CHAPTER IV

AN APPLICATION OF DIFFERENTIAL ALGEBRA

In elementary calculus, there are many unsolvable problems, especially in integration. For a long time mathematicians suspected but were not sure that certain indefinite integrals could not be computed in finite form in terms of the "elementary functions." Until Liouville proved that the indefinite integral of certain functions such as $\int e^{x^2} dx$ cannot be so expressed. Later, Rosenlicht derived a criterion, due to Liouville, that if $f(z)$ and $g(z)$ are two given rational functions of a complex variable z, $f(z)$ being nonzero, and $g(z)$ being nonconstant, then $\int f(z)e^{g(z)} dz$ is elementary if and only if there exists a rational function of complex variable, say a, such that $f = a' + ag'$, where a' and g' are derivatives of a and g respectively.

This final chapter contains results proved by Liouville and Rosenlicht as well as some examples of elementary functions with nonelementary indefinite integrals other than those given by Rosenlicht. To start with, we are not interested in arbitrary functions, but in "elementary functions." Intuitively, an elementary function is a function which is obtained from polynomials, exponentials, logarithms, trigonometric or inverse trigonometric functions by using the operations of addition, subtraction, multiplication, division, radicals and composition a finite number of times. More precise definitions are given later.

Notation Throughout this chapter, a' will denote the derivative of a.

Definition 4-1 Let $0 \neq d$ and b be two elements of the differential field $\mathcal{F}$, d is said to be an exponential of b, or b a logorithm of d, if $b' = \frac{d'}{d}$.

Definition 4-2 By an integral of an element b of a differential field, we shall mean any solution of the differential equation y' = a.

Remark Two integrals of b which are contained in the same extension of the differential field containing b differ by a constant.

Proof Let y_1 and y_2 be two integrals of b. That is,

$$y_1' = b, \quad y_2' = b.$$

Since $(y_1 - y_2)' = y_1' - y_2' = b - b = 0$, $y_1 - y_2$ is a constant. Hence $y_1 = y_2 + \text{constant}$.

Notation The integral of an element b in $\mathbb{C}(z)$ is denoted by $\int b\,dz$.

We now establish the following convenient result which will be needed later.

Lemma 4-1

$$\frac{(a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n})'}{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}} = k_1 \frac{a_1'}{a_1} + k_2 \frac{a_2'}{a_2} + \cdots + k_n \frac{a_n'}{a_n}.$$

where $a_1, a_2, \ldots, a_n$ are nonzero elements of the differential field $\mathcal{F}$ and $k_1, k_2, \ldots, k_n$ are integers.

We call this identity the "logarithmic derivative identity."

<u>Proof</u> The proof is by induction on the number of term n. It is clear that if n = 1, $\dfrac{(a_1^{k_1})'}{a_1^{k_1}} = k_1 \dfrac{a_1'}{a_1}$. Now we suppose that the lemma is true for $n - 1$, $n \geqslant 2$, that is,

$$\frac{(a_1^{k_1} a_2^{k_2} \ldots a_{n-1}^{k_{n-1}})'}{a_1^{k_1} a_2^{k_2} \ldots a_{n-1}^{k_{n-1}}} = k_1 \frac{a_1'}{a_1} + k_2 \frac{a_2'}{a_2} + \ldots + k_{n-1} \frac{a_{n-1}'}{a_{n-1}} .$$

We have to prove that the equation is true for n. Consider

$$\frac{(a_1^{k_1} a_2^{k_2} \ldots a_{n-1}^{k_{n-1}} a_n^{k_n})'}{a_1^{k_1} a_2^{k_2} \ldots a_{n-1}^{k_{n-1}} a_n^{k_n}} = \frac{(a_1^{k_1} a_2^{k_2} \ldots a_{n-1}^{k_{n-1}})' . a_n^{k_n}}{a_1^{k_1} a_2^{k_2} \ldots a_{n-1}^{k_{n-1}} . a_n^{k_n}} + \frac{(a_1^{k_1} a_2^{k_2} \ldots a_{n-1}^{k_{n-1}}) . (a_n^{k_n})'}{a_1^{k_1} a_2^{k_2} \ldots a_{n-1}^{k_{n-1}} . a_n^{k_n}} .$$

By the induction hypothesis, we obtain

$$\frac{(a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n})'}{a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n}} = k_1 \frac{a_1'}{a_1} + k_2 \frac{a_2'}{a_2} + \ldots + k_{n-1} \frac{a_{n-1}'}{a_{n-1}} + k_n \frac{a_n'}{a_n} .$$

This completes the proof.

<u>Lemma 4-2</u> Let $\mathcal{F}$ be a differential field and $\mathcal{F}[X]$ be the polynomial ring in one indeterminate X. (Do not confuse this with the set of forms.) If the maps D_o and D_1 of $\mathcal{F}[X]$ into itself are defined by

$$D_o \left(\sum_{i=o}^{n} a_i X^i \right) = \sum_{i=o}^{n} a_i' X^i$$

$$D_1 \left(\sum_{i=o}^{n} a_i X^i \right) = \sum_{i=o}^{n} i \, a_i X^{i-1} ,$$

for $a_o, a_1, \ldots, a_n \in \mathcal{F}$. Then

i) $D_j(A + B) = D_jA + D_jB$

ii) $D_j(AB) = (D_jA)B + A(D_jB)$

for $j = 0,1$ and $A,B \in \mathcal{F}[X]$.

<u>Proof</u> The proof of this lemma for the case $j = 1$ is very similar to the proof of theorem 1-6 and is omitted.

Now we prove the lemma for the case $j = 0$. Let $A = \sum_{i=o}^{n} a_iX^i$, $B = \sum_{j=o}^{m} b_jX^j$ where a_n and $b_m \neq 0$. To prove (i), suppose that $n > m$.

$$A + B = \sum_{i=o}^{m}{}'(a_i + b_i)X^i + \sum_{i=m+1}^{n}{}' a_iX^i$$

$$D_o(A + B) = \sum_{i=o}^{m}{}'(a_i + b_i)'X^i + \sum_{i=m+1}^{n}{}' a_i'X^i$$

$$= \sum_{i=o}^{m}{}'(a_i' + b_i')X^i + \sum_{i=m+1}^{n}{}' a_i'X^i$$

$$= \sum_{i=o}^{m}{}' a_i'X^i + \sum_{i=m+1}^{n}{}' a_i'X^i + \sum_{i=o}^{m}{}' b_i'X^i$$

$$= \sum_{i=o}^{n}{}' a_i'X^i + \sum_{i=o}^{m}{}' b_i'X^i$$

$$= D_oA + D_oB.$$

In order to prove (ii), let $F = bX^k$. Consider

$$A.F = \left(\sum_{i=o}^{n} a_i X^i\right) bX^k = \sum_{i=o}^{n} a_i bX^{i+k}$$

$$D_o(A.F) = \sum_{i=o}^{n} (a_i b)' Y^{i+k} = \sum_{i=o}^{n} (a_i' b + a_i b') X^{i+k}$$

$$= \sum_{i=o}^{n} a_i' bX^{i+k} + \sum_{i=o}^{n} a_i b' X^{i+k}$$

$$= \left(\sum_{i=o}^{n} a_i' X^i\right) bX^k + \left(\sum_{i=o}^{n} a_i X^i\right) b' X^k$$

$$= (D_o A)F + A.D_o F.$$

In the general case, let $F_i = b_i X^i$, $i = 0,1,\ldots, m$, that is, $B = F_o + F_1 + \ldots + F_m$.

