CHAPTER II

PRELIMINARIES

2.1 Basic Definitions and Notations.

Throughout this thesis we shall denote the set of all non-negative integers and the set of all positive integers by $\mathbb{N}$ and $\mathbb{P}$ respectively. The cardinality of a set S is denoted by $|S|$.

By a finite sequence having n terms in a set S, we mean a function defined on the set $\{1,\ldots, n\}$ into S. If f is a finite sequence having n terms in a set S and $f(i) = s_i$ for $1 \leq i \leq n$, then it is usually written in the form

$$f = (s_1,\ldots,s_n).$$

By a k-partite finite sequence in a set S, we mean a finite sequence $(\delta_1,\ldots,\delta_k)$ of finite sequences δ_t, $1 \leq t \leq k$, in S. If

$$\delta_t = (\delta_t(1),\ldots,\delta_t(n_t)) \quad \text{for} \quad 1 \leq t \leq k,$$

then $(\delta_1,\ldots,\delta_k)$ will be denoted by

$$(\delta_1(1),\ldots,\delta_1(n_1);\ldots;\delta_k(1),\ldots,\delta_k(n_k)).$$

By a m × n-matrix over a set S, we mean a function defined on the Cartesian product $\{1,\ldots,m\} \times \{1,\ldots,n\}$ into S. If μ is a m × n-matrix over a set S and $\mu(i,j) = s_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq n$, then it is usually written in the form

$$\mu = \begin{bmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \vdots & \vdots \\ s_{m1} & \cdots & s_{mn} \end{bmatrix}$$

2.2 Digraphs and Network Flows.

A digraph D is an ordered pair (V,A), where V is a finite non-empty set, its elements are called vertices, and A is a subset of the Cartesian product $V \times V$, its elements are called arcs. For an arc $a = (x,y)$, the vertex x is called its initial endpoint, and the vertex y is called its terminal endpoint, and we say that the arc a joins x to y.

A digraph can be represented by a geometric diagram in which the vertices are indicated by small circles or dots, while any two of them, say x and y are joined by an arrowheaded continuous curve from x to y if and only if (x,y) is an arc. As an illustration, consider the digraph $D = (V,A)$ where

$$V = \{v, w, x, y, z\},$$

$$A = \{(v,v), (v,w), (w,v), (x,w), (y,y)\}.$$

The geometric diagram of D is shown in the following figure :

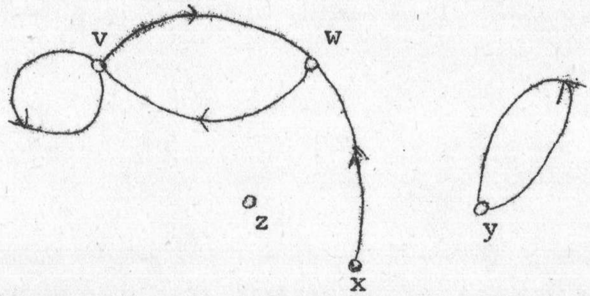

Fig. 2.2.1

A network N is a triple (V,A,α), where (V,A) is a digraph, and α is a function defined on the set A into $\mathbb{N}$. The function α is called the capacity function, and its value at an arc a is called the capacity of a. A function ψ defined on the set A into $\mathbb{N}$ is said to be conservative at vertex x if

$$\sum_{(x,y)\,\varepsilon\, A} \psi(x,y) = \sum_{(z,x)\,\varepsilon\, A} \psi(z,x).$$

Here, and in the sequel, we denote the value of any function ψ at (x,y) simply by ψ(x,y). If ψ is conservative at every vertex in N, then ψ is called a flow in N. We shall say that a flow ψ saturates an arc a in N if $\psi(a) = \alpha(a)$. If ψ is a flow in N such that $\psi(a) \leq \alpha(a)$ for all arcs a, then ψ is said to be compatible.

By a bipartite transportation network, we mean a network in which the vertices form four disjoint sets X, Y,{u},{v}; the vertex u is called the source vertex, the vertex v is called the sink vertex; the arcs are of the following types:

type 1. (x,y) with $x \in X$, $y \in Y$; called the intermediate arcs,

type 2. (u,x) with $x \in X$; called the source arcs,

type 3. (y,v) with $y \in Y$; called the sink arcs,

type 4. (v,u); called the return arc,

and the capacity of the return arc is not less than the sum of the capacities of the source arcs. We shall denote such a bipartite transportation network in which α is the capacity function by $(\{u\}, X, Y, \{v\}; \alpha)$. A compatible flow ψ in a bipartite transportation network N is called a maximum flow if there is no compatible flow ψ' in N such that at the return arc the value of ψ' is greater than the value of ψ

A network can be represented by a diagram of its digraph together with the capacities written on the curves representing the arcs. The following diagram represents a bipartite transportation network.

