CHAPTER V

FIXED POINT THEOREM FOR LOCALLY CONTRACTIVE MAPS

The materials of this chapter are drawn from reference [1].

We have seen in Banach's contraction theorem that if (E, d) is a complete metric space and $T : E \longrightarrow E$ is a contraction map, then T has the unique fixed point.

It is natural to ask whether this theorem could be modified so as to be valid when contraction map T is assumed to hold for sufficiently closed point only. To be more specific we introduce the following definitions.

5.1.1. Definition. Let (E, d) be a generalized semi-metric space. Let $T : E \longrightarrow E$ be a map.

T is called locally contractive map if for each $x \in E$, there exist real numbers ε_x, q_x with $\varepsilon_x > 0$ and $q_x \in [0, 1)$ such that

$$y, z \in S(x, \varepsilon_x) \text{ implies } d(T(y), T(z)) \leq q_x \, d(y, z)$$

T is called an (ε, q) - uniformly locally contractive map if it is locally contractive and both ε_x and q_x do not depend on x.

5.1.2 Remark. (a) If T is q-contraction for some $q \in [0,1)$, then T is (ε, q) - uniformly locally contractive for every $\varepsilon > 0$.

(b) There are examples of uniformly locally contractive map that are not contraction.

(c) Any locally contraction map T is continuous.

First, we prove (a). Let T be a q-contractive map for some $q \in [0, 1)$, then for any $x, y \in E$

$$d(T(x), T(y)) \leqslant q\, d(x, y).$$

Given any $\varepsilon > 0$, for each $x \in E$ and $y, z \in S(x, \varepsilon)$

$$d(T(y), T(z)) \leqslant q\, d(y, z)$$

and since ε and q do not depend on x. Therefore T is (ε, q)-uniformly locally contractive.

To prove (b), we need to construct a counter example.

Consider metric (X, d), where $X = \{0, 1\}$ and

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for any $x, y \in X$. Let $T : X \longrightarrow X$, defined by $T(0) = 1$ and $T(1) = 0$. Then T is (ε, q)-uniformly locally contractive, but T is not q-contraction.

<u>Proof.</u> Given $\varepsilon = \frac{1}{2}$ and $q = \frac{1}{2}$, there are only $0 \in S(0, \frac{1}{2})$ and $1 \in S(1, \frac{1}{2})$ such that

$$d(T(0), T(0)) \leqslant \frac{1}{2} d(0, 0) \text{ and } d(T(1), T(1)) \leqslant \frac{1}{2} d(1,1)$$

and since ε and q do not depend on any element of X. Therefore T is (ε, q)-uniformly locally contractive. But T is not q-contraction. For $0, 1 \in X$, we have

$$d(T(0), T(1)) = 1 > q = q\, d(0, 1)$$

for all $q \in [0, 1)$.

We prove (c). Let x be any point in E and $\varepsilon > 0$ be arbitrary. Since T is a locally contractive map, there exist $\varepsilon_x > 0$ and

$q_x \in [0, 1)$ such that $y, z \in S(x, \varepsilon_x)$ implies $d(T(y), T(z)) \leqslant q_x d(y, z)$. Choose $\delta_x = \min\left\{\varepsilon_x, \frac{\varepsilon}{2q_x+1}\right\}$. Observe that for any $y \in E$ if $d(x, y) < \delta_x$ then $x, y \in S(x, \varepsilon_x)$. It follows that $d(T(x), T(y)) \leqslant q_x\, d(x, y) \leqslant q_x \frac{\varepsilon}{(2q_x+1)} < \varepsilon$. Therefore T is a continuous map.

5.1.3 Definition. Let (E, d) be a semi-metric space and $\varepsilon > 0$ a real number. If , for any $x, y \in E$ there exists a finite set of point $\{x_0, x_1, \ldots, x_n\}$ such that $x = x_0$, $y = x_n$ and $d(x_i, x_{i+1}) < \varepsilon$ for $i = 0, 1, \ldots, n-1$. (E, d) is called a semi-metric ε - chainable space and the set $\{x_0, x_1, \ldots, x_n\}$ will be called an ε - chain.

5.1.4 Theorem. Let (E, d) be a semi-metric ε-chainable space such that for any $x, y \in E$, there exists an ε-chain $\{x_0, x_1, \ldots, x_n\}$ such that $d(x, y) = \sum_{i=1}^{n-1} d(x_i, x_{i+1})$. If $T : E \longrightarrow E$ is an (ε, q)-uniformly locally contractive map, then T is q-contraction.

