CHAPTER I

FACTORIZABLE INVERSE SEMIGROUPS

In this chapter, we first introduce many important theorems related to factorizable inverse semigroups which have been given by S.Y. Chen and S.C. Hsieh in [6]. From their paper, we study more about factorizable inverse semigroups and we get many results concerning to minimum group congruences, the property of being proper and the property of being F-inverse.

Let S be a semigroup. The relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on S are defined as follows :

$$a \mathcal{L} b \iff S^1 a = S^1 b.$$
$$a \mathcal{R} b \iff aS^1 = bS^1.$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}.$$

The relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are called <u>Green's relations</u> on S, and they are equivalence relations on S. Moreover, $\mathcal{L}$ is right compatible and $\mathcal{R}$ is left compatible. For each $a \in S$, let

$$L_a = \{ x \in S \mid x \mathcal{L} a \} ;$$

and R_a, H_a are defined similarly.

In any semigroup S, an $\mathcal{H}$ - class of S containing an idempotent e of S is a subgroup of S [[1], Theorem 2.16], and

$$H_e = \{a \in S \mid ae = ea = a \text{ and } aa' = e = a'a \text{ for some } a' \in S\}$$

which is the maximum subgroup of S having e as its identity. If S has an identity 1, then H_1 is the group of units of S.

The following theorem shows various properties of a factorizable inverse semigroup :

1.1 <u>Theorem</u> [6]. Let S be an inverse semigroup. If S is factorizable as S = GE, then the following hold :

(i) S = EG.

(ii) S has an identity 1 which is the identity of G.

(iii) $G = H_1$, the group of units of S.

(iv) For any $g, h \in G$ and $e, f \in E(S)$, if $ge = hf$, then $e = f$.

(v) E = E(S), the set of all idempotents of S.

The next theorem gives equivalent definitions of an inverse semigroup to be factorizable.

1.2 <u>Theorem</u> [6]. Let S be an inverse semigroup, G be a subgroup of S. Then the following conditions are equivalent :

(i) S = G.E(S).

(ii) $L_e = Ge$, for every $e \in E(S)$.

(iii) $R_e = eG$, for every $e \in E(S)$.

(iv) $S = \omega G$, where $\omega G = \{x \in S \mid x = ge \text{ for some } g \in G, e \in E(S)\}$.

In fact, by Introduction page 3 , ωG in (iv) of theorem 1.2 is the set $\{x \in S \mid x \leq g \text{ for some } g \in G\}$

$= \{x \in S \mid x=eg$ for some $g \in G, e \in E(S)\}$ and so $\omega G = G.E(S) = E(S).G$.

Let X be a set. A one-to-one map α from a subset of X onto a subset of X is called a <u>one-to-one partial transformation of</u> X, and let $\Delta\alpha$ and $\nabla\alpha$ denote the domain and the range of α; respectively. Let I_X be the set of all one-to-one partial transformations of X. If $\alpha \in I_X$ and $\Delta\alpha = \nabla\alpha = \phi$, then α is called the <u>empty partial transformation</u> and is denoted by 0. The product on I_X is defined as follows : For $\alpha, \beta \in I_X$, if $\nabla\alpha \cap \Delta\beta = \phi$, define $\alpha\beta = 0$, otherwise; define $\alpha\beta$ to be the composite map of $\alpha\big|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ and $\beta\big|_{(\nabla\alpha \cap \Delta\beta)\beta}$.

Then I_X is an inverse semigroup with zero and identity and call it the <u>symmetric inverse semigroup on the set</u> X; moreover, for each $\alpha \in I_X$, α^{-1} (the inverse map of α) is the unique inverse of α in I_X, and

$$E(I_X) = \{\alpha \in I_X \mid \alpha \text{ is the identity map on } A \text{ for some subset } A \text{ of } X\}$$
$$= \{\alpha \in I_X \mid \alpha \text{ is the identity map on } \Delta\alpha\}.$$

[[1], page 29]. Let G_X denote the permutation group on X. Then G_X is clear the group of units of I_X.

For any set A, the notation $|A|$ denotes the cardinality of the set A.