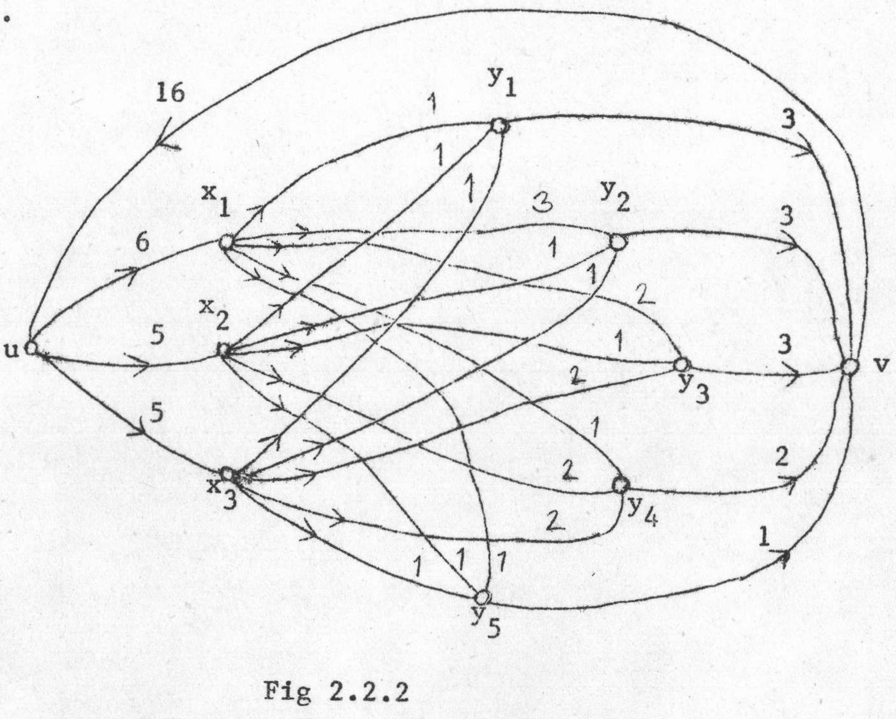

Fig 2.2.2

In Fig. 2.2.2, the number appearing on each curve refers to the capacity of the arc it represents. For example, the capacity of the arc (u,x_1) is 6.

Let $N = (\{u\}, X, Y, \{v\}; \alpha)$ be a bipartite transportation network. Let $B \subseteq Y$. Suppose ψ is a compatible flow in N. Since ψ is conservative at every vertex y in B, we have

$$\sum_{y \in B} \sum_{x \in X} \psi(x,y) = \sum_{y \in B} \psi(y,v).$$

Since ψ is conservative at the sink vertex v, we have

$$\sum_{y \in Y} \psi(y,v) = \psi(v,u)$$

Hence the quantity $\sum_{y \in B} \sum_{x \in X} \psi(x,y)$ equals to the value of ψ at the return arc if $\alpha(y,v) = 0$ for all $y \in Y \setminus B$. We see that the value of ψ at each sink arc (y_o,v) is not greater than the sum of the capacities of the source arcs since

$$\begin{aligned} \psi(y_o,v) &= \sum_{x \in X} \psi(x,y_o) \\ &\leq \sum_{x \in X} \sum_{y \in Y} \psi(x,y) \\ &= \sum_{x \in X} \psi(u,x) \\ &\leq \sum_{x \in X} \alpha(u,x) \end{aligned}$$

Now, let $N_B = (\{u\}, X, Y, \{v\}; \alpha_B)$ be a bipartite transportation network obtained from N by changing α to α_B in the following way :

$$\alpha_B(y,v) = 0 \text{ if } y \in Y \setminus B,$$

$$\alpha_B(y,v) = \sum_{x \in X} \alpha(u,x) \text{ if } y \in B,$$

$$\alpha_B(a) = \alpha(a) \text{ for all other arcs } a.$$

The <u>maximum quantity of flow that can be sent into B</u>, denoted by $F_N(B)$, is defined by

$$F_N(B) = \max_{f \in \mathcal{F}(B)} f(v,u),$$

where $\mathcal{F}(B)$ is the set of all compatible flows in N_B.

