^* has a unique superharmonic extension U to Ω_1 if and only if

$$\liminf_{M \to N} \frac{1}{1-\|M\|}\, u(M) > -\infty \qquad (N \in \omega').$$

Before proving this theorem we need the following remarks.

3.2.3 Remark For any $x = (x_1,\ldots,x_{n+2}) \in \mathbb{R}^{n+2} \setminus E$, the distance of x from E is $\mathrm{dist}(x,E) = \sqrt{(1-r)^2 + x_{n+1}^2 + x_{n+2}^2}$ where $r^2 = x_1^2 + \ldots + x_n^2$.

Proof For $y = (y_1,\ldots,y_n,0,0) \in E$, we have

$$\|x-y\|^2 = \|(x_1,\ldots,x_n)-(y_1,\ldots,y_n)\|^2 + x_{n+1}^2 + x_{n+2}^2 .$$

$$\geq \Big|\|(x_1,\ldots,x_n)\| - \|(y_1,\ldots,y_1)\|\Big|^2 + x_{n+1}^2 + x_{n+2}^2$$

$$= |r-1|^2 + x_{n+1}^2 + x_{n+2}^2 .$$

There exists $z \in E$ such that $\|x-z\|^2 = (r-1)^2 + x_{n+1}^2 + x_{n+2}^2$

Then $$\inf_{y \in E} \|x-y\|^2 = (r-1)^2 + x_{n+1}^2 + x_{n+2}^2$$

Hence $$[\mathrm{dist}\,(x,E)]^2 = (r-1)^2 + x_{n+1}^2 + x_{n+2}^2$$

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3.2.4 Remark If $x = (x_1, \ldots, x_{n+2}) \in \mathbb{R}^{n+2} \setminus E$, λ is the function as defined in lemma 2.2.1 and $d = \operatorname{dist}(x,E)$,

then
$$\lambda(x) \geq \frac{1}{(d+1)^2}$$

Proof Since $d^2 = (r-1)^2 + x_{n+1}^2 + x_{n+2}^2$, we have $d \geq r-1$ and $1 + \|x\|^2 = 1 + r^2 + x_{n+1}^2 + x_{n+2}^2 = d^2 + 2r$.

Then
$$\lambda(x) = \frac{2}{d^2 + 2r + \sqrt{(d^2+2r)^2 - 4r^2}}$$
$$= \frac{4}{(d + \sqrt{d^2 + 4r})^2}$$
$$\geq \frac{4}{(d + \sqrt{d^2 + 4d + 4})^2} = \frac{1}{(d+1)^2} \qquad \#$$

3.2.5 Remark If U is superharmonic in an open set D, then U is locally bounded below in D.

Proof For $z_o \in D$, there is an open ball $B(z_o,\rho)$ with compact closure $\bar{B}(z_o,\rho) \subset D$. Since $U > -\infty$ and U is ℓ.s.c. on compact $\bar{B}(z,\rho)$, we get U is bounded below in $\bar{B}(z_o,\rho)$.
That is, for some k, $U(x) \geq k \qquad (x \in B(z_o,\rho))$.
Hence U is locally bounded below in D #

Proof of theorem 3.2.2 We first prove the "if" part.
Assume $\liminf_{M \to N} \frac{1}{1-\|M\|} u(M) > -\infty \qquad (N \in \omega')$.
Claim that u^* is locally bounded below in Ω_1.

Let $z_o \varepsilon \Omega_1$. Since Ω is open, the case $z_o \varepsilon \Omega$ follows directly from remark 3.2.5.

If $z_o \not\varepsilon \Omega$, then $z_o \varepsilon E \cap \Omega_1$.

Let $z_o = (\xi_1, \ldots, \xi_n, 0, 0)$. Then $N_o = (\xi_1, \ldots, \xi_n) \varepsilon \omega'$.

By assumption, $\lim_{M \to N_o} \inf \frac{1}{1-\|M\|} u(M) > -\infty$.

