CHAPTER I

PRELIMINARIES

In this chapter we collect some definitions and theorems for later chapters of this thesis. However, we will not prove the theorems that can be found in references [1], [2], [3] and [4].

Semicontinuous Functions

1.1 <u>Definition</u>. Let u be an extended real-valued function with domain $D \subset R^n$. For each $y \in R^n$ let $\mathcal{N}_y$ be the collection of neighborhood of y. If x is any point of $\bar{D}$, we define

$$\liminf_{y \to x} u(y) = \sup_{U \in \mathcal{N}_x} \{\inf_{y \in U \cap D} u(y)\}$$

$$\limsup_{y \to x} u(y) = \inf_{U \in \mathcal{N}_x} \{\sup_{y \in U \cap D} u(y)\} .$$

We can show that $\liminf_{y \to x} u(y) \leq u(x) \leq \limsup_{y \to x} u(y)$.

1.2 <u>Definition</u>. The function u is <u>upper</u> <u>semicontinuous</u> at $x \in D$ if $u(x) = \limsup_{y \to x} u(y)$ and <u>lower</u> <u>semicontinuous</u> at $x \in D$ if $u(x) = \liminf_{y \to x} u(y)$.

1.3 <u>Definition</u>. The function u is upper semicontinuous on D if u is upper semicontinuous at each point of D and lower semicontinuous on D if it is lower semicontinuous at each point of D.

We can conclude that u is upper semicontinuous at x if $\lim_{y \to x} \sup u(y) \leq u(x)$. Likewise, u is lower semicontinuous at x if $\lim_{y \to x} \inf u(y) \geq u(x)$.

1.4 <u>Definition</u>. The <u>support</u> of an extended real-valued function u with domain $D \subset R^n$ is defined by

$$\text{support of } u = \text{closure of } \{x \in D \mid u(x) \neq 0\}.$$

The support of u is compact, then u is said to be of <u>compact support</u>.

We now turn to some elementary properties of semicontinuous functions :

(i) If u and v are upper semicontinuous on D and $c \in R$, then cu is upper semicontinuous or lower semicontinuous on D according as $c \geq 0$ or $c \leq 0$. In particular, -u is lower semicontinuous if u is upper semicontinuous. The function max (u, v) is upper semicontinuous on D and the function u + v is upper semicontinuous on D if it is defined on D.

(ii) A function u is upper semicontinuous at x_o if and only if for each $\alpha \in [-\infty, \infty]$ such that $\alpha > u(x_o)$ then there exists a neighborhood V of x_o such that $\alpha > u(x)$ for all $x \in V$.

(iii) A necessary and sufficient condition that u be upper semicontinuous on D that is, for each $c \in R$, the set $\{x \mid u(x) < c\} \cap D$ is relatively open in D.

(iv) If $\mathcal{U}$ is a nonempty collection of upper semicontinuous functions with common domain D, then $u^* = \inf_{u \in \mathcal{U}} u$ is upper semicontinuous on D.

(v) If u is any extended real-valued function on $D \subset R^n$, we define for $x \in \bar{D}$ by

$$\hat{u}(x) = \lim_{y \to x} \sup u(y).$$

Then $\hat{u}$ is upper semicontinuous on $\bar{D}$.

(vi) If u is upper semicontinuous on the compact set D, then u attains its maximum on D.

(vii) If u is upper semicontinuous on $D \subset R^n$ and there is a continuous function f on R^n such that $u \leq f$ on D, then there is a decreasing sequence of continuous functions $\{f_j\}$ on R^n such that $\lim_{j \to \infty} f_j = u$ on D.

Note that a function which is both lower semicontinuous and upper semicontinuous at x is continuous there.

Measure and Integral

1.5 Definition. Suppose that X is a locally compact Hausdorff space, and let $\mathcal{B}$ be the class of Borel subsets of X; that is, suppose that $\mathcal{B}$ is the smallest family of subsets of X such that

(a) $X \in \mathcal{B}$

(b) If $B \in \mathcal{B}$, then $X \setminus B \in \mathcal{B}$.

(c) If $B_i \in \mathcal{B}$ $(i = 1,2,\ldots)$, then $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}$.

(d) If $G \subset X$ is open, then $G \in \mathcal{B}$.

