CHAPTER III

MAL'CEV VARIETIES

When working with arbitrary algebras, there is always the danger that the kind of algebra under consideration might be too special, and thus trivial, or else too general, and thus lack the possibility of interesting results. Therefore, one is always searching for appropriate levels of generality.

One proprosal (see [3]) is to consider special classes of algebras called Mal'cev Varieties. In this chapter we define this kind of class of algebras and investigate the various classes of algebraic lattices and commutative groupoids, introduced in Chapter II, in this setting.

3.1. Definition. Let $\mathfrak{U} = \langle A; F \rangle$ be an algebra, the <u>n-ary polynomials</u> of $\mathfrak{U}$ are certain mappings from ${}^{n}A$ into A, defined as follows :

(i) The projections $e_i^n : {}^{n}A \to A$; $(a_0,\ldots,a_{n-1}) \to a_i$, are n-ary polynomials.

(ii) If $p_0,\ldots,p_{n_\gamma-1}$ are n-ary polynomials, then so is $f_\gamma(p_0,\ldots,p_{n_\gamma-1})$, defined by

$$f_\gamma(p_0,\ldots,p_{n_\gamma-1})(x_0,\ldots,x_{n-1}) = f_\gamma(p_0(x_0,\ldots,x_{n-1}),\ldots,p_{n_\gamma-1}(x_0,\ldots,x_{n-1}));$$

where $f_\gamma \in F$

(iii) n-ary polynomials are those and only those which we get from (i) and (ii) in a finite number of steps.

Let $P^{(n)}(\mathfrak{U})$ denote the set of n-ary polynomials of $\mathfrak{U}$.

3.2. <u>Definition</u>. The <u>n-ary polynomial symbols</u> of type τ are defined as follows :

(i) $x_0, \ldots, x_{n-1}$ are n-ary polynomial symbols.

(ii) if $P_0, \ldots, P_{n_\gamma -1}$ are n-ary polynomial symbols, and $\gamma < O(\tau)$, then $f_\gamma(P_0, \ldots, P_{n_\gamma -1})$ is an n-ary polynomial symbol ;

(iii) n-ary polynomial symbols are those and only those which we get from (i) and (ii) in a finite number of steps.

Let $P^{(n)}(\tau)$ denote the set of all n-ary polynomial symbols.

3.3. <u>Definition</u>. The n-ary polynomial p over the algebra $\mathfrak{U}$ <u>induced by</u> the n-ary symbol P is defined as follow :

(i) X_i induces e_i^n,

(ii) if $P = f_\gamma(P_0, \ldots, P_{n_\gamma -1})$ and P_i induces p_i for $0 \leqslant i < n$, then P induces $f_\gamma(p_0, \ldots, p_{n_\gamma -1})$.

3.4. <u>Definition</u>. Let $P, Q \in P^{(n)}(\tau)$. The n-ary identity $P = Q$ is said to be <u>satisfied</u> in a class K of algebras of type τ if P and Q induce the same polynomials, in each algebra in K, or, equivalently, P induces p, Q induces q, and $p(a_0, \ldots, a_{n-1}) = q(a_0, \ldots, a_{n-1})$ for all $a_0, \ldots, a_{n-1} \in A$, $\mathfrak{U} \in K$.

If K is a class of algebras, Id(K) denotes the set of all identities satisfied in K.

Let Σ be a set of identities in $P^{(n)}(\tau)$, then we get a class of algebras Σ^* satisfying these identities.

3.5. <u>Definition</u>. A class K of algebras is <u>a variety</u> if $K = \Sigma^*$ for some set of identities Σ.

3.6. <u>Theorem</u>. A class K is a variety if and only if K is closed under taking subalgebras, direct products and homomorphic images.

<u>Proof</u>. (see [1], **page 171**).

3.7. <u>Definition</u>. Let U be an equivalence relation on an algebra $\mathfrak{A} = \langle A;F \rangle$, i.e., a reflexive, symmetric, and transitive subset of ${}^2A$.

If U is also a subalgebra of ${}^2A$, then it is called a <u>congruence</u> on A.

3.8. <u>Definition</u>. Let U, V be equivalence relations on a set A, let $U \circ V = \{(x,y) \in {}^2A \mid \exists t \in A, x \, U \, t \, V \, y\}$.