The following formula is derived from the above definition.

(2.2.1) $$F_N(B) = \sum_{x \in X} \min\{\alpha(u,x), \sum_{y \in B} \alpha(x,y)\}.$$

To show this, first we show that

(1) $$\max_{f \in \mathcal{F}(B)} f(v,u) \leq \sum_{x \in X} \min\{\alpha(u,x), \sum_{y \in B} \alpha(x,y)\}.$$

Let f be any compatible flow in N_B. Then, for each $x \in X$, we have

(2) $$f(u,x) = \sum_{y \in Y} f(x,y)$$

$$= \sum_{y \in B} f(x,y)$$

$$\leq \sum_{y \in B} \alpha(x,y),$$

and

(3) $$f(u,x) \leq \alpha(u,x).$$

Therefore, it follows from (2) and (3) that

(4) $$f(u,x) \leq \min\{\alpha(u,x), \sum_{y\varepsilon B} \alpha(x,y)\}.$$

By the conservation property of f at u and (4), we have

(5) $$f(v,u) = \sum_{x\varepsilon X} f(u,x)$$
$$\leq \sum_{x\varepsilon X} \min\{\alpha(u,x), \sum_{y\varepsilon B} \alpha(x,y)\}.$$

Hence (1) holds.

Now, we show that

(6) there exists a compatible flow f' in N_B such that

$$f'(u,x) = \min\{\alpha(u,x), \sum_{y\varepsilon B} \alpha(x,y)\} \text{ for all } x \varepsilon X.$$

We shall prove this by induction on the cardinality of B. For $|B| = 0$, we have $B = \phi$. It can be seen that the function f', defined by

$$f'(a) = 0 \text{ for all arcs } a \text{ in } N_B,$$

is a compatible flow in N_B that has the properties as required. Assume that (6) holds for $|B| = n < |Y|$. Let C be any subset of Y such that $|C| = n + 1$. Suppose that $C = \{y_1,\dots,y_{n+1}\}$. By the induction hypothesis, there exists a compatible flow f'' in $N_{C\setminus\{y_{n+1}\}}$ such that

(7) $$f''(u,x) = \min\{\alpha(u,x), \sum_{j=1}^{n} \alpha(x,y_j)\} \text{ for all } x \in X.$$

Also, for all $x \in X$, we have

(8) $$f''(u,x) = \sum_{y \in Y} f''(x,y)$$

$$= \sum_{y \in Y \setminus \{y_{n+1}\}} f''(x,y).$$

Construct a function f' on the set of arcs in N_C as follows :

<u>Step 1</u>. For any $x \in X$, put

(9) $$f'(u,x) = \min\{\alpha(u,x), \sum_{j=1}^{n+1} \alpha(x,y_j)\}.$$

<u>Step 2</u>. Put

(10) $$f'(v,u) = \sum_{x \in X} f'(u,x).$$

<u>Step 3</u>. For any $x \in X$, put

(11) $$f'(x,y) = f''(x,y) \text{ for all } y \in Y \setminus \{y_{n+1}\},$$

and

(12) $$f'(x,y_{n+1}) = \begin{cases} f''(x,y_{n+1}) & \text{if } \alpha(u,x) \leq \sum_{j=1}^{n} \alpha(x,y_j), \\ \alpha(u,x) - \sum_{j=1}^{n} \alpha(x,y_j) & \text{if } \sum_{j=1}^{n} \alpha(x,y_j) < \alpha(u,x) \leq \sum_{j=1}^{n+1} \alpha(x,y_j), \\ \alpha(x,y_{n+1}) & \text{if } \sum_{j=1}^{n+1} \alpha(x,y_j) < \alpha(u,x). \end{cases}$$

Step 4. For any $y \in Y$, put

$$(13) \qquad f'(y, v) = \sum_{x \in X} f'(x,y).$$

Now, we shall show that f' is conservative at every vertex in N_C. Let $x \in X$.