The following theorem shows that any symmetric inverse semigroup on a set has the largest factorizable inverse subsemigroup :

1.3 Theorem [6]. Let X be a set, G_X be the permutation group on X and let $T_X = \{\alpha \in I_X \mid |X - \Delta\alpha| = |X - \nabla\alpha|\}$. Then $T_X = \omega G_X$ is the largest factorizable inverse subsemigroup of I_X.

1.4 Corollary [6]. Let X be a set. Then I_X is factorizable if and only if X is finite.

A subsemigroup T of a semigroup S is said to be full if $E(S) \subseteq T$.

Let S be an inverse semigroup with identity 1 and G be the group of units of S. Then ωG is a full inverse subsemigroup of S. To prove this, let $a \in \omega G$. Since

$$\omega G = \{x \in S \mid x = ge \text{ for some } g \in G \text{ and } e \in E(S)\}$$
$$= \{x \in S \mid x = eg \text{ for some } g \in G \text{ and } e \in E(S)\},$$

$a = hf$ for some $h \in G$, $f \in E(S)$, so $a^{-1} = f^{-1}h^{-1} = fh^{-1}$ [Introduction, page 2] and hence $a^{-1} = fh^{-1} \in \omega G$ because $h^{-1} \in G$. From $\omega G = G.E(S) = E(S).G$, we have

$$\begin{aligned}(\omega G).(\omega G) &= (G.E(S)).(E(S).G) \\ &= G.(E(S)^2).G \\ &= G.E(S).G \\ &= G.G.E(S) = G.E(S) = \omega G.\end{aligned}$$

Since $1 \in G$, $E(S) \subseteq G.E(S) = \omega G$. Hence ωG is a full inverse subsemigroup of S.

The next theorem shows that ωG is the largest full factorizable inverse subsemigroup of S, and it becomes the largest factorizable inverse subsemigroup of S if and only if ωG contains all subgroups of S.

1.5 Theorem [6]. Let S be an inverse semigroup with identity and G be the group of units of S. Then ωG is the largest full factorizable inverse subsemigroup of S. It is the largest factorizable inverse subsemigroup of S if and only if $H_e \subseteq \omega G$ for all $e \in E(S)$.

Using Theorem 1.3 and Corollary 1.4, an embedding theorem has been obtained in [6] as follows :

1.6 Theorem [6]. Every inverse semigroup can be embedded in a factorizable inverse semigroup.

Recall that in any inverse semigroup S, the minimum group congruence $\sigma(S)$, or σ, always exists and

$$\sigma = \{(a, b) \in S \times S \mid ae = be \text{ for some } e \in E(S)\} .$$
$$= \{(a, b) \in S \times S \mid ea = eb \text{ for some } e \in E(S)\} .$$

We show a relation between the minimum group congruence and the group of units of a factorizable inverse semigroup in the following proposition :

1.7 Proposition. Let S be a factorizable inverse semigroup as S = G.E. Then every σ - class of S intersects G, and hence

$$S/_{\sigma} = \{ g\sigma \mid g \in G\} .$$

Proof: Let $a\sigma$ be a σ - class of S. Then there exist $e \in E(S)$, $g \in G$ such that $a = ge$. Therefore

$$ae = gee = ge.$$

Thus $a \sigma g$ so that $g \in a\sigma$. Hence $a\sigma \cap G \neq \phi$. #

The following corollary follows clearly from Proposition 1.7:

1.8 Corollary. In any factorizable inverse semigroup S, the maximum group homomorphic image of S is a homomorphic image of the group of units of S.

1.9 Proposition. Let S be a factorizable inverse semigroup as G.E. Define a relation δ on G by

$$g \,\delta\, h \Longleftrightarrow ge = he \text{ for some } e \in E(S).$$

Then δ is a congruence on G, and hence G/δ is a group homomorphic image of G. Moreover;

$$S/\sigma \cong G/\delta ,$$

That is ; G/δ is the maximum group homomorphic image of S.