$$A.B = AF_o + AF_1 + \ldots AF_m$$

$$D_o(A.B) = D_o(AF_o) + D_o(AF_1) + \ldots + D_o(AF_m)$$

$$= (D_o A)F_o + ADF_o + (D_o A)F_1 + AD_o F_1 + \ldots + (D_o A)F_n + AD_o F_m$$

$$= D_o A(F_o + F_1 + \ldots + F_m) + A(D_o F_o + D_o F_1 + \ldots + D_o F_m)$$

$$= (D_o A).B + AD_o B.$$

This completes the proof of the lemma.

Our first theorem establishes a basic connection between the ground differential field and its extension field.

Theorem 4-1 Let $\mathcal{F}$ be a differential field of characteristic zero and $\mathcal{K}$ a countably infinite algebraic extension field of $\mathcal{F}$. (That is, $\mathcal{K}$ can be written in the form $\mathcal{K} = \mathcal{F}(x_1, x_2, \ldots)$ where $x_1, x_2, \ldots \in \mathcal{K}$).

Then the derivation on $\mathcal{F}$ can be extended to a derivation on $\mathcal{K}$, and this extension is unique.

<u>Proof</u> Let $\mathcal{F}[X]$ be the polynomial ring in one indeterminate X. Define the maps D_o and D_1 of $\mathcal{F}[X]$ into itself by

$$D_o \left(\sum_{i=o}^{n} a_i X^i \right) = \sum_{i=o}^{n} a_i' X^i ,$$

$$D_1 \left(\sum_{i=o}^{n} a_i X^i \right) = \sum_{i=o}^{n} i a_i X^{i-1}$$

for $a_o, a_1, \ldots, a_n \in \mathcal{F}$. If $\mathcal{K}$ has a differential field structure extending that of $\mathcal{F}$, then for any $x \in \mathcal{K}$ and $A(X) = \sum_{i=o}^{n} a_i X^i \in \mathcal{F}[X]$,

we have

$$\begin{aligned} (A(x))' &= a_o' + (a_1 x)' + \ldots + (a_n x^n)' \\ &= a_o' + a_1' x + a_1 x' + \ldots + a_n' x^n + a_n n x^{n-1} x' \\ &= a_o' + a_1' + a_2' x^2 + \ldots + a_n' x^n + (a_1 + 2a_2 x + \ldots + n a_n x^{n-1}) . x' \\ &= (D_o A)(x) + (D_1 A)(x) . x' . \end{aligned}$$

If we replace A(X) by the minimal polynomial f(X) of x over $\mathcal{F}$, then

$$0 = (D_o f)(x) + (D_1 f)(x) . x' .$$

We have that $(D_1 f)(x) \neq 0$, otherwise x is a root of the polynomial which is of degree less than the minimal polynomial f(X). Then

$$x' = - \frac{(D_o f)(x)}{(D_1 f)(x)} .$$

According to theorem 1-12, there exists a unique minimal polynomial $f(X)$ of x over $\mathcal{F}$, it then follows that the differential structure on $\mathcal{K}$ extending that on $\mathcal{F}$ is unique whenever it exists. We now show that such a structure on $\mathcal{K}$ exists. Assume first that $\mathcal{K}$ is a finite extension of $\mathcal{F}$, then $\mathcal{K} = \mathcal{F}(x)$ (theorem 1-14).

By the preceeding argument, we already know that if the derivation D on $\mathcal{K}$ that extends the one on $\mathcal{F}$ exists, then Dx is equal to $-\dfrac{(D_0 f)(x)}{(D_1 f)(x)}$.

Since $\mathcal{F}[x] = \mathcal{F}(x) = \mathcal{K}$, by theorem 1-12, there exists a polynomial $g(X) \in \mathcal{F}[X]$ such that $g(x) = -\dfrac{(D_0 f)(x)}{(D_1 f)(x)}$. Moreover, any element in $\mathcal{K}$ is of the form $A(x) = \sum_{i=0}^{n} a_i x^i$ where $a_i \in \mathcal{F}$ and

$(A(x))' = (D_0 A)(x) + (D_1 A)(x).Dx$. Now we define the map D of $\mathcal{F}[X]$ into into itself by

$$DA = D_0 A + g(X) D_1 A$$

for any $A \in \mathcal{F}[X]$. By lemma 4-2, we have

$$D(A + B) = DA + DB$$

$$D(AB) = (DA)B + A(DB)$$

and for any $a \in \mathcal{F}$, $Da = D_0 a = a'$.

We claim that D induces a derivation on $\mathcal{K}$ extending that on $\mathcal{F}$. To prove this, it is sufficient to show that D is well-defined. Thus if $A, B \in \mathcal{F}[X]$. $A \sim B$ i.e. $A - B$ has x as its root or $A - B = C(X)f(X)$ for some $C(X) \in \mathcal{F}[X]$, we shall prove that $DA - DB$ also has x as its root. Consider

$$
\begin{aligned}
DA - DB &= D(A - B) = D(C(X)f(X)) \\
&= C(X)(Df)(X) + (DC)(X)f(X) \\
&= C(X)((D_o f)(X) + g(X)(D_1 f)(X)) + (DC)(X)f(X) \\
(DA)(x) - (DB)(x) &= C(x)((D_o f)(x) + g(x)(D_1 f)(x)) \\
&= C(x)\left((D_o f)(x) - \frac{(D_o f)(x)}{(D_1 f)(x)}(D_1 f)(x)\right) \\
&= 0
\end{aligned}
$$

This proves that D is well-defined and thus proves the theorem for the case in which $\mathcal{K}$ is a finite extension of $\mathcal{F}$.

We come now to the case in which $\mathcal{K}$ is countably infinite extension of $\mathcal{F}$, that is $\mathcal{K} = \mathcal{F}(x_1, x_2, \ldots)$. The proof of this case is by induction. Assume that the theorem is true for $m < n$, what we must show is it is also true for n. Let $\mathcal{K}_1 = \mathcal{F}(x_1, x_2, \ldots, x_{n-1})$, by the induction hypothesis, $\mathcal{K}_1$ is a differential field. Since $\mathcal{F}(x_1, x_2, \ldots, x_n) = \mathcal{K}_1(x_n)$, we are back to the case of a finite extension of $\mathcal{F}$. This completes the proof of the theorem.

<u>Definition 4-3</u> Let $\mathcal{F}$ be a differential field, A differential field $\mathcal{K}$ is said to be a <u>differential extension field</u> of $\mathcal{F}$ if $\mathcal{F}$ is a differential subfield of $\mathcal{K}$.

