CHAPTER II

THE REPRESENTATION OF CLASSES OF ALGEBRAIC LATTICES

In section 2.1, we study the relationship between properties of commutative groupoids G and properties of their lattices of subalgebras Su(G).

In section 2.2, we prove representation theorems for various classes of algebraic lattice using the construction of Theorem 1.21.

2.1. Commutative groupoids and their lattices of subalgebras.

Let G be a commutative groupoid and $x, y \in G$. By " $\exists n \in \mathbb{N}$, $x^n = y$ " we mean " $y \in Sg(\{x\})$ " and by " $\forall m, n \in \mathbb{N}$, $x^m \neq y^n$ " we mean " $Sg(\{x\}) \cap Sg(\{y\}) \neq \phi$ "

2.1.1 Theorem. Let G be a commutative groupoid. Then Su(G) has $m > 0$ minimal elements and $\forall X \in Su(G)$, X contains some minimal element in Su(G) if and only if there exists $B \subseteq G$ such that $|B| = m$ and for all $x \in G-B$, there exist $b \in B$, $n \in \mathbb{N}$ such that $x^n = b$ and for all $b_1, b_2 \in B$, $m, n \in \mathbb{N}$, $(b_1)^n \neq (b_2)^m$.

Proof. $(\Rightarrow)$ Let $C = \{S_i \mid S_i$ is minimal in $Su(G), i \in \{1,\ldots,m\}\}$. For each $S_i \in C$, let b_i be an element of S_i. Since $\phi \subsetneq Sg(\{b_i\}) \subseteq S_i$, and S_i is minimal in Su(G), we have $Sg(\{b_i\}) = S_i$. Let $B = \{b_i \mid b_i \in S_i, i \in \{1,\ldots,m\}\}$. $|B| = m$.

Let $x \in G-B$, there exist $b_i \in B$ such that $Sg(\{b_i\}) \subseteq Sg(\{x\})$. That is $b_i \in Sg(\{x\})$, then there exists $n \in \mathbb{N}$, such that $x^n = b_i$

Let $b_i, b_j \in B$, we have $Sg(\{b_i\}) \cap Sg(\{b_j\}) = \phi$. Therefore, for all m, n in $\mathbb{N}$, $(b_i)^m \neq (b_j)^n$.

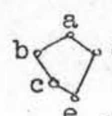
($\Leftarrow$) Let $b \in B$, to show $Sg(\{b\})$ is minimal in $Su(G)$, let $X \in Su(G)$ such that $\phi \subsetneq X \subseteq Sg(\{b\})$, let $x_0 \in X$. If $m = 1$, then there exists $n \in \mathbb{N}$ such that $x_0^n = b$. If not, let $b' \in B-\{b\}$, we have $Sg(\{b\}) \cap Sg(\{b'\}) = \phi$, then $b' \notin X$, that is, for all $n \in \mathbb{N}$, for all $b' \in B-\{b\}$, $x_0^n \neq b'$. Therefore, there exists $n \in \mathbb{N}$ such that $x_0^n = b$, so $b \in X$ and $X = Sg(\{b\})$. Since B has m elements, we have $Su(G)$ has at least m minimal elements. If $X \in Su(G)$ and is minimal then $X = Sg(\{b\})$ for some $b \in B$. That is, $Su(G)$ has m minimal elements.

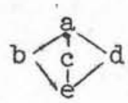
2.1.2. <u>Theorem</u>. $Su(G)$ is a chain iff for all $x, y \in G$, there exists $n \in \mathbb{N}$ such that $x^n = y$ or $y^n = x$.

<u>Proof</u>. ($\Rightarrow$). Let $x, y \in G$, we have $Sg(\{y\}) \subsetneq Sg(\{x\})$ or $Sg(\{x\}) \subsetneq Sg(\{y\})$. Therefore, there exists $n \in \mathbb{N}$ such that $x^n = y$ or $y^n = x$.

($\Leftarrow$). Let $S_1, S_2 \in Su(G)$ such that $S_1 \nsubseteq S_2$, then there exists $x \in S_1$ such that $x \notin S_2$.

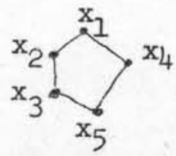
To show $S_2 \subseteq S_1$, let $y \in S_2$, there exists $n \in \mathbb{N}$ such that $x^n = y$. Therefore $y \in S_1$, and $Su(G)$ is a chain.