Proof : Because for any $e, f \in E(S)$, $ef = fe \in E(S)$, it follows obviously that δ is an equivalence relation on G. Let $a, b, c \in G$ such that $a \,\delta\, b$. Then $ae = be$ for some $e \in E(S)$. Therefore $cae = cbe$, so that $ca\delta cb$. Since $cc^{-1} \in E(S)$,

$$ac(c^{-1}ec) = aecc^{-1}c = becc^{-1}c = bc(c^{-1}ec).$$

But $c^{-1}ec \in E(S)$, so $ac\delta bc$. Thus δ is a congruence on G. Since G/δ is a homomorphic image of the group G, G/δ is a group homomorphic image of G.

Next, define $\theta : G/\delta \to S/\sigma$ by

$$(g\delta)\theta = g\sigma \qquad (g \in G).$$

To show θ is well - defined, let $g_1\delta$, $g_2\delta \in G/\delta$ such that $g_1\delta = g_2\delta$. Then $g_1e = g_2e$ for some $e \in E(S)$ and so $g_1\sigma = g_2\sigma$ which implies $(g_1\delta)\theta = (g_2\delta)\theta$.

Let $g_1\delta$, $g_2\delta \in G/\delta$. Then

$$\begin{aligned}((g_1\delta)(g_2\delta))\theta &= ((g_1g_2)\delta)\theta \\ &= (g_1g_2)\sigma \\ &= (g_1\sigma)(g_2\sigma) \\ &= ((g_1\delta)\theta)((g_2\delta)\theta).\end{aligned}$$

This proves θ is a homomorphism.

To show θ is onto, let $a\sigma \in S/\sigma$. Since S is factorizable as G.E, by Proposition 1.7, there exists $g \in G$ such that $g \in a\sigma$. Then $g\sigma = a\sigma$, and therefore

$$(g\delta)\theta = g\sigma = a\sigma .$$

Next, to show θ is one-to-one, let $g_1\delta$, $g_2\delta \in G/\delta$ such that $(g_1\delta)\theta = (g_2\delta)\theta$. Therefore $g_1\sigma = g_2\sigma$. Then $g_1e = g_2e$ for some $e \in E(S)$ and hence $g_1\delta = g_2\delta$. Hence θ is an onto isomorphism. Therefore

$$S/\sigma \cong G/\delta ,$$

as required. #

We show in the next theorem that the group of units of a factorizable inverse semigroup S becomes the maximum group homomorphic image of S if S is proper.

1.10 Theorem. Let S be a factorizable inverse semigroup as G.E(S). If S is proper, then $G \cong S/\sigma$.

Proof. To prove this theorem, it suffices to show that the congruence δ defined on G as in Proposition 1.9 is the identity congruence. Recall that $g_1, g_2 \in G$,

$$g_1 \delta\, g_2 \Longleftrightarrow g_1 e = g_2 e \quad \text{for some } e \in E(S).$$

Let $g_1, g_2 \in G$ such that $g_1 \delta\, g_2$. Then $g_1 e = g_2 e$ for some $e \in E(S)$. Thus $(g_2^{-1} g_1)e = 1e = e$ where 1 is the identity of S. Since S is proper, $g_2^{-1} g_1 \in E(S)$. But $g_2^{-1} g_1 \in G$. Then $g_2^{-1} g_1 = 1$ which implies $g_1 = g_2$.

This proves that δ is the identity congruence on G. #

The following example shows that in any factorizable inverse semigroup S which is not proper, the maximum group homomorphic image of S is not necessarily isomorphic to the group of units of S :

Example : Let $Y = \{\alpha, \beta\}$ be a semilattice with its Hasse diagram

and let $G_\alpha = \alpha \times \mathbb{Z} = \{(\alpha, n) \mid n \in \mathbb{Z}\}$,

$G_\beta = \{(\beta, o)\}$,

where $\mathbb{Z}$ denotes the set of all integers.

Set $S = G_\alpha \cup G_\beta$, and define an operation $*$ on S by

$$(\lambda, n) * (\lambda', n') = \begin{cases} (\lambda, n + n') & \text{if } \lambda = \lambda' \\ (\beta, o) & \text{if } \lambda \neq \lambda' . \end{cases}$$

Then $(S,*)$ is a semilattice Y of groups G_α and G_β , and hence $(S,*)$ is an inverse semigroup. It is clearly seen that

$$E(S) = \{(\alpha, o), (\beta, o)\}$$

and (α,o) is the identity of S. Therefore G_α is the group of units of S and G_α is clearly isomorphic to the group $(\mathbb{Z}, +)$.