So we have $k < 0$ and $\rho > 0$ such that

$$\frac{1}{1-\|M\|} u(M) \geqslant k \qquad (M \varepsilon B(N_o, \rho) \cap \omega). \tag{1}$$

Now $B(z_o, \rho)$ is a neighbourhood of z_o and

$$u^*(x) = \frac{\lambda^{n/2}(x)}{1-\|Q(x)\|^2} u(Q(x)) \qquad (x \varepsilon B(z_o, \rho) \cap \Omega).$$

By lemma 2.2.1 and let $M = Q(x)$, $\|M-N_o\| = \lambda^{1/2}(x)\|x-z_o\| < \rho$.

And by (1) we get

$$u^*(x) = \frac{\lambda^{n/2}(x)}{1+\|M\|} \cdot \frac{1}{1-\|M\|} u(M) \geqslant \frac{\lambda^{n/2}(x)}{1+\|M\|} k .$$

Since $\frac{\lambda^{n/2}(x)}{1+\|M\|} < 1$, $\quad u^*(x) \geqslant k \qquad (x \varepsilon B(z_o, \rho) \cap \Omega)$.

That is $\lim_{x \to z_o} \inf u^*(z_o) > -\infty$.

Hence u^* is locally bounded below in Ω_1.

From (3.2.1) $E \cap \Omega_1$ is a polar set and relatively closed in Ω_1, then by theorem 1.4.4 the superharmonic function u^* in $\Omega = \Omega_1 \setminus (E \cap \Omega_1)$ has a unique superharmonic extension U to Ω_1, i.e. $U = u^*$ on Ω.

To prove the "only if" part, we assume that u^* has a superharmonic extension U to Ω_1.

Then U is locally bounded below in Ω_1 by remark 3.2.5.

Since U is an extension of u^*, i.e. $U = u^*$ on Ω, we get

$$\liminf_{x \to z} u^*(x) > -\infty \qquad (z \in \Omega_1).$$

For any $N = (\xi_1, \ldots, \xi_n) \in \omega'$, $z = (\xi_1, \ldots, \xi_n, 0, 0) \in \Omega_1 \cap E$.

Then there exist $k < 0$ and $\rho > 0$ such that

$$u^*(x) \geq k \qquad (x \in B(z,\rho) \cap \Omega) \qquad (2)$$

Let $M \in B(N, \frac{\rho}{\rho+1}) \cap \omega \subset \mathbb{R}^n$.

Then $M = Q(x_0)$ for some x_0 in Ω. By lemma 2.2.1 and (3.2.4)

$$\frac{\rho}{\rho+1} > \|M-N\| = \|x_0 - z\| \lambda^{1/2}(x_0) \geq \frac{\|x_0 - z\|}{d_0 + 1} \geq \frac{\|x_0 - z\|}{\|x_0 - z\| + 1}$$

where $d_0 = \text{dist}(x_0, E)$, $z \in E$.

That is $\|x_0 - z\| < \rho$.

From (2) we have $\dfrac{\lambda^{n/2}(x_0)}{1 - \|M\|^2} u(M) = u^*(x_0) \geq k$,

$$\frac{1}{1 - \|M\|} u(M) \geq k(1 + \|M\|) \lambda^{-n/2}(x_0).$$

Since $\qquad \rho > \|x_0 - z\| > d_0$, $\lambda^{1/2}(x_0) > \dfrac{1}{d_0 + 1} > \dfrac{1}{\rho + 1}$.

We have $\frac{1}{1-\|M\|} \; u(M) \geqslant 2k\,\lambda^{-n/2}(x_0) \geqslant 2k(\rho+1)^n$.

Hence $\displaystyle \liminf_{M\to N} \frac{1}{1-\|M\|} \; u(M) > -\infty \qquad (N \in \omega').$

That is the theorem is completely proved #

3.2.6 <u>Corollary</u> Let u be superharmonic in B and

$$u^*(x) = f(x)u(Q(x)) \qquad (x \in \mathbb{R}^{n+2} \setminus E).$$

Then the superharmonic function u has a unique superharmonic extension U to $\mathbb{R}^{n+2}$ if and only if $u \geqslant 0$ on B.