2.1.3. <u>Lemma</u>. Let $\mathcal{L}$ be a non-distributive algebraic lattice such that for all compact elements x, y in L, $x \wedge y$ is compact. Then $\mathcal{L}$ contains 

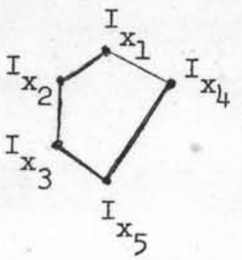
or 

as a sublattice where a, b, c, d are compact in $\mathcal{L}$.

<u>Proof</u>. Let $\mathcal{F} = \langle F; \vee \rangle$ be the semilattice of all compact elements of $\mathcal{L}$, and $I(\mathcal{F})$ be the set of all ideals of $\mathcal{F}$. Then as in Lemma 1.18, $\langle I(\mathcal{F}), \subseteq \rangle \cong \mathcal{L}$.

Suppose $\mathcal{L}$ has a sublattice 

, then I_{x_1}, I_{x_2}, I_{x_3}, I_{x_4}, I_{x_5} in $I(\mathcal{F})$ form the sublattice



, where $I_{x_1} = \{x | x \varepsilon F,\ x \leq x_1\}$, $I_{x_2} = \{x | x \varepsilon F,\ x \leq x_2\}$,

$I_{x_3} = \{x | x \varepsilon F,\ x \leq x_3\}$, $I_{x_4} = \{x | x \varepsilon F,\ x \leq x_4\}$,

$I_{x_5} = \{x | x \varepsilon F,\ x \leq x_5\}$.

Choose $x_2' \varepsilon I_{x_2} - I_{x_3}$, $x_4' \varepsilon I_{x_4} - I_{x_3} = I_{x_4} - I_{x_2}$, then $x_2' \neq x_4'$.

Let $a = x_2' \vee x_4'$. Then $a \varepsilon I_{x_1} - (I_{x_2} \cup I_{x_4}) \subseteq I_{x_1} - (I_{x_3} \cup I_{x_4})$, and there exist $x_3'' \varepsilon I_{x_3} - I_{x_4}$ and $x_4'' \varepsilon I_{x_4} - I_{x_3}$ such that

$a = x_2' \vee x_4' = x_3'' \vee x_4''$. That is a contains x_2', x_3'', x_4', x_4''.

Let $b = x_2' \vee x_3''$, $d = x_4' \vee x_4''$. Then $b \varepsilon I_{x_2} - I_{x_3}$, $d \varepsilon I_{x_4} - I_{x_3}$ and a contains b, d.

Let $b \wedge d = e$, then $I_b \wedge I_d = I_e$, and $e \varepsilon I_{x_2} \wedge I_{x_4} = I_{x_3} \wedge I_{x_4}$.

Let $c = x_3'' \vee e$, we have $c \varepsilon I_{x_3} - I_{x_4}$, $e < c < b$.

$e = e \wedge d \leq c \wedge d = (x_3'' \vee e) \wedge d \leq b \wedge d = e$, implies $c \wedge d = e$.

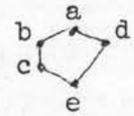
$b \vee d = (x_2' \vee x_3'')\vee(x_4' \vee x_4'')$

$= (x_2' \vee x_3'')\vee(x_4'' \vee x_4')$

$= ((x_2' \vee x_3'')\vee x_4'')\vee x_4'$

$= (x_2' \vee(x_3'' \vee x_4''))\vee x_4'' = (x_2' \vee a)\vee x_4') = a$, and

$c \vee d = (x_3'' \vee e)\vee(x_4' \vee x_4'') = a.$

Therefore, {a, b, c, d, e} forms a sublattice 

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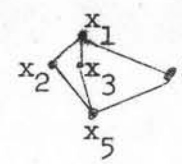
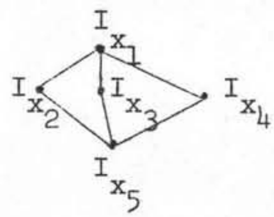
where a, b, c, d, e are compact in $\mathcal{L}$.