The semigroup S is obviously factorizable as $G_\alpha \cdot E(S)$.

From the definition of *, $(S, *)$ has (β, o) as its zero. Then the minimum group congruence of S, σ, is S x S, and so $|S/\sigma| = 1$. Hence S/σ is not isomorphic to G_α .

We have $(\alpha, 1) \notin E(S)$, $(\beta, o) \in E(S)$ and $(\alpha, 1) * (\beta, o) = (\beta,o)$. Therefore S is not proper. #

The above example also shows that a factorizable inverse semigroup need not be proper.

An inverse semigroup S is called an <u>F - inverse semigroup</u> if each σ - class of S has a maximum element under the natural partial order. Any F - inverse semigroup is proper and has an identity [Introduction, page 8] . Then any semilattice without identity is proper but not F - inverse. In fact, a proper inverse semigroup with identity need not be F - inverse.

We will show in the next theorem that a proper and factorizable inverse semigroup is F - inverse.

1.11 <u>Theorem</u>. If S is proper and factorizable inverse semigroup, then

S is F - inverse.

Proof: Let S be a factorizable as G.E(S). Let $a\sigma$ be a σ - class of S. By Proposition 1.7, there exists $g \in G$ such that $g \in a\sigma$. Let $h \in G$ such that $h \in a\sigma$. Then $(h,g) \in \sigma$, so there exists $f \in E(S)$ such that $hf = gf$. Therefore $g^{-1}hf = f$. Since S is proper, $g^{-1}h \in E(S)$. But $g^{-1}h \in G$. Then $g^{-1}h = 1$ and hence $g = h$. This proves that for each $a \in S$, There exists a unique $g \in G$ such that $g \in a\sigma$.

Let $a \in S$ and $g \in G \cap a\sigma$. We claim that g is the maximum element of the σ - class $a\sigma$. Let $x \in a\sigma$. Then $xf = gf$ for some $f \in E(S)$. Since $x \in S = GE(S)$, $x = ke$ for some $k \in G$, $e \in E(S)$. Therefore $(ke)f = gf$, and so $kef = gef$. Thus $(k, g) \in \sigma$. From above proof, we have $k = g$. Hence $x = ge$ so that $x \leq g$ [Introduction, page 3]. This shows that g is the maximum element of $a\sigma$.

Hence each σ - class of S has a maximum element. Therefore S is F - inverse. #

The following example shows that an F - inverse semigroup need not be factorizable. Hence, the converse of Theorem 1.11 is not true in general.

Example. Let $Y = \{\alpha, \beta\}$ be a semilattice with its Hasse diagram

and let $G_\alpha = 2\mathbb{Z} \times \alpha$,

$G_\beta = \mathbb{Z} \times \beta$.

Set $S = G_\alpha \cup G_\beta$, and define an operation o on S by

$$(n,\lambda)o(m,\lambda') = (n+m,\lambda\lambda') \qquad ((n,\lambda),\ (m,\lambda') \in S).$$

Then (S, o) is a semilattice Y of groups G_α and G_β , so S is an inverse semigroup. Moreover,

$$E(S) = \{(0,\alpha),\ (0,\beta)\}.$$

It is easily seen that for any (n, λ) and $(m, \lambda') \in S$,

$$(n, \lambda)\ \sigma\ (m, \lambda') \iff n = m .$$

Hence for any $n \in \mathbb{Z}$, $(n, \alpha)\sigma = \{(n, \alpha),(n, \beta)\}$ if n is even,

$(n, \beta)\sigma = \{(n, \alpha),(n, \beta)\}$ if n is even,

$(n, \beta)\sigma = \{(n, \beta)\}$ if n is odd,

and if n is even, then $(n, \alpha)o(0, \beta) = (n, \beta)$ so that $(n, \beta) \leq (n, \alpha)$. Therefore every σ - class of S has a maximum element. Since $(0, \alpha)$ is the identity of S, G_α is the group of units of S. Because $(3, \beta) \in S$ but $(3, \beta) \notin G_\alpha . E(S)$. Then $S \neq G_\alpha . E(S)$. Therefore S is not a factorizable inverse semigroup. ⋇