<u>Proof</u> We apply the theorem 3.2.2 in the case $\omega = B$, then $\Omega = \mathbb{R}^{n+2} \setminus E$, $\Omega_1 = \mathbb{R}^{n+2}$ and $\omega' = \partial B$.

If $u \geqslant 0$ on B, then

$$\liminf_{M\to N} \frac{1}{1-\|M\|} \; u(M) \geqslant 0 > -\infty \qquad (N \in \partial B).$$

Hence u^* has a unique superharmonic extension U to $\mathbb{R}^{n+2}$.

Conversely if u^* has a superharmonic extension U to $\mathbb{R}^{n+2}$, we have $\displaystyle \liminf_{M\to N} \frac{1}{1-\|M\|} \; u(M) > -\infty \qquad (N \in \partial B)$,

i.e. there exist $k \in \mathbb{R}$, $\rho > 0$ such that

$$\frac{1}{1-\|M\|} \; u(M) \geqslant k \qquad (M \in B(N,\rho) \cap B).$$

Hence $\displaystyle \liminf_{M\to N} u(M) \geqslant \liminf_{M\to N} (1-\|M\|)k = 0 \qquad (N \in \partial B).$

By theorem 1.3.3 $u \geqslant 0$ on B #

3.3 Integral Representation

If u is superharmonic in an open set D, h is harmonic in D and $h \leq u$ on D, then h is called a harmonic minorant of u. The function h is the greatest harmonic minorant of u if h is a harmonic minorant of u and $h \geq v$ whenever v is a harmonic minorant of u.

The following theorem is drawn from [5], page 47.

3.3.1 Theorem (Riesz Decomposition) If U is superharmonic in $\mathbb{R}^m$ $(m \geq 3)$, then

$$U(x) = \int_{R^m} \frac{1}{\|x-y\|^{m-2}} \, d\mu(y) + G(x) \qquad (x \in \mathbb{R}^m)$$

where G is the greatest harmonic minorant of U and the measure μ is given by $\mu = \frac{-\Delta U}{(m-2)\sigma_m}$, ΔU is regarded as the distribution Laplacian of U.

Now return to theorems on the passage. The following theorem has shown that a harmonic function in B can be represented by the certain integral.

3.3.2 Theorem If h is positive and harmonic in the unit ball B of $\mathbb{R}^n$, then

$$h(M) = \int_{\partial B} \frac{1-\|M\|^2}{\|M-N\|^n} \, d\mu(N) \qquad (M \in B)$$

where μ is a positive Radon measure on ∂B.

The measure μ, understood as a measure on $E = \partial B$, is given by $\mu = -\frac{\Delta U}{n\sigma_{n+2}}$ where ΔU is the distribution Laplacian of the superharmonic extension U of $u^* = h^*$ as in corollary 3.2.6.

To prove this we need a lemma.

3.3.3 Lemma Let h be a continuous **real-valued function on B**,

$$h^*(x) = f(x)h(Q(x)) \qquad (x \in \mathbb{R}^{n+2}).$$

Then $\lim_{x\to y} h^*(x) = 0$ where y is a point of infinity of $\mathbb{R}^{n+2}$, i.e. for $\varepsilon > 0$, there exists $b > 0$ such that

$$|h^*(x)| < \varepsilon \qquad (\|x\| > b) .$$

Proof Let $\varepsilon > 0$. Choose $c = 1 - \frac{1}{\sqrt{5}}$.

Since h is continuous on the compact closure $\bar{B}(0,c) \subset B$, there exists $k > 0$ such that

$$|h(M)| \leqslant k \qquad (M \in \bar{B}(0,c)) \qquad (1)$$

Choose $b = \max\left\{2,\ 2\left[\frac{k}{\varepsilon(1-c^2)}\right]^{2/n}\right\}$.

Let $x = (x_1,\ldots,x_{n+2}) \in \mathbb{R}^{n+2}$ with $\|x\| > b$.

Since $N_o = (\frac{x_1}{r},\ldots,\frac{x_n}{r}) \in \partial B$ where $r^2 = \sum_{i=1}^{n} x_i^2$ and by lemma 2.2.1, $\|x-z_o\|\lambda^{1/2} = \|Q(x) - N_o\|$ where $\psi(z_o) = N_o$.