The following result will be the principal tool for proving the main theorem of this chapter.

<u>Lemma 4-3</u> Let $\mathcal{F}$ be a differential field and $\mathcal{F}(t)$ a differential extension field of $\mathcal{F}$ having the same subfield of constants, with t transcendental over $\mathcal{F}$. Then

i) if $t' \in \mathcal{F}$, then for any polynomial $f(t) \in \mathcal{F}[t]$ of positive degree, $(f(t))'$ is a polynomial in $\mathcal{F}[t]$ of the same degree as $f(t)$ or degree one less, according as the highest coefficient of $f(t)$ is, or is not a constant,

ii) if $\frac{t'}{t} \in \mathcal{F}$, then for any nonzero $a \in \mathcal{F}$ and any nonzero integer n we have

$$(at^n)' = ht^n ,$$

for some nonzero $h \in \mathcal{F}$, implying that, for any polynomial $f(t) \in \mathcal{F}[t]$ of positive degree, $(f(t))'$ is a polynomial in $\mathcal{F}[t]$ of the same degree. Furthermore, $(f(t))'$ is a multiple of $f(t)$ only if $f(t)$ is a monomial.

Proof For a proof of statement (i), set $t' = b \in \mathcal{F}$. Let the degree of $f(t)$ be $n > o$, so that

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_o$$

with $a_o, a_1, \ldots, a_n \in \mathcal{F}$, $a_n \neq 0$. Then

$$\begin{aligned}(f(t))' &= a_n' t^n + (na_n t' + a_{n-1}')t^{n-1} + \ldots + a_o' \\ &= a_n' t^n + (na_n b + a_{n-1}')t^{n-1} + \ldots + a_o' .\end{aligned}$$

This is a polynomial in $\mathcal{F}[t]$ of degree n if a_n is not constant. If a_n is constant and $na_n b + a_{n-1}' = 0$, then

$$\begin{aligned}(na_n t + a_{n-1})' &= na_n' t + na_n t' + a_{n-1}' \\ &= na_n b + a_{n-1}' \\ &= 0.\end{aligned}$$

Since $a_n' = 0$, so $na_n t + a_{n-1}$ is a constant and hence is in $\mathcal{F}$. This implies that $t \in \mathcal{F}$, contrary to the assumption that t is transcendental over $\mathcal{F}$. Thus if a_n is constant, $(f(t))'$ has degree n-1.

In order to prove (ii), set $\frac{t'}{t} = d \in \mathcal{F}$. Let $a \in \mathcal{F}$, $a \neq 0$ and let n be a nonzero integer then

$$(at^n)' = a't^n + na\, t^{n-1}t' = a't^n + na\frac{t'}{t}t^n$$

$$= (a' + nad)t^n .$$

If $a' + nad = 0$, then at^n is a constant, therefore an element of $\mathcal{F}$, contradicting the fact that t is transcendental over $\mathcal{F}$. Therefore $(a' + nad) \neq 0$. Hence $(f(t))'$ has the same degree as f(t).

We now prove that if $(f(t))'$ is a multiple of f(t), then f(t) is a monomial. To prove this, suppose that f(t) is not a monomial, let $a_n t^n$ and $a_m t^m$ be two of its non-zero terms, $n > m \geqslant 0$. We have

$$(a_n t^n)' = (a_n' + na_n d)t^n$$

$$(a_m t^m)' = (a_m' + ma_m d)t^m .$$

We already know that $(f(t))'$ has the same degree as that of f(t), so since $(f(t))'$ is a multiple of f(t) we must have that

$$(f(t))' = g\, f(t).$$

with $g \in \mathcal{F}$, $g \neq 0$. Therefore,

$a_n' + na_n d = g\, a_n$ and $a_m' + ma_m d = g\, a_m$, implying that

$$\frac{a_n' + na_n d}{a_n} = \frac{a_m' + ma_m d}{a_m}$$

which gives

$$\frac{a_n'}{a_n} + \frac{nt'}{t} = \frac{a_m'}{a_m} + \frac{mt'}{t},$$

and thus

$$\frac{a_n'}{a_n} - \frac{a_m'}{a_m} + \frac{nt'}{t} - \frac{mt'}{t} - = 0.$$

By the "logarithmic derivative identity",

$$\frac{(a_n a_m^{-1} t^n t^{-m})'}{a_n a_m^{-1} t^n t^{-m}} = 0.$$

which gives us $\left(\frac{a_n t^n}{a_m t^m}\right)' = 0$. This implies that $\frac{a_n t^n}{a_m t^m}$ is a constant and

hence is in $\mathcal{F}$, again contradicting the fact that t is transcendental over $\mathcal{F}$. This completes the proof.

<u>Example 4-1</u> Let $\mathcal{F} = \mathbb{R}(x)$, $t = \log x$ with $t' = \frac{1}{x} \in \mathcal{F}$. t is transcendental over $\mathcal{F}$; otherwise there exists a minimal polynomial $p(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_o \in \mathcal{F}[X]$ such that log x is a root, that is

$$(\log x)^n + a_{n-1} (\log x)^{n-1} + \ldots + a_o = 0.$$

Differentiating this to get

$$n(\log x)^{n-1} \cdot \frac{1}{x} + a_{n-1}'(\log x)^{n-1} + a_{n-1}(n-1)(\log x)^{n-2} \cdot \frac{1}{x} + \ldots + a_o' = 0.$$

or

$$(1) \qquad \left(\frac{n}{x} + a_{n-1}'\right)(\log x)^{n-1} + \left(\frac{(n-1)a_{n-1}}{x} + a_{n-2}'\right)(\log x)^{n-2} + \ldots + a_o' = 0.$$

We shall prove that $a_{n-1}' \neq \frac{-n}{x}$. Let $a_{n-1} = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$ and

$f(x) \in R[x]$ and they have no common factor.

$$a'_{n-1} = \frac{g(x)(f(x))' - f(x)(g(x))'}{(g(x))^2} .$$

We see that the denominator and numerator of a'_{n-1} have no common factor, thus the denominator of a'_{n-1} cannot be of degree one, that is, $a'_{n-1} \neq \frac{-n}{x}$. Hence $a'_{n-1} + \frac{n}{x} \neq 0$. Dividing both sides of (1) by $\frac{n}{x} + a'_{n-1}$ yields

$$(\log x)^{n-1} + \frac{1}{a}\left(\frac{(n-1)a_{n-1}}{x} + a'_{n-2}\right)(\log x)^{n-2} + \dots + \frac{a'_o}{a} = 0$$

where $a = a'_{n-1} + \frac{n}{x}$, therefore the polynomial

$$X^{n-1} + \frac{1}{a}\left(\frac{(n-1)a_{n-1}}{x} + a'_{n-2}\right) X^{n-2} + \dots + \frac{a'_o}{a}$$

has log x as its root, contrary to the assumption that f(X) is the minimal polynomial of log x. Hence t is transcendental over $\mathcal{F}$.