Because every F - inverse semigroup is proper, the above example also shows that a proper inverse semigroup is not necessarily factorizable. However, a certain condition for a proper inverse semigroup with identity to become factorizable is given in term of its minimum group congruence as follows :

1.12 Proposition. Let S be a proper inverse semigroup with identity, and G be the group of units of S. If every σ - class of S intersects G, then S is factorizable.

Proof: Let $x \in S$. Then $x\sigma$ intersects G. Let $g \in x\sigma \cap G$. Then $g\, \sigma\, x$, and so $xe = ge$ for some $e \in E(S)$. Therefore $g^{-1} xe = 1e = e$. Since S is proper, $g^{-1}x = f$ for some $f \in E(S)$. Thus $x = gf \in G.\,E(S)$. Hence $S = G.\,E(S)$ as required. #

Any F - inverse semigroup is proper and has identity. Then the following corollary follows :

1.13 Corollary. Let S be an F - inverse semigroup. If every σ - class of S intersects the group of units G of S, then S is factorizable.

Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. For each $\alpha \in Y$, let e_α denote the identity of the group G_α. Then

$$E(S) = \{ e_\alpha \mid \alpha \in Y \}$$

and it is contained in the center of S [[1], Lemma 4.8]. For each pair $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, the map $\psi_{\alpha,\beta} : G_\alpha \to G_\beta$ which is defined by

$$g\psi_{\alpha,\beta} = ge_\beta \qquad (g \in G_\alpha)$$

is a homomorphism; moreover, if $\alpha \geq \beta \geq \gamma$, $\psi_{\alpha,\beta}\, \psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$ [[1], Theorem 4.11]. We call the maps $\psi_{\alpha,\beta}$ the corresponding homomorphisms of the semilattice Y of groups G_α. It is easy to see that E(S) is isomorphic to Y by the isomorphism : $e_\alpha \to \alpha$ ($\alpha \in Y$). It then follows that $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta \in Y$. Hence if S has an identity, then Y has an identity. Also, if Y has an identity 1, then e_1 is the identity of S. To show this, let $\alpha \in Y$ and $x \in G_\alpha$. Then

$$xe_1 = (xe_\alpha)e_1 = x(e_\alpha e_1) = xe_{\alpha 1} = xe_\alpha = x$$

and

$$e_1 x = e_1(e_\alpha x) = (e_1 e_\alpha)x = e_{1\alpha}x = e_\alpha x = x.$$

Thus, if S is factorizable, then Y has an identity 1 and G_1 is the group of units of S.

1.14 Proposition. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α with corresponding homomorphisms $\psi_{\alpha,\beta}$. Then S is factorizable if and only if Y has an identity 1 and $\psi_{1,\alpha}$ is an epimorphism for all $\alpha \in Y$.

Proof: Assume S is factorizable as $G \cdot E(S)$. From above explanation, Y has an identity 1.

Let α be any element of Y. To show $\psi_{1,\alpha} : G_1 \to G_\alpha$ is onto, let $x \in G_\alpha$. Since $S = G \cdot E(S)$, there exist $g \in G_1$, $\beta \in Y$ such that $x = ge_\beta$. Claim that $\alpha = \beta$. Since $x = ge_\beta$, $x \in G_1 . G_\beta \subseteq G_\beta$. But $x \in G_\alpha$, then $\alpha = \beta$. Therefore

$$x = ge_\beta = ge_\alpha = g\psi_{1,\alpha}.$$

Thus $\psi_{1,\alpha}$ is onto.