Then we have $\|Q(x)\| = 1 - \|x-z_o\|\lambda^{1/2}$. (2)

By the fact that $\lambda^{1/2}(x) \geqslant \frac{1}{\sqrt{1+\|x\|^2}}$ and $\|x\| \leqslant 1 + \|x-z_o\|$, we get $\|x-z_o\| \lambda^{1/2}(x) \geqslant (\|x\|-1)\lambda^{1/2}(x) \geqslant \dfrac{\|x\|-1}{\sqrt{1+\|x\|^2}}$.

From (2), $\|Q(x)\| \leqslant 1 - \dfrac{\|x\|-1}{\sqrt{1+\|x\|^2}} < 1 - \dfrac{1}{\sqrt{5}} = c.$

By (1) $\qquad |h(Q(x))| \leqslant k$.

Therefore $|h^*(x)| \leqslant \dfrac{\lambda^{n/2}(x)}{1-\|Q(x)\|^2} |h(Q(x))| \leqslant \dfrac{\lambda^{n/2}(x)k}{1-c^2}$

Since $\lambda(x) \leqslant \dfrac{2}{1+\|x\|^2}$, we obtain

$$|h^*(x)| \leqslant \frac{k\, 2^{n/2}}{(1-c^2)(1+\|x\|^2)^{n/2}} < \varepsilon .$$

The lemma is completely proved #

We end this chapter with the proof of theorem 3.3.2.

Suppose that h is positive and harmonic in B. By theorem 2.3.1 and corollary 3.2.6, the harmonic function $h^* = f(h \circ Q)$ in $\mathbb{R}^{n+2} \setminus E$ possesses the superharmonic extension U to $\mathbb{R}^{n+2}$.

Since $U = h^*$ on $\mathbb{R}^{n+2} \setminus E$, $U > 0$ on $\mathbb{R}^{n+2} \setminus E$.

Let $y \in E$. By theorem 1.3.6,

$$U(y) \geqslant A(U; y, \delta) \qquad (\delta > 0).$$

Since (n+2)-dimensional Lebesgue measure of E is zero and $U > 0$ on $B(y, \delta) \setminus E$, we have

$$A(U; y,\delta) = \frac{1}{\nu_n \delta^n} \int_{B(y,\delta)} U(z)dz = \frac{1}{\nu_n \delta^n} \int_{B(y,\delta)\setminus E} U(z)dz > 0.$$

Hence $U > 0$ on $\mathbb{R}^{n+2}$.

Since the zero function is a harmonic minorant of U, the greatest harmonic minorant of U, say G, must be non-negative. From theorem 1.2.5, G is constant on $\mathbb{R}^{n+2}$. By lemma 3.3.3 and $0 \leq G \leq h^*$, we have $G = 0$ on $\mathbb{R}^{n+2}$.

Recall the theorem 3.3.2 (Riesz decomposition),

$$U(x) = \int_{\mathbb{R}^{n+2}} \frac{1}{\|x-z\|^n} \, d\mu(z) + 0$$

where the measure μ is the distribution Laplacian of U multiplied by the constant $-\frac{1}{n\sigma_{n+2}}$, i.e. $\mu = -\frac{\Delta U}{n\sigma_{n+2}}$.

Since U is harmonic in $\mathbb{R}^{n+2} \setminus E$, $\mu = 0$ on $\mathbb{R}^{n+2} \setminus E$.

Then $h^*(x) = U(x) = \int_E \frac{1}{\|x-z\|^n} \, d\mu(z)$.

Let $M = (x_1, \ldots, x_n) \in B$ and $x = (x_1, \ldots, x_n, 0, 0)$.

Then $Q(x) = M$, $\lambda(x) = 1$ and

$$\frac{1}{1-\|M\|^2} h(M) = f(x)h(Q(x)) = h^*(x)$$

$$= \int_E \frac{1}{\|x-z\|^n} \, d\mu(z)$$

$$= \int_{\partial B} \frac{1}{\|M-N\|^n} \, d\mu(N),$$

the theorem is now completely proved. #