If $t = e^x$ with $t' = e^x$, then $\frac{t'}{t} \in \mathcal{F}$. t is also transcendental over $\mathcal{F}$ (The proof is given later). Let $a = (1 + x)^5$. Then $at^n = (1 + x)^5 e^{nx}$

$$\begin{aligned} (at^n)' &= ((1 + x)^5 e^{nx})' \\ &= 5x^4 e^{nx} + n(1 + x)^5 e^{nx} \\ &= (5x^4 + n(1 + x)^5)e^{nx} \\ &= (5x^4 + n(1 + x)^5)t^n \end{aligned}$$

which satisfies the result of this lemma.

<u>Definition 4-4</u> Let $\mathcal{F}$ be a differential field. By <u>an elementary extension of $\mathcal{F}$</u> we mean a differential extension field of the form $\mathcal{F}(t_1, t_2, \ldots, t_N)$ where for each $i = 1,2,\ldots,N$, the element t_i is either algebraic over the field $\mathcal{F}(t_1, t_2, \ldots, t_{i-1})$, or the logarithm or exponential of an element of $\mathcal{F}(t_1, t_2, \ldots, t_{i-1})$.

We now come to the main theory which provides the key to our investigation.

<u>Theorem 4-2</u> Let $\mathcal{F}$ be a differential field of characteristic zero and $\alpha \in \mathcal{F}$. If the equation $y' = \alpha$ has a solution in some elementary differential extension field of $\mathcal{F}$ having the same subfield of constants, then there are constants $c_1, c_2, \ldots, c_n \in \mathcal{F}$ and elements $u_1, u_2, \ldots, u_n$, v in $\mathcal{F}$ such that

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v' .$$

<u>Proof</u> Assume that the equation $y' = \alpha$ has a solution in an elementary differential extension field $\mathcal{F}(t_1, t_2, \ldots, t_N)$ of $\mathcal{F}$ having the same subfield of constants. Then there is a sequence of differential fields

$$\mathcal{F} \subset \mathcal{F}(t_1) \subset \quad \ldots \quad \subset \mathcal{F}(t_1, t_2, \ldots, t_N)$$

all with the same subfield of constants, each t_i being algebraic over $\mathcal{F}(t_1, \ldots, t_{i-1})$, or the logarithm or exponential of an element of this field.

We shall prove the theorem by induction on N. The case $N = 0$ is trivial. (Since $y' = \alpha$, $y \in \mathcal{F}$ is an expression for α of the desired form.) So assume that $N > 0$ and the theorem is true for $N - 1$.

Applying the case N - 1 to the differential fields $\mathcal{F}(t_1) \subset \mathcal{F}(t_1,\ldots,t_N)$,

therefore we can write α in the form

$$(2) \qquad \alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'$$

but with $u_1, u_2, \ldots, u_n$, v in $\mathcal{F}(t_1)$ and $c_1, \ldots, c_n \in \mathcal{F}$ (If N = 1, then $\alpha = y'$, $y \in \mathcal{F}(t_1)$. The proof of this case starts from now on.) Setting $t_1 = t$, we have t algebraic over $\mathcal{F}$, or the logarithm or exponential of an element of $\mathcal{F}$. What we now have to do is to find a similar expression for α, possibly with a different n, but with the elements $u_1, \ldots, u_n$, v belonging to $\mathcal{F}$.

First suppose that t is algebraic over $\mathcal{F}$. Then, by theorem 1-12 there are polynomials $U_1(X), \ldots, U_n(X)$, $V(X) \in \mathcal{F}[X]$ such that

$$(3) \qquad U_1(t) = u_1,\ U_2(t) = u_2, \ldots, U_n(t) = u_n,\ V(t) = v.$$

Let the distinct conjugates of t over $\mathcal{F}$ in the algebraic closure of $\mathcal{F}(t)$ be τ_1 (= t), $\tau_2, \ldots, \tau_s$. According to (2) and (3), we obtain

$$\alpha = \sum_{i=1}^{n} c_i \frac{(U_i(t))'}{U_i(t)} + (V(t))' .$$

Multiplying this equation by $U_1(t)U_2(t)\ldots\ldots U_n(t)$, gives us

$$\sum_{i=1}^{n} c_i\, U_1(t)\ldots(U_i(t))'\ldots U_n(t) + ((V(t))' - \alpha)U_1(t)U_2(t)\ldots U_n(t) = 0.$$

Since $U_i(X), (U_i(X))'$ and $(V(X))'$ are polynomials in $\mathcal{F}[X]$ and $c_1, \ldots, c_n$, $\alpha \in \mathcal{F}$, it follows that

$$\sum_{i=1}^{n} c_i\, U_1(X)\ldots(U_i(X))'\ldots U_n(X) + ((V(X))' - \alpha)U_1(X)\ldots U_n(X)$$

is a polynomial in $\mathcal{F}[X]$ such that t is a root. As $\tau_2, \ldots, \tau_s$ are conjugates of t over $\mathcal{F}$, we must have that $\tau_2, \ldots, \tau_s$ are also the roots of this polynomial, then we get

$$\sum_{i=1}^{n} c_i\, U_1(\tau_j)\ldots(U_i(\tau_j))'\ldots U_n(\tau_j) + ((V(\tau_j))' - \alpha\,)U_1(\tau_j)U_2(\tau_j)\ldots U_n(\tau_j)$$

$$= 0$$

for every $j = 1,2,\ldots, s$. $((U_i(\tau_j))'$ makes sense for each $j = 1,2,\ldots, s$ $i = 1,2,\ldots, n$ since the derivation on $\mathcal{F}(t)$ can be extended to a derivation on $\mathcal{F}(\tau_1, \tau_2, \ldots, \tau_s))$. So

$$\alpha = \sum_{i=1}^{n} c_i \frac{(U_i(\tau_1))'}{U_i(\tau_1)} + (V(\tau_1))'$$

$$\alpha = \sum_{i=1}^{n} c_i \frac{(U_i(\tau_2))'}{U_i(\tau_2)} + (V(\tau_2))'$$

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$$\alpha = \sum_{i=1}^{n} c_i \frac{(U_i(\tau_s))'}{U_i(\tau_s)} + (V(\tau_s))' .$$

Adding all these equations yields

$$s\,\alpha = \sum_{j=1}^{s} c_1 \frac{(U_1(\tau_j))'}{U_1(\tau_j)} + \sum_{j=1}^{s} c_2 \frac{(U_2(\tau_j))'}{U_2(\tau_j)} + \ldots + \sum_{j=1}^{s} c_n \frac{(U_n(\tau_j))'}{U_n(\tau_j)} .$$

$$+ \sum_{j=1}^{s} (V(\tau_j))' .$$

Using the "logarithm derivative identity", we get

$$\sum_{j=1}^{s} c_i \frac{(U_i(\tau_j))'}{U_i(\tau_j)} = \frac{c_i(U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_s))'}{U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_s)}$$

for every $i = 1,2,\ldots, n$. Hence we can write α in the form

$$\alpha = \sum_{i=1}^{n} \frac{c_i}{s} \frac{(U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_s))'}{U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_s)} + \frac{(V(\tau_1) +\ldots+ V(\tau_s))'}{s}.$$

Since $U_i(\tau_1)U_i(\tau_2)\ldots U_i(\tau_s)$ and $V(\tau_1) + \ldots V(\tau_s)$ are symmetric polynomials in $\tau_1, \tau_2,\ldots, \tau_s$ with coefficients in $\mathcal{F}$, by theorem 1-10, each of these expressions is in $\mathcal{F}$. Hence the last equation is an expression for α of the required form.