1.6 Definition. A mapping $\mu : \mathcal{B} \to R^+$ is said to be Radon measure on X if

(i) $\mu(B) \geq 0$ for all $B \in \mathcal{B}$

(ii) If $B_i \in \mathcal{B}$ $(i = 1,2,\ldots)$ and B_i are disjoint then
$\mu(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i)$

(iii) $\mu(B) = \inf \{\mu(G) | G \supset B$ and G is open$\}$

(iv) If $K \subset X$ is compact, then $\mu(K) < +\infty$.

1.7 Definition. A function $f : X \to R$ is said to be Borel measurable if for every $\lambda \in R$ the set $\{x \in X | f(x) > \lambda\} \in \mathcal{B}$.

It follows that semicontinuous functions are Borel measurable.

1.8 Definition. Given a measure μ, a property is said to hold almost everywhere in X if it holds in a set $X \setminus N$ where N is a set such that $\mu(N) = 0$.

For any measurable function f we define its integral with respect to a measure μ as follows :

First suppose that f is non-negative on X. Then say that the sets $\{A_k\}_{k=1,\ldots,n}$ are a partition of X if $\bigcup_{k=1}^{n} A_k = X$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A_i \in \mathcal{B}$. Let

$$S = \sum_{k=1}^{n} \inf_{x \in A_k} f(x)\mu(A_k) .$$

Then the integral of f over X with respect to μ, $\int f(x)d\mu(x)$, is defined by

$$\int f(x)d\mu(x) \quad = \quad \sup S,$$

the supremum being taken over all partitions of X. Natually, we allow the integral to have value ∞ .

Next if f may have arbitrary sign over X, we define

$$f_+(x) \; = \; \max \{f(x),0\} \; ; \; f_-(x) \; = \; -\min \{f(x),0\} \; ,$$

so that f_+, f_- are non-negative. Furthermore, f_+ and f_- are measurable, and so the integral of each with respect to μ is defined. If not both $\int f_+(x)d\mu(x)$ and $\int f_-(x)d\mu(x)$ have the value ∞ , we define the integral of f by

$$\int f(x)d\mu(x) \quad = \quad \int f_+(x)d\mu(x) - \int f_-(x)d\mu(x),$$

and if both are finite, we say that f is integrable with respect to μ.

Since now that X is a locally compact Hausdorff Space. Let $\mathcal{C}_o(X)$ denote the set of all real-valued functions each of which has compact support and is continuous on X. Then $\mathcal{C}_o(X)$ is a linear space over R.

Given any Radon measure μ on X, we define a linear functional ϕ on $\mathcal{C}_o(X)$ by

$$\phi(f) \quad = \int f(x)d\mu(x).$$

It is clear that ϕ is a positive linear functional, that is, if $f \geq 0$, then $\phi(f) \geq 0$. Thus, with each Radon measure we may associate a positive linear functional on $\mathcal{C}_o(X)$.

It is an extremely important result that the converse of this is true. Thus we have

1.9 <u>Theorem</u>. (The Riesz Representation Theorem). Let X be a locally compact Hausdorff space and let ϕ be a positive linear functional on $\mathcal{C}_0(X)$. Then there is one, and only one, Radon measure μ such that

$$\phi(f) = \int f(x)d\mu(x)$$

for all $f \in \mathcal{C}_0(x)$.

<u>Proof</u> : See [3] page 40-47.

Given two locally compact Hausdorff spaces X and Y, and given that λ and μ are Radon measure defined on X and Y respectively, the product measure $\lambda \times \mu$ on $X \times Y$ is defined in the following way:

Suppose that $K \subset X$ and $L \subset Y$ are compact and that $f(x,y)$ is continuous on $X \times Y$ and support of $f \subset K \times L$. Then it may be shown that

$$h(y) = \int f(x,y)d\lambda(x)$$

is continuous in Y and that support of $h \subset L$.

Thus we define ϕ as a linear functional on $\mathcal{C}_0(X \times Y)$ by

$$\phi(f) = \int h(y)d\mu(y).$$

Since it is a positive functional, it has, by Theorem 1.9, a measure associated with it, and this is the product measure $\lambda \times \mu$ of λ and μ. Furthermore, we have Fubini's Theorem, that

$$\int f(x,y)d(\lambda\times\mu)[(x,y)] = \int\int f(x,y)d\lambda(x)d\mu(y).$$

for any function integrable in $X \times Y$.

The Space R^n.

R^n is a locally compact Hausdorff space and, a set is compact in R^n if and only if it is closed and bounded. Among all the measures on R^n one has special importance. This is Lebesgue measure.

