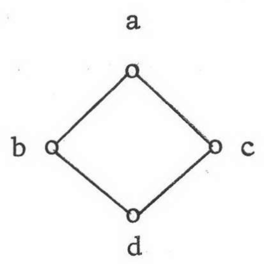
CHAPTER II

IDEALS OF FACTORIZABLE INVERSE SEMIGROUPS

In this chapter, we study ideals and Rees quotient semigroups of factorizable inverse semigroups.

Let A be an ideal of an inverse semigroup S. Then A is a subsemigroup of S. If $a \in A$, then $a^{-1} \in S$ and so $a^{-1} = a^{-1}aa^{-1} \in A$. Thus A is an inverse subsemigroup of S.

An ideal of a factorizable inverse semigroup is not necessarily factorizable. An example is given as follows : Let $S = \{a,b,c,d\}$ be a semilattice with the Hasse diagram



Then S is a semilattice with identity a and so $E(S) = S$ and the group of units of S is $\{a\}$. Hence $S = E(S).\{a\}$ which implies that S is factorizable. In fact, any semilattice with identity is factorizable. Let $A = \{b,c,d\}$. Then A is an ideal of S. But A does not have its identity. Thus A is not factorizable. #

The first theorem of this chapter shows that an ideal with its identity of a factorizable inverse semigroup is factorizable.

2.1 Theorem. Let S be a factorizable inverse semigroup. If A is an ideal of S and A has its identity, then A is factorizable.

Proof: Let 1 be the identity of S and G be the group of units of S. Let 1' be the identity of A. Since A is an ideal of S and $1' \in A$, $1'G \subseteq A$. Claim that 1'G is a subgroup of A, let $1'g \in 1'G$ $(g \in G)$. Then $1'g^{-1} \in 1'G$ and $1' = 1'1 \in 1'G \subseteq A$, so

$$(1'g)(1'g^{-1}) = ((1'g)1')g^{-1} = (1'g)g^{-1} = 1'(gg^{-1}) = 1'(1) = 1'.$$

and for all $x \in 1'G$, $1'x = x1' = x$ because $1'G \subseteq A$. Therefore 1'G is a subgroup of A. Next we show that $A = (1'G).E(A)$, let $x \in A$. Then $x = ge$ for some $g \in G$, $e \in E(S)$. Therefore

$$x = x1' = (ge)1' = (g1')e = 1'(g1')e = (1'g)(1'e).$$

But $1'g \in 1'G$ and 1'e is an idempotent and belongs to A, so $(1'g)(1'e) \subseteq (1'G).(E(A))$. Then $x \in (1'G).(E(A))$. Hence $A = (1'G).(E(A))$. Therefore A is factorizable. #

From the proof of Theorem 2.1 and Theorem 1.1, the following follows : Let G be the group of units of a factorizable inverse semigroup S. If A is an ideal of S and A has its identity 1', then 1'G is the group of units of A.

Now, we have a question whether an inverse subsemigroup with its identity of a factorizable inverse semigroup is factorizable. The following example shows that this is not true in general :

Let $X = \{a,b\}$, and I_X be the symmetric inverse semigroup on X. Let 0 and 1 be the zero and the identity of I_X; respectively, and let

$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in I_X$ such that

$$\Delta\alpha_1 = \nabla\alpha_1 = \{a\} ,$$

$$\Delta\alpha_2 = \nabla\alpha_2 = \{b\} ,$$

$$\Delta\alpha_3 = \{a\}, \quad \nabla\alpha_3 = \{b\} ,$$

$$\Delta\alpha_4 = \{b\}, \quad \nabla\alpha_4 = \{a\} ,$$

and $\Delta\alpha_5 = \nabla\alpha_5 = \{a,b\}$ such that $a\alpha_5 = b$, $b\alpha_5 = a$. Then $I_X = \{0, 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and the multiplication is as follows:

o	0	1	α_1	α_2	α_3	α_4	α_5
0	0	0	0	0	0	0	0
1	0	1	α_1	α_2	α_3	α_4	α_5
α_1	0	α_1	α_1	0	α_3	0	α_3
α_2	0	α_2	0	α_2	0	α_4	α_4
α_3	0	α_3	0	α_3	0	α_1	α_1
α_4	0	α_4	α_4	0	α_2	0	α_2
α_5	0	α_5	α_4	α_3	α_2	α_1	1

Let $T = \{0, 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. From the table, T is a subsemigroup of I_X.