Suppose $\mathcal{L}$ has a sublattice 

, we have I_{x_1}, I_{x_2}, I_{x_3}, I_{x_4}, I_{x_5} in $I(\mathcal{F})$ form the sublattice 

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Choose $x_2' \in I_{x_2} - I_{x_3} = I_{x_2} - I_{x_4}$, $x_4' \in I_{x_4} - I_{x_3} = I_{x_4} - I_{x_2}$.

Let $a = x_2' \vee x_4'$. Then $a \in I_{x_1} - (I_{x_2} \cup I_{x_3} \cup I_{x_4})$, there exist $x_3'' \in I_{x_3} - I_{x_4}$, $x_4'' \in I_{x_4} - I_{x_3}$, $x_2''' \in I_{x_2} - I_{x_4}$, $x_3''' \in I_{x_3} - I_{x_2}$ such that $a = x_2' \vee x_4' = x_3'' \vee x_4'' = x_2''' \vee x_3'''$, that is, a contains x_2', x_4', x_3'', x_4'', x_2''', x_3'''. Let $b' = x_2' \vee x_2'''$, $d' = x_4' \vee x_4''$, $c' = x_3'' \vee x_3'''$, then $b' \in I_{x_2} - I_{x_3}$, $d' \in I_{x_4} - I_{x_3}$, $c' \in I_{x_3} - I_{x_4}$.

Suppose $I_{b'} \cap I_{d'} = I_e$ and $I_{b'} \cap I_{c'} = I_f$ and $I_{c'} \cap I_{d'} = I_h$, e, f, h are compact, $e \vee f \vee h \in I_{x_5}$.

$$\text{Let } I_b = \vee(I_{b'} \cup I_e \cup I_f \cup I_h) = I_{b' \vee e \vee f \vee h},$$

$$I_c = \vee(I_{c'} \cup I_e \cup I_f \cup I_h) = I_{c' \vee e \vee f \vee h},$$

$$I_d = \vee(I_{d'} \cup I_e \cup I_f \cup I_h) = I_{d' \vee e \vee f \vee h},$$

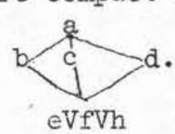
Since

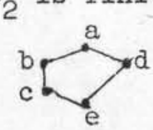
$$I_b \wedge I_d = I_{b \wedge d} = \vee(I_e \cup I_f \cup I_h) = I_{e \vee f \vee h},$$

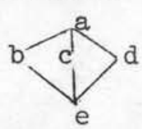
$$I_c \wedge I_d = I_{c \wedge d} = \vee(I_e \cup I_f \cup I_h) = I_{e \vee f \vee h},$$

$$I_b \wedge I_c = I_{b \wedge c} = \vee(I_e \cup I_f \cup I_h) = I_{e \vee f \vee h}, \text{ we have}$$

$$b \wedge d = c \wedge d = b \wedge c = e \vee f \vee h.$$

And similarly to the previous proof, we can show that $b \vee d = c \vee d = b \vee c$. Therefore, a, b, c, d, $e \vee f \vee h$, are compact in $\mathcal{L}$, and $\{a, b, c, d, e \vee f \vee h\}$ forms a sublattice 

2.1.4. Theorem. Let G be a commutative groupoid such that Su(G) is non-distributive and for all S_1, S_2 in Su(G) which are finitely generated, $S_1 \cap S_2$ is finitely generated. Then Su(G) contains either a sublattice 

or 

where a, b, c, d, e are finitely generated subalgebras of G.

Proof. (By Lemma 2.1.3 and Lemma 1.19)

2.2. Representation theorems for classes of algebraic lattices.

2.2.1. Definition. A class K of algebraic lattices is represented by a class C of commutative groupoids if

(R_1) for all $\mathcal{L}$ in K, there exists G in C such that $\mathcal{L} \cong Su(G)$.

(R_2) for all G in C, there exists $\mathcal{L}$ in K such that $Su(G) \cong \mathcal{L}$.

2.2.2. Theorem. Let K be the class of non-trivial algebraic lattices which have the property that each compact element contains only countably many compact elements. And let C be the class of commutative groupoids.

Then K is represented by C.

Proof. (By Theorem 1.21).

Note: For the rest of this chapter, K, C denotes the same class of lattices and groupoids it denotes in Theorem 2.2.2.