1.10 Definition. Given any $f \in \mathcal{C}_0(R^n)$, the Riemann integral $\int_{R^n} f(x)dx$ is well-defined. There is thus defined on $\mathcal{C}_0(R^n)$ a positive linear functional. By Theorem 1.9 this gives rise to a Radon measure, and this measure is said to be Lebesgue measure, which we usually denote by m. We shall simply denote the integral $\int f(x)dm(x)$ by $\int f(x)dx$.

By repeated application of Fubini's Theorem we may show that, for any f which is Lebesgue integrable over R^n,

$$\int f(x)dx = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(x_1, \dots, x_n)dx_1 \dots dx_n .$$

The following theorem is fundamental for any discussion of change of variable.

1.11 Theorem. Let $G \subset R^n$ be open and let $h : G \to R^n$ be a mapping which has continuous first partial derivatives. Suppose that

$$h(z) = (h_1(z), \dots, h_n(z))$$

and that $J(z) = \det[\frac{\partial h_i}{\partial z_i}(z)] \neq 0$ in G. Then the function f(x) is integrable over h(G) if and only if $f(h(z))\, J(z)$ is integrable over G and

$$\int_{h(G)} f(x)dx = \int_G f(h(z))|J(z)|dz.$$

Let, in particular

$$G = \{(\rho\ \theta_1, \ldots, \theta_{n-1}) | 0 < \rho < \infty,\ 0 < \theta_i < \pi \quad \text{and } 0 < \theta_{n-1} < 2\pi\},$$

and let $x = h(\rho, \theta_1, \ldots, \theta_{n-1})$ be given by

$$x_i = \rho \sin\theta_1 \ldots \sin\theta_{i-1} \cos\theta_i \qquad (i = 1,2,\ldots,n-1)$$

(1-1) $$x_n = \rho \sin\theta_1 \ldots \sin\theta_{n-1}.$$

Then $h(G) = R^n \setminus \{x \in R^n | \ x_{n-1} = 0 \text{ and } x_n = 0\}$ and $J(z) = \rho^{n-1} \sin^{n-2}\theta_1 \ldots$ $\sin\theta_{n-2}$. So, using Fubini's Theorem, we have

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)dx_1 \ldots dx_n = \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} \ldots \int_0^{\pi} F(\rho_1\theta_1, \ldots, \theta_{n-1})\rho^{n-1} \sin^{n-2}\theta_1$$
$$\ldots \sin\theta_{n-2} d\theta_1 \ldots d\theta_{n-1} d\rho,$$

where $F(\rho, \theta_1, \ldots, \theta_{n-1}) = f(h(z))$.

We now define two other particular measures which we shall be using constantly in later chapters.

1.12 <u>Definition</u>. If $y = (y_1, \ldots, y_n) \in R^n$ and $r > 0$, then $\partial B_{y,r}$ is the surface defined by the equation

$$(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2 = r^2$$

Given a function f defined on R^n, let the restriction of f to $\partial B_{y,r}$ be denoted by f_r, and define the function $F(\rho, \theta_1, \ldots, \theta_{n-1})$ by

$$F(\rho, \theta_1, \ldots, \theta_{n-1}) = f_r(x_1, \ldots, x_n)$$

where $\rho, \theta_1, \ldots, \theta_{n-1}$ and $x_1, \ldots, x_n$ are related by the variant of (1-1) used a moment ago.

Let ψ be defined on $\mathcal{C}_o(R^n)$ by

$$(1\text{-}2) \qquad \psi(f) = \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} F(r,\theta_1,\ldots,\theta_{n-1}) r^{n-1} \sin^{n-2}\theta_1 \cdots$$

$$\cdots \sin\theta_{n-2}\, d\theta_1 \cdots d\theta_{n-1} \, .$$

Then ψ is a positive linear functional on $\mathcal{C}_o(R^n)$, R^n is locally compact Hausdorff space and so, by Theorem 1.9, ψ determines a unique Radon measure 6 such that

$$\psi(f) = \int f(x) d\sigma(x)$$

This measure is called <u>the surface area measure</u> on $\partial B_{y,r}$ and f is called a function integrable relative to surface area measure on $\partial B_{y,r}$.