Because

$$0^{-1} = 0,\ 1^{-1} = 1,\ \alpha_1^{-1} = \alpha_1,\ \alpha_2^{-1} = \alpha_2,\ \alpha_3^{-1} = \alpha_4,\ \alpha_4^{-1} = \alpha_3 ,$$

it follows that T is an inverse subsemigroup of S and T has the identity 1. It is clearly seen that the group of units of T is $\{1\}$ and the set of all idempotents of T is $\{0, 1, \alpha_1, \alpha_2\}$ so $E(T) = \{0,1,\alpha_1,\alpha_2\}$. Since $\alpha_3 \notin E(T) = \{1\}.E(T)$, T is not factorizable. #

Let S be a semigroup with identity 1 and G be the group of units of S. Then

$$G = \{a \in S \mid aa' = a'a = 1 \text{ for some } a' \in S\} .$$

If A is an ideal of S, then either $A \cap G = \phi$ or $A = S$. To prove this, assume $A \cap G \neq \phi$. Then there exists $g \in G$ such that $g \in A$, so $1 = g^{-1}g \in A$ where g^{-1} is the group inverse of g in G. Hence for all $x \in S$, $x = x1 \in A$. Thus, $A = S$.

Let A be an ideal of a semigroup S. Let ρ_A denote the Rees congruence on S induced by the ideal A, that is,

$$a\rho_A = \begin{cases} \{a\} & \text{if } a \notin A. \\ A & \text{if } a \in A. \end{cases}$$

Recall that the semigroup S/ρ_A is called the Rees quotient semigroup of S induced by A, and denoted by S/A. Because a homomorphic image of an inverse semigroup is an inverse semigroup, S/A is an inverse semigroup if S is an inverse semigroup.

To show that a Rees quotient semigroup of a factorizable inverse semigroup is a factorizable inverse semigroup, we need the following lemma :

2.2 <u>Lemma</u>. Let S be a semigroup with identity 1 and let G be the group of units of S. If A is an ideal of S, then the set $\{a\rho_A \mid a \in G\}$ is the group of units of S/A.

<u>Proof</u>: It is clear that $1\rho_A$ is the identity of S/A. Because A is an ideal of S and G is the group of units of S, it follows that

either $A = S$ or $G \cap A = \phi$. If $A = S$, then S/A is a trivial semigroup, so $S/A = \{1\rho_A\}$ which is a trivial group.

Assume $G \cap A = \phi$. Let $\overline{G} = \{a\rho_A \mid a \in G\}$. Then for $a \in G$, $a\rho_A = \{a\}$, so $\overline{G}$ is obvious to be a subgroup of S/A since G is a subgroup of S. Let $\overline{H}$ be the group of units of S/A. Then

$$\overline{H} = \{x\rho_A \mid (x\rho_A)(x'\rho_A) = (x'\rho_A)(x\rho_A) = 1\rho_A \text{ for some } x' \in S\}.$$

Because $\overline{H}$ is the greatest subgroup of S/A having $1\rho_A$ as its identity and $\overline{G}$ is a subgroup and $1\rho_A \in \overline{G}$, it follows that $\overline{G} \subseteq \overline{H}$.

Next, let $x\rho_A \in \overline{H}$. Then

$$(x\rho_A)(x'\rho_A) = (x'\rho_A)(x\rho_A) = 1\rho_A ,$$

so

$$(xx')\rho_A = (x'x)\rho_A = 1\rho_A ,$$

Because $1 \notin A$, $xx' = x'x = 1$ and hence $x \in G$. Then $x\rho_A \in \overline{G}$. Hence, we have $\overline{G} = \overline{H}$ as desired. #

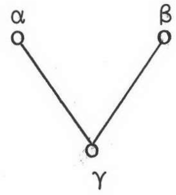
2.3 <u>Theorem</u>. Let A be an ideal of an inverse semigroup S. If S is factorizable, the Rees quotient semigroup S/A is factorizable.