Proof. Let y, z be points in E. Since T is (ε, q)-uniformly locally contractive, we have

$$d(T(x_i), T(x_{i+1})) \leqslant q\, d(x_i, x_{i+1})$$

for $i = 0, 1, \ldots, n-1$, so that

$$\begin{aligned} d(T(y), T(z)) &\leqslant \sum_{i=0}^{n-1} d(T(x_i), T(x_{i+1})) \\ &\leqslant q \sum_{i=0}^{n-1} d(x_i, x_{i+1}) \\ &= q\, d(y, z). \end{aligned}$$

Therefore T is q-contraction. Our proof is complete.

5.1.5 Theorem. (Edelstien) Let (E, d) be a complete semi-metric ε -chainable space. T : E $\longrightarrow$ E is a map which is (ε, q)-uniformly locally contractive. For any $x \in E$, $\{T^n(x)\}$ d-converges to a point $x^* \in E$ and $d(T(x^*), x^*) = 0$. Moreover, if (E, d) is a complete metric ε -chainable space, then x^* is the unique fixed point of T.

Proof. Let x be any point in E. Let $\{x_o, x_1, \ldots, x_n\}$ be an ε -chain such that $x = x_o$, $T(x) = x_n$. Since

$$d(x, T(x)) \leqslant \sum_{i=0}^{n-1} d(x_i, x_{i+1})$$

and T is a (ε, q)-uniformly locally contractive

$$\begin{aligned} d(T(x_i), T(x_{i+1})) &\leqslant q\, d(x_i, x_{i+1}) \\ &< q\,\varepsilon . \end{aligned}$$

By induction

$$d(T^m(x_i), T^m(x_{i+1})) < q^m \varepsilon .$$

It follows that

$$\begin{aligned} d(T^m(x), T^{m+1}(x)) &\leqslant \sum_{i=0}^{n-1} d(T^m(x_i), T^m(x_{i+1})) \\ &< n\, q^m \varepsilon \end{aligned}$$

so that for any positive integer ℓ, m and $\ell \geqslant m$

$$\begin{aligned} d(T^m(x), T^{\ell}(x)) &\leqslant \sum_{i=m}^{\ell-1} d(T^i(x), T^{i+1}(x)) \\ &< \sum_{i=m}^{\ell-1} n\, q^i \varepsilon \end{aligned}$$

$$= n\,\varepsilon\,(q^m + q^{m+1} + \ldots, + q^{\ell-1})$$

$$< n\,\varepsilon\,q^m(1 + q + \ldots\, q^{\ell-(1+m)} + q^{\ell-m} + \ldots$$

$$= n\,\varepsilon\,\frac{q^m}{1-q}$$

The quantity on the right is equal to 0 when $q = 0$ and d-converges to 0 when $0 < q < 1$, hence the sequence $\{T^m(x)\}$ is d-Cauchy. Since (E, d) is a complete metric space, hence $\{T^m(x)\}$ d-converges to a point $x^* \in E$. Since T is continuous, the sequence $\{T^{m+1}(x)\}$ d-converges to $T(x^*)$. But $\{T^{m+1}(x)\}$ is a subsequence of $\{T^m(x)\}$ and hence also d-converges to x^*. Since

$$0 \leqslant d(T(x^*), x^*) \leqslant d(T(x^*), T^{m+1}(x)) + d(T^{m+1}(x), x^*)$$

and the quantities on the right converges to 0, so that we have

$$d(T(x^*), x^*) = 0.$$

Moreover, if d is a metric, it follows that $x^* = T(x^*)$ and if there exists $y^* \in E$ such that $y^* = T(y^*)$. Then there exists an ε-chain $\{x_0, x_1, \ldots, x_{n'}\}$ such that $x^* = x_0$, $y^* = x_{n'}$ and

$$d(x^*, y^*) = d(T(x^*), T(y^*))$$

$$= d(T^m(x^*), T^m(y^*))$$

$$\leqslant \sum_{i=0}^{n'-1} d(T^m(x_i), T^m(x_{i+1}))$$

$$= n'\,q^m\,\varepsilon\,.$$

Since $q \in [0, 1)$, the quantity on the right converges to 0, hence $d(x^*, y^*) = 0$. Therefore $x^* = y^*$. The proof is complete.