Conversely, assume that $\psi_{1,\alpha}$ is an epimorphism for all $\alpha \in Y$. Let $x \in S$. Then there exists $\alpha \in Y$ such that $x \in G_\alpha$. Since $\psi_{1,\alpha}$ is an epimorphism, there exists $g \in G_1$ such that $x = g\psi_{1,\alpha} = ge_\alpha \in G_1 . E(S)$. Therefore $S = G_1 . E(S)$. Hence S is factorizable. #

The following corollary follows directly from Proposition 1.14:

1.15 <u>Corollary</u>. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then S is factorizable if and only if Y has an identity 1 and $G_1 e_\alpha = G_\alpha$ for all $\alpha \in Y$.

1.16 <u>Theorem</u>. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. If S is factorizable, then all the homomorphisms $\psi_{\alpha,\beta}$ are epimorphisms. The converse is true if S has an identity.

<u>Proof</u>: Assume that S is factorizable. By Proposition 1.14, Y has an identity 1 and $\psi_{1,\gamma}$ is an epimorphism for all $\gamma \in Y$. Let $\alpha, \beta \in Y$ such that $\alpha \geq \beta$. Then $1 \geq \alpha \geq \beta$ and so

$$\psi_{1,\beta} = (\psi_{1,\alpha})(\psi_{\alpha,\beta}).$$

Since $\psi_{1,\beta}$ is onto, $\psi_{\alpha,\beta}$ is onto.

Conversely, assume all the homomorphisms $\psi_{\alpha,\beta}$ are onto and S has an identity. Then Y has identity 1, and then for each $\alpha \in Y$, $\psi_{1,\alpha}$ is onto. By Proposition 1.14, S is factorizable. #

1.17 <u>Lemma</u>. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α and $\psi_{\alpha,\beta} (\alpha \geq \beta)$ be its corresponding homomorphisms. Then the following are equivalent :

(i) S is proper.

(ii) All homomorphisms $\psi_{\alpha,\beta}$ are one-to-one.

<u>Proof</u>: Assume S is proper. Let $\alpha \geq \beta$. Then $\beta = \alpha\beta = \beta\alpha$. To show that $\psi_{\alpha,\beta}$ is one-to-one, let $a,b \in G_\alpha$ such that $a\psi_{\alpha,\beta} = b\psi_{\alpha,\beta}$. Then $ae_\beta = be_\beta$ and so $b^{-1}ae_\beta = b^{-1}be_\beta = e_\alpha e_\beta = e_{\alpha\beta}$ which implies $b^{-1}ae_\beta = e_\beta$. Since S is proper, $b^{-1}a \in E(S)$. But $b^{-1}a \in G_\alpha$. Then

$b^{-1}a = e_\alpha$ and hence $a = b$.

Conversely, assume that all the homomorphisms $\psi_{\alpha,\beta}$ are one-to-one. Let $a \in S$, $e \in E(S)$ such that $ae = e$. Assume $a \in G_\alpha$ and $e \in G_\beta$. Then $e_\beta = ae_\beta \in G_{\alpha\beta}$ and so $\beta = \alpha\beta$ which implies $\alpha \geq \beta$. Hence $\psi_{\alpha,\beta}$ is defined. From $\beta = \alpha\beta$, we have $e_\beta = e_{\alpha\beta} = e_\alpha e_\beta$. Therefore $ae_\beta = e_\beta = e_\alpha e_\beta$ which implies $a\psi_{\alpha,\beta} = e_\alpha\psi_{\alpha,\beta}$. Since $\psi_{\alpha,\beta}$ is one-to-one, we have $a = e_\alpha$. Hence $a \in E(S)$. This proves that S is proper.#

1.18 Proposition. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α with corresponding homomorphisms $\psi_{\alpha,\beta}$.

Then, if S is factorizable and proper, then all the groups G_α are isomorphic and for each $\alpha \in Y$, G_α is the maximum group homomorphic image of S and $S \cong G_\alpha \times Y$.

Proof: Assume S is factorizable and proper. By Lemma 1.17, all the homomorphisms $\psi_{\alpha,\beta}$ are one-to-one. Since S is factorizable, Y has an identity 1. All the homomorphisms $\psi_{\alpha,\beta}$ are onto by Theorem 1.16. Hence $\psi_{1,\alpha}$ is an onto isomorphism for all $\alpha \in Y$. Therefore $G_1 \cong G_\alpha$ for all $\alpha \in Y$. By Theorem 1.10, $G_1 \cong S/\sigma$ which implies $G_\alpha \cong S/\sigma$ for all $\alpha \in Y$ and hence for each $\alpha \in Y$, G_α is the maximum group homomorphic image of S.