Case 1. Assume that $\alpha(u,x) \leq \sum_{j=1}^{n} \alpha(x,y_j)$. Then, by (9) and (7), we have

$$(14) \qquad f'(u,x) = \alpha(u,x)$$

$$= f''(u, x).$$

By (8), (11) and (12), we have

$$(15) \qquad f''(u,x) = \sum_{y \in Y} f''(x,y)$$

$$= \sum_{y \in Y} f'(x,y).$$

Hence, by (14) and (15), we get

$$f'(u,x) = \sum_{y \in Y} f'(x,y).$$

i.e. f' is conservative at x,

Case 2. Assume that $\sum_{j=1}^{n} \alpha(x,y_j) < \alpha(u,x) \leq \sum_{j=1}^{n+1} \alpha(x,y_j)$.

Then, by (11) and (12), we have

(16) $$\sum_{y \in Y} f'(x,y) = \sum_{y \in Y \setminus \{y_{n+1}\}} f''(x,y) + \alpha(u,x) - \sum_{j=1}^{n} \alpha(x,y_j).$$

By (8) and (7), we have

(17) $$\sum_{y \in Y \setminus \{y_{n+1}\}} f''(x,y) = f''(u,x) = \sum_{j=1}^{n} \alpha(x,y_j).$$

Hence, by (16), (17) and (9), we get

$$\sum_{y \in Y} f'(x,y) = \alpha(u,x) = f'(u,x).$$

Therefore f' is conservative at x.

<u>Case 3</u>. Assume that $\sum_{j=1}^{n+1} \alpha(x,y_j) < \alpha(u,x)$. Then, by (11) and (12), we have

(18) $$\sum_{y \in Y} f'(x, y) = \sum_{y \in Y \setminus \{y_{n+1}\}} f''(x,y) + \alpha(x,y_{n+1}).$$

In this case, we also have (17). Hence, by (18), (17) and (9), we get

$$\sum_{y \in Y} f'(x,y) = \sum_{j=1}^{n+1} \alpha(x,y_j) = f'(u,x).$$

Therefore f' is conservative at x.

This proves that f' is conservative at every vertex $x \in X$. Observe, from (10) and (13), that f' is conservative at the vertex u and every vertex $y \in Y$. Hence we have

$$\begin{aligned} f'(v,u) &= \sum_{x \in X} f'(u,x) \\ &= \sum_{x \in X} \sum_{y \in Y} f'(x,y) \\ &= \sum_{y \in Y} \sum_{x \in X} f'(x,y) \\ &= \sum_{y \in Y} f'(y,v). \end{aligned}$$

Therefore f' is conservative at the vertex v.

Hence f' is a flow in N_C. We can see that f' is compatible. This completes the proof of (6). Note that

$$\begin{aligned} f'(v,u) &= \sum_{x \in X} f'(u,x) \\ &= \sum_{x \in X} \min\{\alpha(u,x), \sum_{y \in B} \alpha(x,y)\} \end{aligned}$$

Therefore, by (1) and the fact that $f' \in \mathcal{F}(B)$, we get

$$\max_{f \in \mathcal{F}(B)} f(v,u) = f'(v,u).$$

i.e. $$F_N(B) = f'(v,u).$$

Hence we obtain the equation (2.2.1).

In a bipartite transportation network $N = (\{u\}, X, Y, \{v\}; \alpha)$, the <u>demand</u> of a subset B of Y, denoted by $d_N(B)$, is defined by

$$(2.2.2) \qquad d_N(B) = \sum_{y \in B} \alpha(y,v).$$

As an illustration, consider the bipartite transportation network N represented by the diagram in Fig. 2.2.2. Let $B = \{y_1, y_3, y_4\}$. Then, by the equations (2.2.1) and (2.2.2),
we have

$$\begin{aligned} F_N(B) &= \sum_{x \in X} \min\{\alpha(u,x), \sum_{y \in B} \alpha(x,y)\} \\ &= \min\{6,4\} + \min\{5,4\} + \min\{5,5\} \\ &= 4 + 4 + 5 \\ &= 13 \end{aligned}$$

and

$$\begin{aligned} d_N(B) &= \sum_{y \in B} \alpha(y,v) \\ &= 3 + 3 + 2 \\ &= 8. \end{aligned}$$

In [1] (see page 84), the following theorem, due to D. Gale, provides a necessary and sufficient condition for the existence of a compatible flow that saturates all the sink arcs in a bipartite transportation network.