2.2.3. Theorem. Let K_0^m be the class of algebraic lattices in K which have m minimal elements. And let C_0^m be the class of commutative groupoids G with the following properties :

i) G has m idempotent elements,

ii) let B be the set of all idempotents of G,

then for all x in G-B, there exist $b \in B$, $n \in \mathbb{N}$, such that $x^n = b$.

Then K_0^m is represented by C_0^m.

Proof. (R_1)

Let $\mathcal{L} \in K_0^m$, and let $\{b_1, b_2, \ldots\}$ be the set of all minimal elements of $\mathcal{L}$.

To show $b_i \in \{b_1, b_2, \ldots\}$ is compact in $\mathcal{L}$, since there exists only one set, $\{b_i, 0\}$, in L such that $b_i = V\{b_i, 0\}$, b_i is obviously compact.

Follow the construction of G in theorem 1.21., $\mathcal{L} \cong Su(G)$, and $\{b_1, b_2, \ldots\} \subseteq V$ where $b_i' = \langle b_i, b_i, \ldots \rangle$, for all $b_i \in \{b_1, b_2, \ldots\}$, we have $b_i * b_i = b_i$. So, b_i is idempotent of G.

Let $B = \{b_1, b_2, \ldots\}$. B contains m idempotents of G. Let $x \in G-B$. Since $x \in L$, there exists $b_i \in B$ such that x contains b_i and $x' = \langle x, x_1', x_2', \ldots \rangle$ where x_i' is contained in x, we have $x_{n-1}' = b_i$ for some $n \in \mathbb{N}$ and $(x)^n = x_{n-1}' = b_i$. Therefore, x is not an idempotent of G.

That is, G contains m idempotent elements and for all x in G-B, there exist $b \in B$, $n \in \mathbb{N}$, such that $x^n = b$. Consequently, $G \in C_0^m$.

(R_2). Let $G \in C_0^m$, G has properties i), ii). Therefore, Su(G) has m minimal elements, by Theorem 2.1.1., where B = {idempotents of G}, and so $Su(G) \in K_0^m$.

Hence K_0^m is represented by C_0^m.

2.2.4. <u>Theorem</u>. Let K_1 be the class of algebraic lattices $\mathcal{L}$ in K which are chains. And let C_1 be the class of groupoids G in C with the property that for all x, y in G, there exists $n \in \mathbb{N}$ such that $x^n = y$ or $y^n = x$.

Then K_1 is represented by C_1.

<u>Proof</u>. (R_1). Let $\mathcal{L} \in K_1$, $\mathcal{L}$ is a chain.

Consider G, which is constructed as in Theorem 1.21., such that $Su(G) \cong \mathcal{L}$.

Let $x, y \in G$, so $x, y \in L$. Since $\mathcal{L}$ is a chain, we have $y \leq x$ or $x \leq y$, i.e. x contains y or y contains x.

Therefore, there exists $n \in \mathbb{N}$ such that $x^n = y$ or $y^n = x$, and so $G \in C_2$.

(R_2). Let $G \in C_2$, consider Su(G).

Since G satisfies the property in Theorem 2.1.2., we have Su(G) is a chain. Therefore, $Su(G) \in K_1$.

Hence, K_1 is represented by C_1.

<u>Remark</u>. If $\mathcal{L}$ is a chain of 2 or 3 elements, then G, constructed from Theorem 1.21., is a semigroup.

<u>Proof</u>. Let $\mathcal{L}$ be a chain $\begin{array}{c} 1 \\ | \\ 0 \end{array}$. $G = \{1\}$, so G is a semigroup with only one element.

Let $\mathcal{L}$ be a chain $\begin{array}{c} 1 \\ | \\ x \\ | \\ 0 \end{array}$. G has two elements 1, x, which $1 * 1 = x$, $1 * x = x$, $x * x = x$, i.e., G is a semigroup with two elements, one is the generator, and the other is an idempotent.

2.2.5. <u>Theorem</u>. Let K_2 be the class of algebraic lattices $\mathcal{L}$ in K with the property that there exists x in $\mathcal{L}$, such that x is compact and contains all compact elements. And let C_2 be the class of groupoids G in C which are finitely generated.

Then K_2 is represented by C_2.

Proof. (R_1) Let $\mathcal{L} \varepsilon K_2$, and $x \varepsilon L$ such that x is compact and contains all compact elements of $\mathcal{L}$.