Next, we define $\theta : S = G_1.E(S) \rightarrow G_1 \times Y$ by

$$(ge_\alpha)\theta = (g,\alpha) \qquad (g \in G_1,\ \alpha \in Y).$$

To show θ is well-defined, let g_1e_α, $g_2e_\beta \in S = G_1.E(S)$ such that $g_1e_\alpha = g_2e_\beta$. Since $g_1e_\alpha \in G_\alpha$ and $g_2e_\beta \in G_\beta$, we have $\alpha = \beta$. Therefore

$g_2^{-1}g_1e_\alpha = e_\alpha$. Since S is proper, $g_2^{-1}g_1 \in E(S)$ which implies $g_2^{-1}g_1 = e_1$ so that $g_1 = g_2$. Therefore

$$(g_1e_\alpha)\theta = (g_1,\alpha) = (g_2,\beta) = (g_2e_\beta)\theta \quad .$$

Let g_1e_α and $g_2e_\beta \in S$. Then

$$\begin{aligned}((g_1e_\alpha)(g_2e_\beta))\theta &= ((g_1g_2)(e_\alpha e_\beta))\theta \text{ (since } E(S) \subseteq C(S))\\ &= ((g_1g_2)e_{\alpha\beta})\theta\\ &= (g_1g_2,\ \alpha\beta)\\ &= (g_1,\alpha)(g_2,\beta)\\ &= ((g_1e_\alpha)\theta)((g_2e_\beta)\theta).\end{aligned}$$

Therefore θ is a homomorphism and θ is clearly onto and one-to-one. Hence θ is an onto isomorphism. This proves that $S \cong G_1 \times Y$. Since for each $\alpha \in Y$, $G_\alpha \cong G_1$, it follows that $S \cong G_\alpha \times Y$ for all $\alpha \in Y$. #

Let G be a group and Y be a semilattice. Then the semigroup $G \times Y$ is a semilattice Y of groups G_α where $G_\alpha = \{(g,\alpha) \mid g \in G\}$. A proof is given as follows : It is clear that $S = \bigcup_{\alpha \in Y} G_\alpha$ is a disjoint union, and for each $\alpha \in Y$, G_α is a group. Let $(g,\alpha) \in G_\alpha$ and $(h,\beta) \in G_\beta$. Then $(g,\alpha)(h,\beta) = (gh,\alpha\beta) \in G_{\alpha\beta}$.

If e is the identity of G, then

$$E(G \times Y) = \{(e,\alpha) \mid \alpha \in Y\}$$

which is isomorphic to Y.

1.19 <u>Proposition</u>. Let G be a group and Y be a semilattice with identity 1. Assume that a semigroup S is isomorphic to the semilattice Y of groups, $G \times Y$. Then S is factorizable and proper, and hence S is

F-inverse and G is the maximum group homomorphic image of S.

Proof: Clearly, $G_1 = \{(g,1) \mid g \in G\}$ is the group of units of G x Y. Let $\alpha \in Y$ and $(g,\alpha) \in G_\alpha$. Let e be the identity of G. Then $(g,\alpha) = (g,1)(e,\alpha) \in G_1 . E(G \times Y)$. Hence G x Y is factorizable.

Let $(g,\alpha) \in G \times Y$ and $(e,\beta) \in E(G \times Y)$ such that

$$(g,\alpha)(e,\beta) = (e,\beta) .$$

Then $(g,\alpha\beta) = (e,\beta)$, so $g = e$ and hence $(g,\alpha) = (e,\alpha) \in E(G \times Y)$. Hence G x Y is proper.

By Theorem 1.10, G_1 is the maximum group homomorphic image of G x Y. But $G_1 \cong G$. Then G is the maximum group homomorphic image of G x Y. Since G x Y is proper and factorizable, by Theorem 1.11, G x Y is F - inverse.

Because S is isomorphic to G x Y, the proposition follows. #