2.2.3 Theorem. A bipartite transportation network $N = (\{u\}, X, Y, \{v\}; \alpha)$ has a compatible flow that saturates all the sink arcs if and only if

$$F_N(B) \geq d_N(B) \quad \text{for all } B \subseteq Y.$$

The next remarks are useful to our study.

2.2.4 Remark. Let N be a bipartite transportation network such that the sum of the capacities of the source arcs equals to the sum of the capacities of the sink arcs. And let ψ be a compatible flow in N. Then, ψ saturates all the sink arcs if and only if ψ saturates all the source arcs.

Proof : Let $N = (\{u\}, X, Y, \{v\}; \alpha)$ be a bipartite transportation network such that

$$(1) \qquad \sum_{x \in X} \alpha(u,x) = \sum_{y \in Y} \alpha(y,v).$$

Let ψ be any compatible flow in N. Hence

$$(2) \qquad \sum_{x \in X} \psi(u,x) = \psi(v,u) = \sum_{y \in Y} \psi(y,v).$$

Suppose that ψ saturates all the sink arcs but does not saturate some source arc. Hence

$$\sum_{x \in X} \psi(u,x) < \sum_{x \in X} \alpha(u,x)$$

$$= \sum_{y \in Y} \alpha(y,v)$$

$$= \sum_{y \in Y} \psi(y,v),$$

which is contrary to (2). Hence, if ψ saturates all the sink arcs, it must also saturate all the source arcs. The converse can be shown similarly. #

2.2.5 <u>Remark</u>. Let δ_1 and δ_2 be finite sequences having n_1 and n_2 terms in $\mathbb{N}$, respectively. Suppose that

(i) $$\sum_{i=1}^{n_1} \delta_1(i) = \sum_{j=1}^{n_2} \delta_2(j), \text{ and}$$

(ii) $$\sum_{i=1}^{n_1} \min\{\delta_1(i), |B|\} \geq \sum_{j \in B} \delta_2(j) \text{ for } B \subseteq \{1,\ldots,n_2\}.$$

Then we have

(iii) $$\sum_{j=1}^{n_2} \min\{\delta_2(j), |C|\} \geq \sum_{i \in C} \delta_1(i) \text{ for } C \subseteq \{1,\ldots,n_1\}$$

<u>Proof</u> : Let $N = (\{u\}, X, Y, \{v\}; \alpha)$ be a bipartite transportation network, where $X = \{x_1, \ldots, x_{n_1}\}$, $Y = \{1, \ldots, n_2\}$, and

$$\alpha(x_i, j) = 1 \text{ for } 1 \leq i \leq n_1, \ 1 \leq j \leq n_2;$$

$$\alpha(u, x_i) = \delta_1(i) \text{ for } 1 \leq i \leq n_1;$$

$$\alpha(j,v) = \delta_2(j) \text{ for } 1 \leq j \leq n_2;$$

$$\alpha(v,u) = \sum_{i=1}^{n_1} \delta_1(i).$$

For $B \subseteq Y$, by the equations (2.2.1) and (2.2.2), we have

(1) $$F_N(B) = \sum_{i=1}^{n_1} \min\{\delta_1(i), |B|\},$$

and

(2) $$d_N(B) = \sum_{j \varepsilon B} \delta_2(j).$$

Hence, by (1) and (2), the assumption (ii) becomes

$$F_N(B) \geq d_N(B) \text{ for all } B \subseteq Y.$$

Therefore, by Theorem 2.2.3, N has a compatible flow ψ that saturates all the sink arcs. By the definition of α and the assumption (i), we have

$$\sum_{x_i \varepsilon X} \alpha(u,x_i) = \sum_{j \varepsilon Y} \alpha(j,v).$$

Hence, by Remark 2.2.4, ψ also saturates all the source arcs. Now, let $N' = (\{v\}, Y, X, \{u\}; \alpha')$ be a bipartite transportation network, where the capacity function α' is given by

$$\alpha'(j,x_i) = 1 \text{ for } 1 \leq j \leq n_2,\ 1 \leq i \leq n_1;$$

$$\alpha'(v,j) = \delta_2(j) \text{ for } 1 \leq j \leq n_2;$$

$$\alpha'(x_i,u) = \delta_1(i) \text{ for } 1 \leq i \leq n_1;$$

$$\alpha'(u,v) = \sum_{j=1}^{n_2} \delta_2(j).$$

Define a function f on the set of arcs in N' by

$$f(a,b) = \psi(b,a) \quad \text{for all arcs } (a,b).$$

Then f is a compatible flow that saturates all the sink arcs in N'. Let $C \subseteq \{1,\ldots,n_1\}$ and $X(C) = \{x_i \in X \,/\, i \in C\}$. Then, by Theorem 2.2.3, we have