In the remaining cases, where t is logarithm or exponential of an element of $\mathcal{F}$, we may assume that t is transcendental over $\mathcal{F}$. Then we have

$$\alpha = \sum_{i=1}^{n} c_i \frac{(u_i(t))'}{u_i(t)} + (v(t))'$$

with $u_1(t),\ldots, u_n(t), v(t) \in \mathcal{F}(t)$. We can assume that $u_1(t),\ldots,$ $u_n(t)$, $v(t)$, are distinct and none of the elements $c_1,\ldots, c_n$ are zero. Each $u_i(t)$ can be written as

$$u_i(t) = \frac{a\, p_1^{r_1}(t) \ldots p_k^{r_k}(t)}{q_1^{s_1}(t) \ldots q_m^{s_m}(t)}$$

where $p_i(t)$, $q_j(t)$ are monic irreducible polynomials in $\mathcal{F}[t]$, r_i, s_j are positive integers and $a \in \mathcal{F}$. Differentiating this equation yields

$$(u_i(t))' = \frac{(q_1^{s_1}(t)\dots q_m^{s_m}(t))(ap_1^{r_1}(t)\dots p_k^{r_k}(t))' - (ap_1^{r_1}(t)\dots p_k^{r_k}(t))(q_1^{s_1}(t)\dots q_m^{s_m}(t))'}{(q_1^{s_1}(t)\dots q_m^{s_m}(t))^2}$$

Dividing this by $u_i(t)$, we obtain

$$\frac{(u_i(t))'}{u_i(t)} = \frac{(ap_1^{r_1}(t)\dots p_k^{r_k}(t))'}{ap_1^{r_1}(t)\dots p_k^{r_k}(t)} - \frac{(q_1^{s_1}(t)\dots q_m^{s_m}(t))'}{q_1^{s_1}(t)\dots q_m^{s_m}(t)}.$$

The "logarithmic derivative identity", gives

$$\frac{(u_i(t))'}{u_i(t)} = \frac{a'}{a} + r_1\frac{(p_1(t))'}{p_1(t)} + \dots + r_k\frac{(p_k(t))'}{p_k(t)} - s_1\frac{(q_1(t))'}{q_1(t)} - \dots - s_m\frac{(q_m(t))'}{q_m(t)}.$$

Consequently, we rewrite $\sum c_i \frac{(u_i(t))'}{u_i(t)}$ in a similar form, but with each $u_i(t)$ either in $\mathcal{F}$ or a monic irreducible of $\mathcal{F}[t]$. We now consider $v(t)$. The partial fraction decomposition of $v(t)$ allows us to express $v(t)$ as the sum of an element of $\mathcal{F}[t]$ plus various terms of the form $\frac{g(t)}{(f(t))^r}$, where $f(t)$ is a monic irreducible, r a positive integer, and $g(t)$ is a non-zero element of $\mathcal{F}[t]$ of degree less than that of $f(t)$. What we must do now is to show that each of $u_1, \dots, u_n$, v does not involve t, that is

$$\alpha = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'$$

where $c_i, u_i, v \in \mathcal{F}$, $i = 1,2,\dots, n$.

To start with, suppose first that t is the logarithm of an element of $\mathcal{F}$, so that $t' = \frac{d'}{d}$ for some $d \in \mathcal{F}$. Let $f(t)$ be an arbitrary monic irreducible element of $\mathcal{F}[t]$. Then $(f(t))'$ is also in $\mathcal{F}[t]$ and it has degree less than that of $f(t)$, so that $f(t)$ does not divide $(f(t))'$. Thus if

$u_i(t) = f(t)$, then $\dfrac{(u_i(t))'}{u_i(t)}$ is already in lowest terms, with denominator $f(t)$. If $\dfrac{g(t)}{(f(t))^r}$ appears in the partial fraction expression for $v(t)$ where $g(t) \in \mathcal{F}[t]$ is of degree less than that of $f(t)$ and $r > o$ is the maximum exponent of $f(t)$, then $(v(t))'$ will consist of

$$\left(\frac{g(t)}{(f(t))^r}\right)' = \frac{(g(t))'}{(f(t))^r} - r\,\frac{g(t)(f(t))'}{(f(t))^{r+1}}$$

and various terms having $(f(t))^k$ in the denominator for $k \leq r$. Since $f(t)$ does not divide $g(t)(f(t))'$ (otherwise, $f(t)$ divides $g(t)$ or $(f(t))'$). We see that a term with denominator $(f(t))^{r+1}$ actually appears in $(v(t))'$. Hence if $f(t)$ appears as a denominator in the partial fraction expression of $f(t)$, it will appear in α , which is impossible. Therefore, $f(t)$ does not appear in the denominator of $v(t)$. So $f(t)$ cannot be one of the $u_i(t)$'s either. We already know that each $u_i(t)$ is either in $\mathcal{F}$ or a monic irreducible element of $\mathcal{F}[t]$. This implies that each $u_i(t) \in \mathcal{F}$ and $v(t) \in \mathcal{F}[t]$. Consequently,

$$(v(t))' = \alpha - \sum_{i=1}^{n} c_i \frac{u_i'}{u_i}$$

belongs to $\mathcal{F}$. Since $(v(t))'$ is of the same degree as $v(t)$ or degree one less, according as the highest coefficient of $v(t)$ is, or is not a constant, by lemma 4-3 (i) so $v(t)$ is of the form

$$v(t) = ct + h$$

where c is a constant and $h \in \mathcal{F}$. Thus

$$\begin{aligned}\alpha &= \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + (ct + h)' \\ &= \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + c\frac{d'}{d} + h'\end{aligned}$$

is an expression for α of the desired form.