If $U \circ V = V \circ U$, then U and V are said to <u>commute</u>.

3.9. <u>Definition</u>. Let $K(\tau)$ be a variety such that each pair of congruences, on each algebra $\mathfrak{A}$ in $K(\tau)$, commutes. Then $K(\tau)$ is said to be a <u>Mal'cev variety</u>, and algebras in $K(\tau)$ are called <u>Mal'cev algebras</u>.

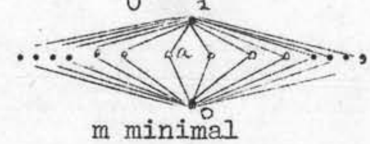
3.10. Theorem. If $K(\tau)$ is a Mal'cev variety, then for each algebra $\mathfrak{A}$ in $K(\tau)$, each subalgebra of ${}^2A$ containing $\hat{A}$, where $\hat{A} = \{(a,a) | a \in A\}$, is a congruence on A.

Proof. (see [3], 19-20;).

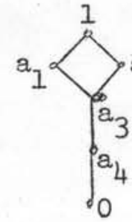
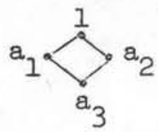
Next we check whether the classes of algebraic lattices and commutative groupoids from Chapter II are Mal'cev varieties.
Recall that K is the class of non-trivial algebraic lattices which have the property that each compact element contains only countably many compact elements. K_0^m is the class of algebraic lattices in K which have m minimal elements. K_1 is the class of algebraic lattices in K which are chains.
K_2 is the class of algebraic lattices $\mathcal{L}$ in K with the property that there exists x in L, such that x is compact and contains all compact elements.
K_3^m is the class of algebraic lattices $\mathcal{L}$ in K, such that $L = \cup \{C_i\}_{i \in I \text{ where } |I| \geq 1}$. C_i is a finite chain of length m, $0_i = 0_j$ and $1_i = 1_j$, for $i \neq j$, $C_i \cap C_j = \{0_i, 1_i\}$.
K_4 is the class of algebraic lattices in K which are complemented lattices and units are compact.

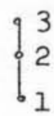
Claim. K, K_0^m, K_1, K_2, K_3^m, K_4 are not varieties and hence not Mal'cev varieties.

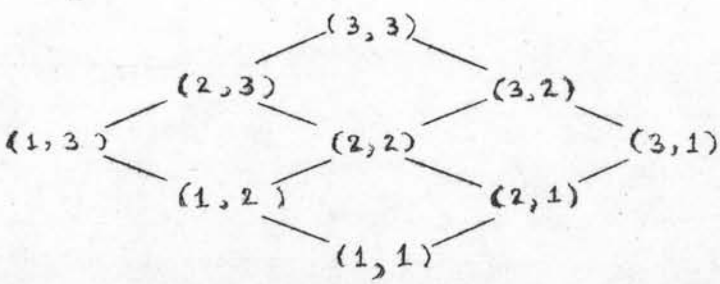
To show K is not a variety, let $\mathcal{L} = \langle \omega+1; \leq \rangle$, $\mathcal{L} \in K$ and $\{1,2,3,\ldots\}$ is sublattice of $\mathcal{L}$ which is not complete. That is, K is not closed under taking a subalgebras. Therefore, K is not a variety.

To show K_0^m is not a variety. Suppose $m > 1$. Let $\mathcal{L}$ have structure , then $\mathcal{L} \in K_0^m$

$\mathcal{L}_0 = \langle \{1,a,0\}; \leq \rangle$ is a sublattice of $\mathcal{L}$ with one minimal element. Hence $\mathcal{L}_0 \notin K_0^m$.

Suppose $m = 1$. Let $\mathcal{L}$ have a structure , $\mathcal{L} \in K_0^m$, but  is a sublattice of $\mathcal{L}$, with two minimal elements. Therefore, K_0^m is not closed under taking subalgebras, and thus, K_0^m is not a variety.

To show K_1 is not a variety, let $\mathcal{L}$ have structure , $\mathcal{L} \in K_1$. $\mathcal{L} \times \mathcal{L}$ has a structure

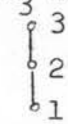


Since $(1,3), (3,1) \in \mathcal{L}\times\mathcal{L}$ such that $(1,3) \nleq (3,1)$ and $(3,1) \nleq (1,3)$, we have $\mathcal{L}\times\mathcal{L}$ is not a chain.