Consider G which is constructed as in Theorem 1.21., such that $Su(G) \cong \mathcal{L}$.

We have $x \varepsilon G$, and x generates G.

Therefore, G is finitely generated, $G \varepsilon C_2$.

(R_2) Let $G \varepsilon C_2$, G is finitely generated.

Then G is compact in Su(G), by Lemma 1.19, and G contains all compact elements of Su(G).

Therefore, $Su(G) \varepsilon K_2$, and K_2 is represented by C_2.

2.2.6. Theorem. Let K_3^m be the class of algebraic lattices $\mathcal{L}$ in K, such that $L = \underset{i \varepsilon I,\ \text{with } |I| \geq 1}{\cup\{C_i\}}$ C_i is a finite chain of length m, $0_i = 0_j$ and $1_i = 1_j$, for $i \neq j$, $C_i \cap C_j = \{0_i, 1_i\}$.

And let C_3^m be the class of groupoids G in C with the properties that

i) there exists $a \varepsilon G$, such that a generates G,

ii) $B = \{e \varepsilon G \mid e * e = e\} \neq \emptyset$ and for $e_1 \neq e_2$ in B, $e_1 * e_2 = a$,

iii) for all e_i in B, there exists $\emptyset \neq X_i \subsetneq G$ such that $X_i = \{x \mid x \varepsilon G, \exists n \varepsilon \mathbb{N}, \ni x^n = e_i\}$, $|X_i| = m-1$ and for all x, y in X_i, $\exists n \varepsilon \mathbb{N}$ such that $x^n = y$ or $y^n = x$,

iv) for all x in G-{a}, $x \varepsilon X_i$ for some i.

Then K_3^m is represented by C_3^m

<u>Proof</u>. (R_1) Let $\mathcal{L} \varepsilon K_3^m$, $L = \cup\{C_i\}_{i\varepsilon I, |I| \geqslant 1}$, C_i is a finite chain of length m, $0_i = 0_j$, $1_i = 1_j$, and for $i \neq j$, $C_i \cap C_j = \{0_i, 1_i\}$.

Since all elements in L are compact, we have that the groupoid G constructed in Theorem 1.21, contains all elements in L and has 1 as its generator.

Let e_i be the minimal element in C_i.
Since $C_i \cap C_j = \{0,1\}$, for all $i \neq j$, we have that e_i is also minimal in $\mathcal{L}$ and $e_i * e_j = e_i \vee e_j$, that is, $e_i * e_j$ generates e_i and e_j, then $e_i * e_j \ \varepsilon\ C_i \cap C_j$ and $e_i * e_j = 1$.

Let $B = \cup\{e_i\}_{i\varepsilon I, |I| \geqslant 1}$. As in Theorem 2.2.3., B is the set of idempotent elements in G. $|I| \geqslant 1$ implies $B \neq \emptyset$. Let $e_i \varepsilon B$, we have $e_i \ \varepsilon\ C_i$, and for all x in $C_i - \{0\}$, $x \ \varepsilon\ G$ and contains e_i, that is, there exists $n \ \varepsilon\ \mathbb{N}$, such that $x^n = e_i$, and since $C_i \cap C_j = \{0,1\}$ for all $i \neq j$, we have $y \notin C_i$ if and only if y does not contain e_i if and only if for all n in $\mathbb{N}$, $y^n \neq e_i$.

Let $X_i = \{x | x \ \varepsilon\ G, \exists\, n \ \varepsilon\ \mathbb{N} \ni x^n = e_i\}$. $|X_i| = |C_i - \{0\}| = m-1$.

Since C_i is a chain, we have for all x, y in X_i, there exists $n \ \varepsilon\ \mathbb{N}$, such that $x^n = y$ or $y^n = x$, by Theorem 2.2.4.

Let $x \ \varepsilon\ G - \{a\}$, $x \ \varepsilon\ L - \{a\}$ implies $x \ \varepsilon\ C_i$, for some i, that is $x \ \varepsilon\ X_i$, for some i.

Therefore, $G \ \varepsilon\ C_3^m$.

(R_2) Let $G \in C_3^m$, then G has the properties i), ii), iii). By Theorem 2.2.3. $\cup\{Sg(\{e_i\})\}$ is the set of all minimal elements of Su(G).