Finally, consider the case where t is the exponential of an element of $\mathcal{F}$, say $\frac{t'}{t} = b'$ with $b \in \mathcal{F}$. By virtue of lemma 4-3 (ii), if f(t) is a monic irreducible element of $\mathcal{F}[t]$ other than t itself, then $(f(t))' \in \mathcal{F}[t]$ and f(t) does not divide $(f(t))'$. By the same reasoning as above, we have that f(t) cannot occur in the denominator of v(t), nor can any $u_i(t)$ equal f(t). This asserts that v(t) can be written as

$$v(t) = \Sigma a_j t^j$$

where j ranges over a finite set of integers, $a_j \in \mathcal{F}$, and each of quantities $U_1(t),\ldots, U_n(t)$ is in $\mathcal{F}$ or equal to t itself. However, if some $u_i(t) = t$, we have

$$\frac{(u_i(t))'}{u_i(t)} = \frac{t'}{t} = b'$$

which is in $\mathcal{F}$. Consequently, $(v(t))' \in \mathcal{F}$. According to lemma 4-3(ii) v(t) and $(v(t))'$ have the same degree, hence v(t) is also in $\mathcal{F}$. If each $u_i(t)$ is in $\mathcal{F}$, we already have α in the required form, and we are done. If not, we may thus limit ourselves to the case that only one $u_i(t)$, say $u_1(t)$ is not in $\mathcal{F}$. Then $u_1(t) = t$ and $u_2(t),\ldots, u_n(t) \in \mathcal{F}$, so we can write

$$\alpha = c_1 \frac{t'}{t} + \sum_{i=2}^{n} c_i \frac{u_i'}{u_i} + v'$$

$$= \sum_{i=2}^{n} c_i \frac{u_i'}{u_i} + c_1 b' + v'$$

$$= \sum_{i=2}^{n} c_i \frac{u_i'}{u_i} + (c_1 b + v)' ,$$

with $u_2, \ldots, u_n$, $c_1 b + v$ all in $\mathcal{F}$. This completes the proof of the theorem.

<u>Remark</u> The condition that $\mathcal{F}$ and its elementary differential extension field have the same subfield of constants is essential. This can be seen from the following example.

<u>Example 4-2</u> Let $\mathcal{F} = \mathbb{R}(x)$ with $x' = 1$ and $\alpha = \frac{1}{1+x^2} \in \mathbb{R}(x)$. Then the equation $y' = \alpha = \frac{1}{1+x^2}$ has a solution in the elementary differential extension field $\mathbb{R}(i, x, \log(x + iy)) = \mathbb{C}(z, \log z)$ of $\mathcal{F}$. We see that $\mathbb{C}(z, \log z)$ has a different subfield of constants from that of $\mathbb{R}(x)$, since the subfield of constants of $\mathbb{C}(z, \log z)$ is $\mathbb{C}$, whereas $\mathbb{R}(x)$ has $\mathbb{R}$ for its subfield of constants. We claim that $\frac{1}{1+x^2}$ cannot be written in the form

$$\frac{1}{1+x^2} = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'$$

where $c_1, \ldots, c_n \in \mathbb{R}$ and $u_1, \ldots, u_n, v \in \mathbb{R}(x)$. To see this, suppose that $\frac{1}{1+x^2}$ can be written in this form. Let

$$u_i = (1 + x^2)^{r_i} Q_i(x)$$

where each r_i is an integer and $1 + x^2$ is not a factor of both numerator and denominator of $Q_i(x)$ in $\mathbb{R}(x)$ $i = 1,2,\ldots, n$. We have

$$\frac{u_i'}{u_i} = \frac{2r_i x}{1+x^2} + \frac{(Q_i(x))'}{Q_i(x)}$$

Therefore,

$$\frac{1}{1+x^2} = \sum_{i=1}^{n} \frac{2rc_i x}{1+x^2} + \sum_{i=1}^{n} c_i \frac{(Q_i(x))'}{Q_i(x)} + v' ,$$

that is,

$$(4) \qquad \frac{1 - \sum_{i=1}^{n} 2rc_i x}{1+x^2} = \sum_{i=1}^{n} \frac{c_i (Q_i(x))'}{Q_i(x)} + v' .$$

It is clear that $1 + x^2$ does not appear in the denominator of $\sum_{i=1}^{n} \frac{c_i(Q_i(x))'}{Q_i(x)}$. Now we consider $v \in \mathbb{R}(x)$. If $1 + x^2$ does not appear in the denominator of v, then the expression on the right hand side of (4) has no $1 + x^2$ in its denominator. This implies that $1 + x^2$ must divide $1 - \sum_{i=1}^{n} 2r_i c_i x$ which is impossible.

If $1 + x^2$ occurs in the denominator of v, it will occur at least twice in the denominator of v', then v' will be balanced neither by $\frac{1 - \sum_{i=1}^{n} 2r_i c_i x}{1+x^2}$ nor by $\sum_{i=1}^{n} \frac{c_i(Q_i(x))'}{Q_i(x)}$. This proves our claim.

From this example if we replaces $\mathbb{R}(x)$ by $\mathbb{C}(z)$, with $z' = 1$, the equation $y' = \frac{1}{1+z^2} \in \mathbb{C}(z)$ has a solution in the elementary differential extension field $\mathbb{C}(z, nz)$. It is not difficult to verify that $\frac{1}{1+z^2}$ can be written in the form

$$\frac{1}{1+z^2} = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'$$

with $c_1, \ldots, c_n \in \mathbb{C}$ and $u_1, \ldots, u_n, v \in \mathbb{C}(z)$. To do this, note that

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i} \cdot \frac{1}{z-i} - \frac{1}{2i} \cdot \frac{1}{z+i}$$

$$= \frac{i}{2} \frac{(z+i)'}{z+i} - \frac{i}{2} \frac{(z-i)'}{z-i}.$$

We see that $\frac{1}{1+z^2}$ can be written in the required form.

In the remainder of this chapter we shall find a simple device which tells us whether or not a function in the elementary differential extension of $\mathbb{C}(z)$ is a nonelementary indefinite integral. The preliminary lemma below provides the key to establishing the needed theorem.

Lemma 4-4 If g is a nonconstant element of $\mathbb{C}(z)$, then e^g is not algebraic over $\mathbb{C}(z)$.

Proof Assume that e^g is algebraic over $\mathbb{C}(z)$, then there is a minimal polynomial $P(X) = X^n + a_1 X^{n-1} + \ldots a_n$ in $\mathbb{C}(z)[X]$ such that e^g is a root, thus

$$e^{ng} + a_1 e^{(n-1)g} + \ldots + a_n = 0.$$

Differentiating this to get

$$ng' e^{ng} + (a_1' + (n-1)a_1 g')e^{(n-1)g} + \ldots + a_n' = 0.$$

Since $g' \neq 0$, dividing this by ng' yields

$$e^{ng} + \frac{(a_1' + (n-1)a_1 g')}{ng'} e^{(n-1)g} + \ldots + \frac{a_n'}{ng'} = 0,$$

therefore, the polynomial

$$X^n + \frac{(a_1' + (n-1)a_1 g')}{ng'} X^{n-1} + \ldots + \frac{a_n'}{ng'}$$

has e^g as its root, then by theorem 1-12

$$X^n + a_1 X^{n-1} + \ldots + a_n = X^n + \frac{(a_1' + (n-1)a_1 g')}{ng'} X^{n-1} + \ldots + \frac{a_n'}{ng'} .$$

This implies that

$$a_n = \frac{a_n'}{ng'} \quad \text{or} \quad ng' = \frac{a_n'}{a_n} .$$

$\frac{a_n'}{a_n}$ may be either zero or non-zero element of $\mathbb{C}(z)$, we now suppose that

