CHAPTER III

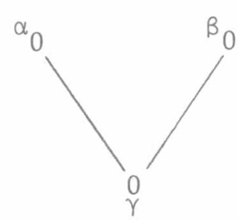
MINIMUM F-INVERSE CONGRUENCES

In chapter II, we show the existence of the minimum proper congruence of any inverse semigroup which has been proved by L. O' Carroll. Since every F-inverse semigroup is a proper, it follows that every F-inverse congruence on an inverse semigroup is a proper congruence. Hence if the minimum F-inverse congruence of an inverse semigroup S exists, then it becomes a proper congruence on S.

In this chapter, we show that the minimum F-inverse congruence on an inverse semigroup need not exist. We also study some inverse semigroups which their minimum F-inverse congruences exist and they are the minimum proper congruences.

The following example shows that the minimum F-inverse congruence on an inverse semigroup need not exist :

<u>Example</u>. Let $S = \{\alpha, \beta, \gamma\}$ be a semilattice with its Hasse diagram :



Then S is an inverse semigroup and S has only following four congruences :

$$\rho_1 = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma)\},$$

$$\rho_2 = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\alpha, \gamma), (\gamma, \alpha)\},$$

$$\rho_3 = \{(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\beta, \gamma), (\gamma, \beta)\},$$

and $\rho_4 = S \times S$.

We can easily check that ρ_2, ρ_3, ρ_4 are all F-inverse congruences of S. But $\rho_2 \subseteq \rho_4$, $\rho_3 \subseteq \rho_4$, $\rho_2 \nsubseteq \rho_3$ and $\rho_3 \nsubseteq \rho_2$. Hence S does not have the minimum F-inverse congruence.

Note that ρ_1 is the minimum proper congruence of S because S is a semilattice which is then proper. #

The inverse semigroup in the above example is commutative and has a zero element. Then the conditions of being commutative and/or of having a zero element of an inverse semigroup S are not sufficient for S to have the minimum F-inverse congruence. It can be easily seen that any semilattice with identity has the identity congruence as its minimum F-inverse congruence .

Throughout this chapter, let $\eta(S)$ or η denote the minimum F-inverse congruence of the inverse semigroup S if it exists. Thus, if η on an inverse semigroup S exists, then $\tau \subseteq \eta$.

The first proposition in this chapter shows the existence of the minimum F-inverse semigroup in any inverse semigroup with zero and identity.

3.1 Proposition. Let S be an inverse semigroup with zero and identity. Then S has the minimum F-inverse congruence and it is the minimum proper congruence of S.

Proof : Let τ be the minimum proper congruence on S. Let 0 and 1 be the zero and the identity of S; respectively. Then S/τ is proper with zero 0τ and identity 1τ. Since S/τ is proper and has zero, every element of S/τ is an idempotent. Since S/τ is an inverse semigroup, it follows that S/τ is a semilattice with identity 1τ. Hence S/τ is F-inverse so that τ is an F-inverse congruence. If ρ is an F-inverse congruence on S, then ρ is a proper congruence so that $\tau \subseteq \rho$. This proves τ is the minimum F-inverse congruence on S. #

Proposition 3.1 shows that for any set X, the symmetric inverse semigroup on the set X, I_X, has the minimum F-inverse congruence which is also its minimum proper congruence.

Since every group is an F-inverse semigroup and every F-inverse semigroup is proper, it follows that for any inverse semigroup, if η exists, we have $\tau \subseteq \eta \subseteq \sigma$.

The following theorem shows that in any inverse semigroup S which its minimum F-inverse congruence η exists, any congruence of S which lies between η and σ is always an F-inverse congruence :

3.2 Proposition. Let S be an inverse semigroup. Assume that S has the minimum F-inverse congruence η. Let ρ be a congruence on S such that $\eta \subseteq \rho \subseteq \sigma$. Then ρ is an F-inverse congruence of S. Hence ρ is a proper congruence of S.

<u>Proof</u> : Assume that ρ is a congruence on S such that $\eta\subseteq\rho\subseteq\sigma$. To show ρ is an F-inverse congruence, let $(a\rho)\sigma(S/\rho)$ be an $\sigma(S/\rho)$-class. Consider $(a\eta)\sigma(S/\eta)$. Since S/η is an F-inverse semigroup, $(a\eta)\sigma(S/\eta)$ has the maximum element, say $b\eta$. Thus, by Proposition 2.2, $(a, b)\in\sigma$. Again, by Proposition 2.2, we have $(a\rho, b\rho)\in\sigma(S/\rho)$. Let $x\rho\in(a\rho)\sigma(S/\rho)$. By Proposition 2.2, $x\in a\sigma$ and so $x\eta\in(a\eta)\sigma(S/\eta)$. Therefore $x\eta = (e\eta)(b\eta) = (eb)\eta$ for some $e\in E(S)$, so $(x, eb)\in\eta\subseteq\rho$, hence $x\rho = (eb)\rho = (e\rho)(b\rho)$. Thus $x\rho\leq b\rho$. This proves S/ρ is F-inverse and hence ρ is an F-inverse congruence.#

<u>Example</u>. Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\ldots\}$

and $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\ldots\}$.

Set $S = \mathbb{N}\times\mathbb{Z}$, and define an operation $*$ on S by

$$(n, a)*(m, b) = (\max(n, m), a+b)$$

where max (n, m) denotes the maximum of the set $\{n, m\}$. Then S is a semigroup, and

$$E(S) = \{(n, 0)\,|\,n\in\mathbb{N}\}$$

which is a commutative subsemigroup of S. For any $(n, a)\in S$, we have

$$(n, a)*(n, -a)*(n, a) = (n, a)$$

and $$(n, -a)*(n, a)*(n, -a) = (n, -a).$$

Hence S is regular. Therefore (S, *) is an inverse semigroup [[1], Theorem 1.17], and for $(n, a)\in S$, $(n, a)^{-1} = (n, -a)$.