Consider $Su(Sg(X_i-\{a\}))$, by Theorem 2.2.4., it is a chain. Since G contains all subalgebras in $Su(Sg(X_i-\{a\}))$, we have that $S_i = Su(Sg(X_i-\{a\})) \cup G$ is also a chain.

For each x in X_i, $Sg(\{e_i\}) \subseteq Sg(\{x\}) \subseteq G$.

Since X_i has m-1 elements, we have m-1 subalgebras,

that is, $Su(Sg(X_i- a)) \cup G$ has at least m elements including $\emptyset$.

Let $S \in Su(G)$ such that $Sg(\{e_i\}) \subset S \subset G$, then $e_i \in S$ and $e_j \notin S$ for all $j \neq i$, and for all x in S, $x \in X_i$. Therefore, $S \subset X_i$

Since X_i is finite, we have S is finite, say $S = \{y_1, \dots, y_k\}_{k<m}$, $y_j \in X_i-\{a\}$, then we can choose y_i in S such that y_i generates S. Thus; $S = Sg(\{y_i\})$. This shows that, S_i has exactly m elements.

$Su(G) = \cup \{S_i\}$, where S_i is a chain of length m and for all $i \neq j$, $0_i = \emptyset = 0_j$, $1_i = G = 1_j$.

To show for $i \neq j$, $S_i \cap S_j = \{\emptyset, G\}$, suppose not, then there exists $S \in Su(G) \ni \emptyset \neq S \subsetneq G$ and $S \in S_i \cap S_j$, this implies that $S \subset X_i$ and $S \subset X_j$, that is, S contains e_i and e_j where $i \neq j$, and $e_i * e_j = a \in S$, a contradiction that $S \neq G$. Therefore $Su(G) \in K_3^m$.

Hence K_3^m is represented by C_3^m.

2.2.7. Theorem. Let K_4 be the class of algebraic lattices $\mathcal{L}$ in K which are complemented lattices and units are compact. And let C_4 be the class of groupoids G in C with the following properties ;

i) there exists a in G such that a generates G.

ii) for all $\emptyset \neq X \subsetneq G$, if for all x, y in **X**, $x * y \in X$ and $Sg(\{x,y\}) \neq G$, then there exist x' in G, x in X such that $x * x' = a$, and for all n in $\mathbb{N}$, for all y in X, $(x')^n \neq y$.

Then K_4 is represented by C_4.

Proof. (R_1) Let $\mathcal{L} \in K_4$, by the construction of G in Theorem 1.21, we have $Su(G) \cong \mathcal{L}$ and G is generated by the unit (1) of $\mathcal{L}$.

Let $\emptyset \neq X \subsetneq G$, and for all x, y in X, $x * y \in X$ and $Sg(\{x,y\}) \neq G$.

$X = Sg(X) \subsetneq G$, then since Su(G) is a complemented lattice, then there exists S in Su(G) such that

$$S \vee X = G, \quad S \cap X = \emptyset \quad \text{and} \quad \emptyset \neq S \subsetneq G.$$

Therefore $1 \in S \vee X$ where $1 \notin S$ and $1 \notin X$, that is there exist $x' \in S$, x in X such that $x' * x = 1$. Since $S \cap X = \emptyset$, we have for all n in $\mathbb{N}$, for all y in X, $(x')^n \neq y$. Hence, $G \in C_4$.

(R_2). Let $G \in C_4$ and G is generated by a. Then G, the unit of Su(G), is compact.

To show $Su(G) \in K_4$, let $\emptyset \neq S \in Su(G)$ and $S \neq G$. Then for all x, y in S, $x * y \in S$ and $Sg(\{x,y\}) \neq G$. From ii), there exists $x' \in G$, x in S such that $x * x' = a$ and for all n in $\mathbb{N}$, for all y in S, $(x')^n \neq y$, consequently, $(x')^n \notin S$.

Consider $Sg(\{x'\})$. $\vee(Sg(\{x'\}), S) = G$,

$$\wedge(Sg(\{x'\}), S) = Sg(\{x'\}) \cap S = \emptyset.$$

That is, $Sg(\{x'\})$ is a complement of S in Su(G). Therefore, Su(G) is a complemented lattice in K_4. Hence, K_4 is represented by C_4.

<u>Note</u> : $\mathcal{L}$ in K_4 is a unique complemented lattice if and only if x' of G in C_4 in Theorem 2.2.7 is unique.