$\frac{a_n'}{a_n} \neq 0$, let $a_n = \frac{r(z)}{g(z)}$ where $0 \neq g(z)$ and $r(z) \in \mathbb{C}[z]$. By the Fundamental theorem of algebra, we can write a_n as

$$a_n = \frac{c\,(z - c_1)^{r_1}(z - c_2)^{r_2} \ldots (z - c_n)^{r_n}}{(z - d_1)^{k_1}(z - d_2)^{k_2} \ldots (z - d_m)^{k_m}}$$

where $c, c_1, \ldots, c_n, d_1, \ldots, d_m \in \mathbb{C}$ and r_i, k_j are positive integers. By the "logarithmic derivative identity", we get

$$\frac{a_n'}{a_n} = \frac{cr_1(z-c_1)'}{z - c_1} + \ldots + \frac{cr_n(z-c_n)'}{z - c_n} - \frac{ck_1(z-d_1)'}{z - d_1} - \ldots - \frac{ck_m(z-d_m)'}{z - d_m}$$

$$= \frac{cr_1}{z-c_1} + \ldots + \frac{cr_n}{z-c_n} - \frac{ck_1}{z-d_1} - \ldots - \frac{ck_m}{z-d_m} .$$

Thus $\frac{a_n'}{a_n}$ is a sum of fractions with constant numerators and linear denominators, whereas ng' can have no linear denominator since if $f(z)$ occurs in the denominator of ng, it will occur at least twice in the denominator of ng'. This yields $\frac{a_n'}{a_n} = ng' = 0$, contradicting the assumption that g is nonconstant. This proves the lemma.

As an application of the foregoing ideas, we now come to the theorem which provides a convenient criterion to tell us whether a certain function is a nonelementary indefinite integral or not.

Theorem 4-3 Let $f(z)$ be a nonzero element of $\mathbb{C}(z)$ and $g(z)$ a nonconstant element of $\mathbb{C}(z)$, then $\int f(z)e^{g(z)}dz$ is elementary if and only if there exists $a(z) \in \mathbb{C}(z)$ such that

$$f = a' + ag'$$

Proof Setting $e^g = t$, we have $\frac{t'}{t} = g'$, by the preceeding lemma, t is transcendental over $\mathbb{C}(z)$. Now let $\mathcal{F} = \mathbb{C}(z)$ and $\mathcal{F}(t) = \mathbb{C}(z,t)$. Suppose first that $\int f(z)e^{g(z)}dz$ is elementary, in other words, the equation $y' = fe^g \in \mathcal{F}(t)$ has a solution in some elementary differential extension field of $\mathcal{F}(t)$, it then follows from theorem 4-2 that we can write

$$tf = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'$$

with $c_1, \ldots, c_n \in \mathbb{C}$ and $u_1, \ldots, u_n, v \in \mathcal{F}(t)$. By factoring numerator and denominator of each u_i as a product of irreducible elements of $\mathcal{F}[t]$ and using the "logarithmic derivative identity", if necessary, we can guarantee that the u_i's which are not in $\mathcal{F}$ are distinct monic irreducible elements of $\mathcal{F}[t]$. We decompose v into partial fractions with respect to $\mathcal{F}[t]$. By the same reasoning as used in the proof of theorem 4-2, we have that the only possible monic irreducible factor of a denominator in v is t, which is also the only possible u_i not in $\mathcal{F}$. That is, v is of the form

$$v = \Sigma b_j t^j$$

where j ranges over a finite set of integers and each $b_j \in \mathcal{F}$. If $u_i = t$ for some i, $\frac{u_i'}{u_i} = \frac{t'}{t} = g' \in \mathcal{F}$, then $\sum c_i \frac{u_i'}{u_i} \in \mathcal{F}$. We let $\sum c_i \frac{u_i'}{u_i}$ = d for some $d \in \mathcal{F}$. Therefore

$$tf = d + v'$$

or

$$v' = tf - d.$$

Lemma 4-3 (ii) implies that v is a polynomial in $\mathcal{F}[t]$ of the same degree as that of v', thus there exist $a,b \in \mathcal{F}$ such that

$$v = at + b.$$

Differentiating this, we get

$$v' = a't + at' + b'$$

$$= (a' + ag')t + b' .$$

This implies that $f = a' + ag'$.

As regards the converse, if there is an $a \in C(z)$ such that

$$f = a' + ag' .$$

Multiplying this by t, we obtain

$$tf = a't + atg' = (at)' .$$

Thus one elementary integral of $\int fe^g$ is ae^g . This completes the proof of the theorem.

Now, at last, we can illustrate the use of these ideas with several examples.

Example 4-3 Consider $\int e^{z^2}\,dz$, if this integral is elementary then there is an $a \neq 0 \in \mathbb{C}(z)$ such that

$$1 = a' + 2az ,$$

or

$$2z = \frac{1}{a} - \frac{a'}{a} .$$

Let $a = \frac{P(z)}{Q(z)}$ where $0 \neq Q(z)$ and $P(z)$ are in $\mathbb{C}[z]$, we have

$$2z = \frac{Q(z)}{P(z)} - \frac{(P(z))'}{P(z)} + \frac{(Q(z))'}{Q(z)} .$$

We can write

$$\frac{Q(z)}{P(z)} = A(z) + \frac{R(z)}{P(z)}$$

with degree $R(z) <$ degree $P(z)$, so

$$(5) \qquad 2z = A(z) + \frac{R(z)}{P(z)} - \frac{(P(z))'}{P(z)} + \frac{(Q(z))'}{Q(z)} .$$

We see that, since $P(z)$ and $Q(z)$ do not divide $(P(z))'$ and $(Q(z))'$ respectively. $P(z)$ and $Q(z)$ occur in the denominator of the expression in the right hand side of (5) which is not balanced by $2z$. We can thus assert that $\int e^{z^2}\,dz$ is a nonelementary indefinite integral.

Example 4-4 For $\int \frac{e^z}{z}\,dz$, if this integral is elementary, then there exists $a \in \mathbb{C}(z)$ such that

$$\frac{1}{z} = a' + a.$$

Assume first that z occurs in the denominator of a, then it occurs at least twice in the denominator of a' and $a' + a$ which is not balanced by $\frac{1}{z}$. If z does not occur in the denominator of a, then it is clear that z does not occur in the denominator of both a' and $a + a'$. This case is impossible. Thus, in either case this equation has no solution. Hence $\int \frac{e^z}{z} dz$ is not an elementary indefinite integral.

As an immediate cousequence of the above examples, we get that $\int e^{e^z} dz$ and $\int \frac{1}{\log z} dz$ are nonelementary. The first assertion proved by replacing z with e^z and the second integral by replacing z with log z.

For $\int \log \log z \, dz$, we use integration by parts to reduce this integral to the previous integral, i.e.

$$\int \log \log z \, dz = z \log \log z - \int \frac{dz}{\log z} .$$

Since $\int \frac{dz}{\log z}$ is nonelementary, this implies that $\int \log \log z \, dz$ is also nonelementary.