Let $(n, a), (m, b)\in S$ such that $(n, a)\sigma(m, b)$. Then there exists $(k, 0) \in E(S)$ such that

$$(n, a)*(k, 0) = (m, b)*(k, 0)$$

and therefore $a = b$. If (n, a) and $(m, a) \in S$, then $(\max(n, m), 0) \in E(S)$ and

$$\begin{aligned}(n, a)*(\max(n, m), 0) &= (\max(n, \max(n, m)), a) \\ &= (\max(n, m), a) \\ &= (\max(m, \max(n, m)), a) \\ &= (m, a)*(\max(n, m), 0)\end{aligned}$$

so that $(n, a)\sigma(m, a)$. This proves that for any $(n, a) \in S$,

$$(n, a)\sigma = \{(m, a) \mid m \in \mathbb{N}\} = (1, a)\sigma.$$

Hence

$$S/\sigma = \{(1, a)\sigma \mid a \in \mathbb{Z}\} \cong (\mathbb{Z}, +).$$

For $(1, a)\sigma \in S/\sigma$ and $(n, a) \in (1, a)\sigma$, we have

$$\begin{aligned}(n, a)*(n, a)^{-1} &= (n, a)*(n, -a) \\ &= (n, 0) \\ &= (n, a)*(1, -a) \\ &= (n, a)*(1, a)^{-1}\end{aligned}$$

which implies $(n, a) \leq (1, a)$.

Therefore every σ-class of S has a maximum element and hence S is F-inverse. Thus η of S is the identity congruence on S. Then by Proposition 3.2, any congruence on S contained in σ is an F-inverse congruence.

For each $k \in \mathbb{N}$, let ρ_k be as follows :

$$\rho_k = \{((n, a), (m, a)) | n, m \in \{1, 2, \ldots, k\}, a \in \mathbb{Z}\} \cup$$
$$\{((n, a), (m, a)) | n, m \in \{k+1, k+2, \ldots\}, a \in \mathbb{Z}\}.$$

It is easily seen that for each $k \in \mathbb{N}$, ρ_k is a congruence on S and $\rho_k \subseteq \sigma$. Hence all the congruences ρ_k are F-inverse congruences. Note that $\rho_k \nsubseteq \rho_e$ and $\rho_e \nsubseteq \rho_k$ if $e \neq k$. #

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then, on S we have $\tau = \mathcal{R} \cap \sigma = \mathcal{L} \cap \sigma = \mathcal{H} \cap \sigma$. We will show that τ becomes the minimum F-inverse congruence η of S under a certain condition of Y.

3.3 Proposition. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. If every ideal of Y is principal, then τ is the minimum F-inverse congruence on S.

Proof : Recall that

$$\tau = \{(a, b) \in G_\alpha \times G_\alpha | \alpha \in Y,\ ae_\beta = be_\beta \text{ for some } \beta \in Y\}$$
$$= \{(a, b) \in G_\alpha \times G_\alpha | \alpha \in Y \text{ and } a\sigma b\}. \qquad (*)$$

To show that S/τ is F-inverse, let $(a\tau)\sigma(S/\tau)$ be an $\sigma(S/\tau)$-class of S/τ. Let

$$Y_a = \{\alpha \in Y | G_\alpha \cap a\sigma \neq \phi\}.$$

Claim that Y_a is an ideal of Y. Let $\alpha \in Y_a$ and $\beta \in Y$. Since $\alpha \in Y_a$, $G_\alpha \cap a\sigma \neq \phi$, so that there exists an element $b \in S$ such that $b \in G_\alpha \cap a\sigma$. Let $e \in E(G_\beta)$. Then $be \in G_{\alpha\beta}$ and

$$(be)\sigma = b\sigma = a\sigma$$

so that $be \in G_{\alpha\beta} \cap a\sigma$. Hence $\alpha\beta \in Y_a$. Therefore Y_a is an ideal of Y. By assumption, there exists $\alpha_a \in Y_a$ such that $Y_a = \alpha_a Y$. Thus for all $\alpha \in Y_a$, $\alpha \leq \alpha_a$. Let $m \in G_{\alpha_a} \cap a\sigma$. To show $m\tau$ is the maximum element of $(a\tau)\sigma(S/\tau)$, let $x\tau \in (a\tau)\sigma(S/\tau)$. Then by Proposition 2.2, $x \in a\sigma = m\sigma$. Let $\alpha \in Y$ such that $x \in G_\alpha$. Therefore $\alpha \in Y_a$ so that $\alpha \leq \alpha_a$. Thus $xx^{-1}m \in G_\alpha G_{\alpha_a} \subseteq G_{\alpha(\alpha_a)} = G_\alpha$ and $x\sigma = m\sigma = (xx^{-1}m)\sigma$. Because of (*), we have

$$x\tau \quad = \quad (xx^{-1}m)\tau \quad = \quad ((xx^{-1})\tau)(m\tau).$$

Hence $x\tau \leq m\tau$. Therefore $m\tau$ is the maximum element of $(a\tau)\sigma(S/\tau)$. Hence S/τ is an F-inverse semigroup.

Let ρ be any F-inverse congruence on S. Then S/ρ is F-inverse so S/ρ is proper, thus $\tau \subseteq \rho$. Hence τ is the minimum F-inverse congruence on S as required. #

Since every F-inverse congruence on an inverse semigroup S is a proper congruence on S, the following remark follows :

3.4 Remark. Let τ be the minimum proper congruence on an inverse semigroup S. If τ is an F-inverse congruence of S, then τ becomes the minimum F-inverse congruence of S.