Hence K_1 is not closed under taking direct products, this implies that K_1 is not a variety.

To show K_2 is not a variety, let $\mathcal{L} = \langle \omega+2; \leq \rangle$, $\omega+1$ is compact, therefore $\mathcal{L} \in K_2$. $\omega = \vee \{0,1,2,\ldots\}$, ω is not compact, and $\langle \omega+1; \leq \rangle$ is a sublattice of $\langle \omega+2; \leq \rangle$ which the greatest compact element does not exist, therefore $\langle \omega+1; \leq \rangle \notin K_2$.

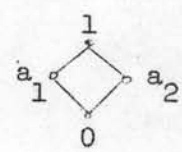
Hence K_2 is not closed under taking subalgebras, and thus K_2 is not a variety.

To show K_3^m is not a variety. For any $\mathcal{L}$ in K_3^m we have $\mathcal{L}_3$ with structure 

as a subalgebra of $\mathcal{L}$

$C_1 = \langle \{(3,3),(2,3),(1,3),(1,2),(1,1)\}; \leq \rangle$ and

$C_2 = \langle \{(3,3),(2,3),(2,2),(1,2),(1,1)\}; \leq \rangle$, we have C_1, C_2 are chain in $\mathcal{L}_3 \times \mathcal{L}_3$ and $C_1 \neq C_2$. But $C_1 \cap C_2 = \{(3,3),(2,3),(1,2),(1,1)\} \neq \{(1,1),(3,3)\}$ where $(1,1)$ is the unit of $\mathcal{L}_3 \times \mathcal{L}_3$ and $(3,3)$ is a zero of $\mathcal{L}_3 \times \mathcal{L}_3$. Therefore $\mathcal{L}_3 \times \mathcal{L}_3 \notin K_3^m$. Since $\mathcal{L}_3 \times \mathcal{L}_3$ is a subalgebra of $\mathcal{L}\times\mathcal{L}$, we have K_3^m is not closed under direct product, and thus K_3^m is not a variety.

To show K_4 is not a variety, let $\mathcal{L}$ have a structure 

; we have $\mathcal{L} \in K_4$

$\mathcal{L}_0 = \langle\{1, a, 0\}, \leq\rangle$ is a sublattice of $\mathcal{L}$ and $\mathcal{L}_0$ is not a complemented lattice, then $\mathcal{L}_0 \notin K_4$.

Therefore, K_4 is not closed under taking subalgebras, and thus K_4 is not a variety. Recall that C is the class of commutative groupoids. C_0^m is the class of commutative groupoids G with the following properties; i) G has m idempotent elements, ii) let B be the set of all idempotents of G, then for all x in G–B, there exist $b \in B$, $n \in \mathbb{N}$ such that $x^n = b$.

C_1 is the class of G in C with the property that for all x, y in G, there exists $n \in \mathbb{N}$ such that $x^n = y$ or $y^n = x$.

C_2 is the class of G in C which are finitely generated.

C_3^m is the class of G in C with the properties that;

i) there exists $a \in G$, such that a generates G,

ii) $B = \{e \in G \mid e * e = e\} \neq \emptyset$ and for $e_1 \neq e_2$ in B, $e_1 * e_2 = a$,

iii) for all e_i in B, there exists $\emptyset \neq X_i \subseteq G$ such that $X_i = \{x \mid x \in G, \exists\, n \in \mathbb{N} \ni x^n = e_i\}$, $|X_i| = m-1$ and for all x, y in X_i, $\exists\, n \in \mathbb{N}$ such that $x^n = y$ or $y^n = x$.

iv) for all x in $G-\{a\}$, $x \in X_i$, for some i.

C_4 is the class of G in C with the following properties,

i) there exists a in G, such that a generates G.

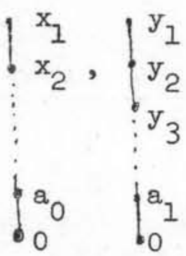
ii) for all $\emptyset \neq X \subsetneq G$, if for all x, y in X, $x * y \in X$ and $Sg(\{x,y\}) \neq G$, then there exist x' in $\dot{G}$, x in X such that $x * x' = a$, and for all n in $\mathbb{N}$, for all y in X, $(x')^n \neq y$.