<u>Proof</u>: Assume that S is factorizable as $G.E(S)$. Then G is the group of units of S. By Introduction, page 5,

$$E(S/A) = E(S/\rho_A) = \{e\rho_A \mid e \in E(S)\}.$$

By Lemma 2.2, $\{a\rho_A \mid a \in G\} = \overline{G}$ is the group of units of S/A. Now, we show that $S/A = \overline{G}.E(S/A)$. Let $x\rho_A \in S/A$. Then $x = ge$ for some $g \in G$, $e \in E(S)$. Therefore $x\rho_A = (g\rho_A)(e\rho_A) \in \overline{G}.E(S/A)$. Hence S/A is factorizable. #

The converse of Theorem 2.3 is not true even though the Rees quotient semigroup S/A is not trivial. For example, let Y be a semilattice with Hasse diagram



For each $\delta \in Y$, let $G_\delta = \mathbb{Z} \times \delta$ and set $S = G_\alpha \cup G_\beta \cup G_\gamma$. Define the operation on S by

$$(n,\delta_1)(m,\delta_2) = (n+m,\delta_1\delta_2).$$

Then S is a semilattice Y of groups G_α, G_β, G_γ and

$$E(S) = \{(0,\alpha), (0,\beta), (0,\gamma)\} .$$

Because Y has no identity, S has no identity, so S is not factorizable.

Let $A = G_\beta \cup G_\gamma$. It is easy to see that A is an ideal of S. The Rees quotient semigroup S/A is isomorphic to G_α^{0*}, the group G_α adjoined the zero 0*. But the group of units of G_α^{0*} is G_α and $E(G_\alpha^{0*}) = \{0^*, (0,\alpha)\}$ and

$$G_\alpha^{0*} = G_\alpha . E(G_\alpha^{0*}) .$$

Then G_α^{0*} is a factorizable inverse semigroup. Hence S/A is factorizable. #

Let Y be a semilattice. Then for each $\alpha \in Y$, αY is the principal ideal of Y generated by α and it is also a semilattice which has α as its identity, so α is the maximum element of αY ; moreover,

$$\alpha Y = \{\beta \in Y \mid \beta \leq \alpha\} .$$

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. For each $\alpha \in Y$, let

$$A_\alpha = \bigcup_{\beta \leq \alpha} G_\beta .$$

Then for each $\alpha \in Y$, $A_\alpha = \bigcup_{\beta \in \alpha Y} G_\beta$. Since αY is a semilattice with identity α, A_α is a semilattice αY of groups G_β, and A_α has the identity e_α, where e_λ denotes the identity G_λ for all $\lambda \in Y$. Moreover, A_α is an ideal of S for all $\alpha \in Y$. To show this, let $\alpha \in Y$. Let $x \in S$ and $a \in A_\alpha$. Then $x \in G_\gamma$ for some $\gamma \in Y$ and $a \in G_\beta$ for some $\beta \leq \alpha$. Then $\beta = \alpha\beta = \beta\alpha$. Thus, $ax \in G_\beta G_\gamma \subseteq G_{\beta\gamma}$ and $xa \in G_\gamma G_\beta \subseteq G_{\gamma\beta} = G_{\beta\gamma}$. Since $\beta\gamma = \alpha\beta\gamma$, $\beta\gamma \leq \alpha$ and hence $ax, xa \in G_{\beta\gamma} \subseteq A_\alpha$. Hence A_α is an ideal of S.

The following proposition follows directly from the above fact and Theorem 2.1 :

2.4 Proposition. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. For any $\alpha \in Y$, let $A_\alpha = \bigcup_{\beta \leq \alpha} G_\beta$. If S is factorizable, then A_α is a factorizable inverse semigroup for all $\alpha \in Y$.

The next proposition follows from Proposition 1.14.

2.5 Proposition. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α with corresponding homomorphisms $\psi_{\alpha,\beta}$. Let $\alpha \in Y$ and $A_\alpha = \bigcup_{\beta \leq \alpha} G_\beta$. If $\psi_{\alpha,\beta}$ is an epimorphism for all $\beta \in Y$, $\beta \leq \alpha$, then A_α is a factorizable inverse subsemigroup of S.