Example 4-5 As another application of these ideas, consider $\int \frac{\sin z}{z} dz$. As we know, $\text{Sin } z = \frac{e^{iz} - e^{-iz}}{2i}$. We first change variables iz = w, thus

$$\int \frac{\text{Sin } z}{z} dz = \int \frac{e^{w} - e^{-w}}{2w} dw .$$

Let $t = e^w$, $\mathbb{C}(w) = \mathcal{F}$. Working in $\mathcal{F}(t)$, if $\int \frac{e^w - e^{-w}}{w} dw$ is elementary, by theorem 4-2, we can write

$$\frac{t^2 - 1}{tw} = \sum_{i=1}^{n} c_i \frac{u_i'}{u_i} + v'$$

with $c_1, c_2, \ldots, c_n \in \mathbb{C}$ and $u_1, \ldots, u_n, v \in \mathcal{F}(t)$. By the same reasoning as before we have that the u_i's which are not in $\mathcal{F}$ are destinct monic irreducible elements of $\mathcal{F}[t]$, and the only possible u_i not in $\mathcal{F}$ is t, and the only possible monic irreducible factor of a denominator in v is t, that is, we can write $v = \Sigma b_j t^j$, where j ranges over a finite set of integers, $b_j \in \mathcal{F}$. If $u_i = t$, then $\frac{u_i'}{u_i} = \frac{t'}{t} = 1 \in \mathcal{F}$. Hence $\sum c_i \frac{u_i'}{u_i} \in \mathcal{F}$, let $\sum c_i \frac{u_i'}{u_i} = d$ for some $d \in \mathcal{F}$. Differentiating v yields

$$v' = \sum t^j (j\, b_j + b_j') .$$

Therefore

$$\frac{t^2 - 1}{tw} = d + \sum t^j (j\, b_j + b_j') ,$$

or

$$\frac{t^2 - 1}{w} = dt + \sum t^{j+1} (j\, b_j + b_j') .$$

We thus see that j must be -1,0 and 1. That is,

$$\frac{t^2 - 1}{w} = dt + (-b_{-1} + b'_{-1}) + b_o' t + (b_1 + b_1')t^2 ,$$

hence $d + b_o' = 0$, $b_1 + b_1' = \frac{1}{w}$, $-b_{-1} + b_{-1}' = \frac{-1}{w}$.

But we have proven that the equation

$$b_1 + b_1' = \frac{1}{w}$$

has no solution. Therefore $\int \frac{\text{Sin } z}{z} dz$ is not elementary.

As an immediate consequence of this example, we have that $\int \mathrm{Sin}\ e^z\, dz$ is also nonelementary. To see this, let $e^z = w$, then

$$\int \mathrm{Sin}\ e^z\, dz = \int \frac{\mathrm{Sin}\ w}{w}\, dw .$$

Since $\int \frac{\mathrm{Sin}\ w}{w}\, dw$ is nonelementary, (see example 4-5) hence $\int \mathrm{Sin}\ e^z\, dz$ is also nonelementary.

Example 4-6 For $\int e^{\mathrm{Sin}\ z}\, dz$, since $\mathrm{Sin}\ z = \frac{e^{iz} - e^{-iz}}{2i}$, let $t = \frac{e^{iz}}{2i}$, then we have

$$\int e^{\left(\frac{e^{iz} - e^{-iz}}{2i}\right)}\, dz = -i \int \frac{e^{t+1/4t}}{t}\, dt .$$

We claim that $\int \frac{e^{z+\frac{1}{4z}}}{z}\, dz$ is nonelementary. To do this, assume that this integral in elementary, by theorem 4-3 we have that there exists an $a \in \mathbb{C}(z)$ such that

$$\frac{1}{z} = a' + a\left(1 - \frac{1}{4z^2}\right)$$

or

$$\text{(6)} \qquad \frac{a'}{a} = \frac{1}{az} + \frac{1}{4z^2} - 1.$$

It is clear that the expression in the right hand side of (6) cannot be written as a sum of fractions with constant numerators and linear denominators, whereas $\frac{a'}{a}$(see the proof of lemma 4-4) is a sum of fractions with constant numerators and linear denominators. This is a contradiction. Hence $\int e^{\sin z}\, dz$ is a nonelementary indefinite integral.

APPENDIX

Definition Let $n_1, n_2, \ldots, n_r$ be non-negative integers such that $n_1 + n_2 + \ldots + n_r = n$. Then the expression $\binom{n}{n_1, n_2, \ldots, n_r}$ will denote $\frac{n!}{n_1! n_2! \ldots n_r!}$.

These numbers are called the multinomial coefficients in view of the following theorem which generalizes the Binomial theorem.

Theorem

$$(a_1 + a_2 + \ldots + a_r)^n = \sum_{n_1+n_2+\ldots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} a_1^{n_1} a_2^{n_2} \ldots a_r^{n_r}$$

Proof The proof is by induction on r. It is trivial for $r = 1$. Assume that the theorem is true for r. Let $a_1 + a_2 + \ldots + a_r = b$. Then, by the Binomial theorem, we have

$$(a_1 + a_2 + \ldots + a_r + a_{r+1})^n = (b + a_{r+1})^n$$

$$= \sum_{k=o}^{n} \binom{n}{k} a_{r+1}^{n-k} b^k$$

$$= \sum_{k=o}^{n} \binom{n}{k} a_{r+1}^{n-k} (a_1 + a_2 + \ldots + a_r)^k .$$

By the induction hypothesis

$$(a_1 + a_2 + \ldots + a_r + a_{r+1})^n = \sum_{k=o}^{n} \binom{n}{k} a_{r+1}^{n-k} \sum_{n_1+n_2+\ldots+n_r=k} \binom{k}{n_1, n_2, \ldots, n_r} a_1^{n_1} a_2^{n_2} \ldots a_r^{n_r} .$$

$$= \sum_{k=o}^{n} \sum_{n_1+\ldots+n_r=k} \binom{n}{k} \binom{k}{n_1, n_2, \ldots, n_r} a_1^{n_1} a_2^{n_2} \ldots a_r^{n_r} \cdot a_{r+1}^{n-k} .$$

$$\binom{n}{k} \binom{k}{n_1, n_2, \ldots, n_r} = \frac{n!}{k!(n-k)!} \quad \frac{k!}{n_1! \ldots n_r!}$$

$$= \frac{n!}{n_1! \ldots n_r! (n-k)!}$$

$$= \binom{n}{n_1, \ldots, n_r, n-k} .$$

Therefore

$$(a_1 + a_2 + \ldots a_{r+1})^n = \sum_{k=o}^{n} \sum_{n_1+\ldots+n_r=k} \binom{n}{n_1, \ldots, n_r, n-k} a_1^{n_1} \ldots a_r^{n_r} a_{r+1}^{n-k} .$$

Let $n_{r+1} = n-k$, then n_{r+1} ranges from 0 to n and hence

$$(a_1 + a_2 + \ldots + a_{r+1})^n = \sum_{n_1+\ldots n_{r+1}=n} \binom{n}{n_1, \ldots, n_r, n_{r+1}} a_1^{n_1} \ldots a_r^{n_r} a_{r+1}^{n_{r+1}} .$$

This completes the proof.