$$(3) \qquad F_{N'}(X(C)) \geq d_{N'}(X(C)).$$

By the equations (2.2.1) and (2.2.2), we have

$$F_{N'}(X(C)) = \sum_{j=1}^{n_2} \min\{\delta_2(j), |C|\},$$

and

$$d_{N'}(X(C)) = \sum_{i \in C} \delta_1(i).$$

Hence (1) becomes

$$\sum_{j=1}^{n_2} \min\{\delta_2(j), |C|\} \geq \sum_{i \in C} \delta_1(i).$$

Therefore (iii) holds. #

2.3 Hypergraphs.

A hypergraph H is an ordered pair $(V, \mathcal{E})$, where V is a finite non-empty set, and $\mathcal{E}$ is a set of non-empty subsets of V such that $\cup \mathcal{E} = V$. The elements in V are called vertices, and the sets in $\mathcal{E}$

are called hyperedges or simply edges. The rank of a hypergraph is the maximum cardinality of the edges in the hypergraph. A hypergraph in which every edge has the same cardinality is called an uniform hypergraph. An uniform hypergraph of rank r will be called an r-uniform hypergraph. A hypergraph $H = (V, \mathcal{E})$ is called a k-partite hypergraph if V can be partitioned into k subsets V_t, $1 \leq t \leq k$, such that $|E \cap V_t| \leq 1$ for every edge E and for $1 \leq t \leq k$. Such an ordered partition $(V_1, \ldots, V_k)$ is called a k-partition of V. We shall often denote a k-partite hypergraph in which $(V_1, \ldots, V_k)$ is a k-partition of the set of vertices, and $\mathcal{E}$ is the set of edges by $(V_1, \ldots, V_k; \mathcal{E})$. By a (k,r) - hypergraph, we mean a k-partite r-uniform hypergraph.

To illustrate the above concepts, let

$$V = \{1,2,3,4,5\},$$
$$\mathcal{E}_1 = \{\{1,2,3\},\{1,3,4\},\{2,3,5\},\{3,4,5\},\{3,5\},\{1\}\},$$
$$\mathcal{E}_2 = \{\{1,2,3\},\{1,3,4\},\{2,3,5\},\{3,4,5\}\},$$
$$\mathcal{E}_3 = \{\{1,2,3\},\{1,2,4\}, \{2,3,4\},\{2,3,5\}\}.$$

Then $H_1 = (V, \mathcal{E}_1)$ is a 3-partite hypergraph since $(\{1,5\},\{2,4\},\{3\})$ is a 3-partition of V. $H_2 = (V, \mathcal{E}_2)$ is a (3,3)-hypergraph. We can see that $H_3 = (V, \mathcal{E}_3)$ is a 3-uniform hypergraph. But H_3 is not a 3-partite hypergraph.

Let $H = (V, \mathcal{E})$ be a hypergraph. For each subset S of V, we define

$$\mathcal{E}(S) = \{E \in \mathcal{E} \;/\; S \subseteq E\}.$$

The degree of S in H, denoted by $d_H(S)$, is defined by

$$d_H(S) \quad = \quad |\mathcal{E}(S)|.$$

We shall write $\mathcal{E}(v)$ and $d_H(v)$ instead of $\mathcal{E}(\{v\})$ and $d_H(\{v\})$ respectively. For every vertex v, since $\cup\mathcal{E} = V$, $\mathcal{E}(v)$ is a non-empty subset of $\mathcal{E}$; hence $d_H(v) \neq 0$.

As an illustration, consider the hypergraph H_1 above. Then we have

$$\mathcal{E}_1(1) \quad = \quad \{\{1,2,3\},\{1,3,4\},\{1\}\},$$
$$\mathcal{E}_1(\{2,3\}) \quad = \quad \{\{1,2,3\},\{2,3,5\}\};$$

hence

$$d_{H_1}(1) \quad = \quad 3 \text{ and } d_{H_1}(\{2,3\}) \quad = \quad 2.$$

In the following study, we shall consider only (3,3)- hypergraphs