If f is a function integrable relative to surface area measure on the boundary ∂B of $B = B_{y,r}$, define

$$L(f\colon y,r) = \frac{1}{\sigma_n r^{n-1}} \int_{\partial B} f(x) d\sigma(x)$$

where σ_n is the surface area measure of the unit ball center 0 in R^n and

$$\sigma_n = \int_{\partial B_{0,1}} d\sigma(x) = \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2}\, d\theta_1 \cdots d\theta_{n-1}.$$

If f is integrable on B relative to Lebesgue measure, define

$$A(f\colon y,r) = \frac{1}{\nu_n r^n} \int_B f(x)\, dx$$

where ν_n is the volume of the unit ball center 0 in R^n and

$$\nu_n = \int_{B_{0,1}} dx = \int_0^1 \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} \rho^{n-1} \sin^{n-2}\theta_1 \cdots \sin\theta_{n-1} d\theta_1 \cdots$$

$$\cdots d\theta_{n-1}\, d\rho$$

$$= \frac{\sigma_n}{n} .$$

There is a useful relation between A(f: y,r) and L(f: y,r). We have

$$A(f\colon y,r) = \frac{1}{\nu_n r^n} \int_B f(x)dx$$

$$= \frac{1}{\nu_n r^n} \int_0^r \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} F(\rho,\theta_1,\ldots,\theta_{n-1})\rho^{n-1} \sin^{n-2}\theta_1 \cdots$$

$$\cdots \sin\theta_{n-2} d\theta_1 \cdots d\theta_{n-1} d\rho$$

$$= \frac{1}{\nu_n r^n} \int_0^r \rho^{n-1}\, \sigma_n\Big(\frac{1}{\sigma_n \rho^{n-1}} \int_0^{2\pi} \int_0^{\pi} \cdots \int_0^{\pi} F(\rho,\theta_1,\ldots,\theta_{n-1})\rho^{n-1} \sin^{n-2}\theta_1 \cdots$$

$$\cdots \sin\theta_{n-2} d\theta_1 \cdots d\theta_{n-1}\Big) d\rho .$$

Using Fubini's Theorem and (1-2), we get

$$\text{(1-3)} \quad A(f\colon y,r) = \frac{\sigma_n}{\nu_n r^n} \int_0^r \rho^{n-1} L(f\colon y,\rho) d\rho .$$

Harmonic Functions

1.13 **Definition**. A real-valued function u defined on R^n and having continuous second partial derivatives is called a harmonic function if

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$$

on R^n.

If h is harmonic on a neighborhood of the closure of the ball $B = B_{y,\rho}$, then by using Green's identity, we get

(i) the value of h at the centre of the ball is equal to the average of h over the boundary of the ball and

$$h(y) = \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B} h(z)d\sigma(z),$$

(ii) the value of h at the center of the ball is equal to the average of h on the ball itself and

$$h(y) = \frac{1}{\nu_n \rho^n} \int_B h(z)dz,$$

(iii) for $x \in B$,

$$h(x) = \frac{1}{\sigma_n \rho} \int_{\partial B} \frac{\rho^2 - \|y-x\|^2}{\|z-x\|^n} h(z)d\sigma(z).$$

If f is Borel measurable on $\partial B = \partial B_{y,\rho}$ and integrable relative to surface area, then we introduce the <u>Poisson Integral Formula</u> as follows :

$$PI(f,B)(x) = \frac{1}{\sigma_n \rho} \int_{\partial B} \frac{\rho^2 - \|y-x\|^2}{\|z-x\|^n} f(z)d\sigma(z)$$

where $x \in B$ and it has the following properties :

(i) $PI(1,B) = 1$

(ii) $PI(f,B)$ is harmonic on B.

(iii) If f is continuous on ∂B, then $PI(f,B)$ is continuous on $\bar{B}$.

1.14 <u>Theorem</u>. If h_j is a monotone increasing sequence of harmonic functions on an open connected set G, then $h(x) = \lim_{j \to \infty} h_j(x)$ is either identically $+\infty$ or harmonic on G.

<u>Proof</u> : See [2] page 33.

1.15 <u>Definition</u>. A family $\mathcal{F}$ of functions defined on G is <u>left-directed</u> if for each pair $u, v \in \mathcal{F}$ there is a $w \in \mathcal{F}$ such that $w \leq u$ and $w \leq v$. There is similar definition of <u>right-directed</u> obtained by reversing the inequalities.

1.16 <u>Theorem</u>. If $\{h_i | i \in I\}$ is a left-directed family of functions harmonic on an open connected set G, then $h = \inf_{i \in I} h_i$ is either identically $-\infty$ or harmonic on G.

Proof : See [2] page 34-35.