<u>Claim</u> C is a variety.

Let Σ be the following set of identities, $\{x_1 * x_2 = x_2 * x_1$, for all x_1, x_2 in $P^n(\langle 2 \rangle)\}$. Then $\Sigma^* = C$. Therefore, C is a variety.

<u>Claim</u> $C_0^m, C_1, C_2, C_3^m, C_4$ are not varieties, and hence not Mal'cev varieties.

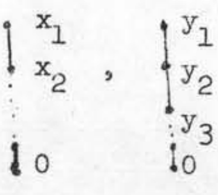
To show C_0^m is not a variety, suppose $m > 1$, let $G \in C_0^m$ and B $B = \{$idempotent of G$\}$. Let $b \in B$, we have $Sg(\{b\}) = \{b\}$ as a subalgebra of G_0 with 1 idempotent element. Therefore, $\{b\} \notin C_0$. Suppose $m = 1$. Let G_0, G_1 come form the lattices of the form



, therefore, a_0, a_1 are the idempotent of G_0, G_1, respectively. We have (a_0, a_1) is the idempotent of $G_0 \times G_1$.

Let $x_1' = \langle x_1, x_2, a_0, \ldots \rangle$, $y_1' = \langle y_1, a_1, y_2, y_3, \ldots \rangle$, we have $(x_1)^3 = a_0$ and $(y_1)^2 = a_1$. Let $n \in \mathbb{N}$, $(x_1, y_1)^n = (x_1^n, y_1^n) \neq (a_0, a_1)$. Therefore $G_0 \times G_1 \notin C_0^m$, and thus, C_0^m is not a variety.

To show C_1 is not a variety, let G_1, G_2 come from lattices of the forms



, respectively. Then $G_1, G_2 \in C_1$.

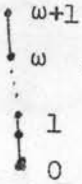
Since $(x_1, y_1), (x_2, y_3) \in G_1 \times G_2$ and $x_1^2 = x_2$, $y_1^3 = y_3$, we have for all $n \in \mathbb{N}$, $(x_1, y_1)^n = (x_1^n, y_1^n) \neq (x_2, y_3)$ and sence $x_2^n \neq x_1$,

$y_3^n \neq y_1$, we have

$$(x_2, y_3)^n = (x_2^n, y_3^n) \neq (x_1, y_1).$$

Therefore $G_1 \times G_2 \notin C_1$, that is C_1 is not closed under taking direct products, and thus C_1 is not a variety.

To show C_2 is not a variety, let G_0 come from lattice $\mathcal{L}$ which has a structure

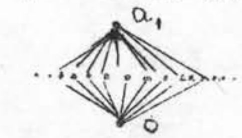


, then $G_0 \in C_2$.

$G_0 \times G_0$ must be generated by $\{((\omega+1)^m, (\omega+1)^n) \mid m, n \in \mathbb{N}\}$ which is infinite. Therefore $G_0 \times G_0 \notin C_2$.

That is, C_2 is not closed under taking direct products, and thus C_2 is not a variety.

To show C_3^m is not a variety, similarly to C_2, for all G in C_3^m must be finitely generated, but for some G in C_3^m, $G \times G$ is not neccessary finitely generated, for example G which comes from a lattice



$G \in C_3^m$, but $G \times G$ is not generated by one element, and hence $G \times G \notin C_3^m$. Therefore C_3^m is not a variety.

To show C_4 is not a variety, since for all G in C_4 must be finitely generated and if G is infinite, then this property is not closed under taking direct products by similarly proof in C_2.

Therefore, C_4 is not a variety.

Claim C is not a Mal'cev variety.

Consider $G = \langle \mathbb{N}; \max \rangle$, G is obviously a commutative groupoid, i.e., $G \in C$.

$\widehat{\mathbb{N}} \cup \{\langle 1,2 \rangle\} = Sg(\widehat{\mathbb{N}} \cup \{\langle 1,2 \rangle\})$ is a subalgebra of $G \times G$ which contains $\widehat{\mathbb{N}}$ but it is not a congruence relation on ${}^2\mathbb{N}$.

That is, C is not a Mal'cev variety, by Theorem 3.10.