CHAPTER III

BANACH'S CONTRACTION THEOREM

The purpose of this chapter is to prove Banach's contraction theorem.

The materials of this chapter are drawn from reference [2].

3.1.1 Definition. Let (E, d) be a generalized semi-metric space and $q \in [0, 1)$. A map

$$T : E \longrightarrow E$$

is said to be q-contraction if

$$d(T(x), T(y)) \leq q\, d(x, y)$$

for any $x, y \in E$.

A map

$$T : E \longrightarrow E$$

is said to be a contraction if there exists $q \in [0, 1)$ such that T is q-contraction.

3.1.2 Lemma. Let (E, d) be a generalized semi-metric space. Any contraction map T is continuous.

Proof. Assume $\{x_n\}$ d-converges to $x \in E$. Given any $\varepsilon > 0$, there exists a positive integer N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. Since T is contraction, there exists $q \in [0, 1)$ such that

$$d(T(x_n), T(x)) \leq q\, d(x_n, x).$$

Therefore $d(T(x_n), T(x)) < \varepsilon$ for all $n \geq N$, i.e. $\{T(x_n)\}$ d-converges to $T(x)$. By proposition 2.1.9, T is continuous. The proof is complete.

3.1.3 Definition. Let T be a map of a set E into itself, then a point $x \in E$ is called a fixed point of T if

$$T(x) = x$$

3.1.4 Definition. Let T be a map of a set E into itself. For each non-negative integer p we define a map T^p by

$$T^o(x) = x \quad \text{for all } x \in E$$

and $$T^{p+1}(x) = T^p(T(x)) \quad \text{for all } x \in E.$$

3.1.5 Theorem. (Banach's contraction theorem)

(1) Let (E, d) be a complete semi-metric space. If $T : E \longrightarrow E$ is a continuous map such that T^p is a contraction map for some positive integer p. Then there exists at least one point $x \in E$ satisfying

$$d(T(x), x) = 0.$$

Moreover, if y is any point in E satisfying

$$d(T(y), y) = 0$$

then $d(x, y) = 0$.

(2) If (E, d) is a complete metric space and T is as in (1) then T has exactly one fixed point.

Proof. Since in a metric space, if $d(x, y) = 0$ then $x = y$. Therefore (2) follows immediately from (1), we only need to prove (1).

By the assumption T^p is a contraction for some positive integer p. There exists $q \in [0, 1)$ such that

$$d(T^p(x), T^p(y)) \leqslant q\, d(x, y)$$

for any $x, y \in E$.

First we consider the case $p = 1$. In this case T is a contraction. Let x_o be any point in E and define recursively

$$x_n = T(x_{n-1}), \text{ then } x_n = T^n(x_o) \text{ for all positive integers } n.$$

If m and n are positive integers then

$$\begin{aligned} d(x_{m+n}, x_n) &= d(T^{m+n}(x_o), T^n(x_o)) \\ &\leqslant q^n\, d(T^m(x_o), x_o) \\ &= q^n\, d(x_m, x_o). \end{aligned}$$

Since

$$\begin{aligned} d(x_m,x_o) &\leqslant d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \dots + d(x_1,x_o) \\ &\leqslant q^{m-1} d(x_1,x_o) + q^{m-2} d(x_1,x_o) + \dots + d(x_1, x_o) \\ &= (q^{m-1} + q^{m-2} + \dots + 1)\, d(x_1,x_o) \\ &\leqslant (1 + q + \dots + q^{m-1} + q^m + \dots)\, d(x_1,x_o) \\ &= \frac{1}{1-q}\, d(x_1, x_o), \end{aligned}$$

hence

$$d(x_{m+n}, x_n) \leqslant \frac{q^n}{1 - q}\, d(x_1, x_o).$$

But the quantity on the right hand side is 0 when $q = 0$ and d-converges to 0 when $0 < q < 1$.

Therefore $\{x_n\}$ is d-Cauchy in E. Since E is complete, $\{x_n\}$ d-converges to a point $x \in E$. Since the sequence $\{T(x_n)\} = \{x_{n+1}\}$ is a subsequence of $\{x_n\}$, hence it is d-convergent to x. By the continuity of T, the sequence $\{T(x_n)\}$ d-converges to T(x).

Since

$$d(T(x), x) \leqslant d(T(x), T(x_n)) + d(T(x_n), x)$$

and the quantities on the right converge to 0. Therefore $d(T(x), x) = 0$.

Now we consider the general case. Let $S = T^p$. Then, by the case p = 1, we have for any $x_o \in E$, the sequence $\{x_n\} = \{S^n(x_o)\}$ d-converges to a point $x \in E$ and $d(S(x),x) = 0$.

By continuity of T, the sequence $\{T(x_n)\}$ d-converges to T(x) so that the subsequence $\{T(x_{n+1})\}$ of $\{T(x_n)\}$ d-converges to T(x). Since S is a contraction map, hence S is continuous, i.e. $\{S(T(x_n))\}$ d-converges to to S(T(x)). Since

$$d(T(x), S(T(x))) \leqslant d(T(x), T(x_{n+1})) + d(T(x_{n+1}), S(T(x))).$$

We have

$$T(x_{n+1}) = T(S(x_n)) = T(T^p(x_n)) = S(T(x_n)).$$

It follows that

$$d(T(x), S(T(x))) \leqslant d(T(x), T(x_{n+1})) + d(S(T(x_n)),S(T(x))),$$

since quantities on the right d-converge to 0, hence $d(T(x),S(T(x))) = 0$. But

$$d(T(x), x) \leqslant d(T(x), S(T(x))) + d(S(T(x)), S(x)) + d(S(x), x)$$

$$= d(S(T(x)), S(x))$$

$$\leqslant q\, d(T(x), x).$$

Therefore $d(T(x), x) = 0$, since $q \in [0, 1)$.

For the final state of (1), suppose that there are $x, y \in E$ such that

$$d(T(x), x) = 0 \text{ and } d(T(y), y) = 0.$$

By proposition 2.1.10, $T(x) \in \overline{\{x\}}$. Since T is continuous

$$T^2(x) = T(T(x)) \in T\overline{\{x\}} \subset \overline{\{T(x)\}} \subset \overline{\overline{\{x\}}} = \overline{\{x\}}.$$

By repetition $T^p(x) \in \overline{\{x\}}$. Similarly $T^p(y) \in \overline{\{y\}}$.

By proposition 2.1.10, $d(T^p(x), x) = 0$ and $d(T^p(y), y) = 0$, hence

$$\begin{aligned} d(x, y) &\leq d(T^p(x), x) + d(T^p(x), T^p(y)) + d(T^p(y), y) \\ &= d(T^p(x), T^p(y)) \\ &\leq q\, d(x, y). \end{aligned}$$

Therefore $d(x, y) = 0$, since $q \in [0,1)$. The proof is complete.

3.1.6 <u>Corollary</u>. Let (E, d) be a complete semi-metric space. For any $x_0 \in E$, let

$$x_n = T(x_{n-1}) \text{ then } x_n = T^n(x_0) \text{ for all positive integers } n.$$

Then $\{x_n\}$ d-converges to a point $x \in E$ and

$$d(T(x), x) = 0.$$

In this case if (E,d) is a complete metric space, then x is a unique fixed point of T.