CHAPTER I

PREPARATION

This chapter gives all the prerequisites which are necessary for chapter II which contains the results on hypergeodesic equations.

If $\vec{v}$ is an n-vector we shall always write its components using superscripts $v^1,\ldots,v^n$, and thus we have $\vec{v} = (v^1,\ldots,v^n)$

The system of ordinary differential equations

$$
(1.1) \qquad
\begin{aligned}
\frac{dy^1}{dt} &= f^1(y^1,\ldots, y^n, t) \\
\frac{dy^2}{dt} &= f^2(y^1,\ldots, y^n, t) \\
&\vdots \\
\frac{dy^n}{dt} &= f^n(y^1,\ldots, y^n, t)
\end{aligned}
$$

is equivalent to the vector ordinary differential equation

$$
(1.2) \qquad \frac{d\vec{Y}}{dt} = \vec{f}(\vec{Y}, t)
$$

where $\vec{f}$ is an n-vector valued function whose components are functions of the real variable t and the n-vector $\vec{Y}$. The domain of $\vec{f}$ is an open set $D \subseteq R^{n+1}$.

Definition 1.3   Let $\vec{f} : D \to R^n$ where D is an open subset of R^{n+1}. The vector valued function $\vec{f}$ is said to be continuous at a point $(\vec{Y}_0, t_0) \in D$ if for any real number $\varepsilon > 0$, there are two real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|\vec{f}(\vec{Y},t) - \vec{f}(\vec{Y}_0,t_0)| < \varepsilon$, whenever $(\vec{Y},t) \in D$, $|t-t_0| < \delta_1$ and $|\vec{Y}-\vec{Y}_0| < \delta_2$.

The vector-valued function $\vec{f}$ is continuous in D if it is continuous at each point of D. Also, it can be easily shown that $\vec{f}$ is continuous on D if and only if each of its components is continuous on D.

Definition 1.4   Let $\vec{f} : U \to R^n$ where U is a non-empty open subset of R^m. If $\vec{f}$ has continuous partial derivatives of the first order on U then we say that $\vec{f}$ is continuously differentiable on U and we will denote this property by saying that $\vec{f}$ is c^1 on U. Also, for all $k \geqq 1$, $\vec{f}$ is c^k on U if $\vec{f}$ has continuous partial derivatives up to and including order k. $\vec{f}$ is infinitely differentiable if $\vec{f}$ is $c^k, \forall k \in N$.

Definition 1.5   Let g be a differentiable function defined on an open subset U of R^n into R. Let $\vec{P}_0 = (x_0^1, \ldots, x_0^n)$ be any point in U. Then the gradient vector of g at $\vec{P}_0$ is denoted by $\nabla g(\vec{P}_0)$ and defined by the formula

$$\nabla g = \left(\frac{\partial g}{\partial x^1}, \ldots, \frac{\partial g}{\partial x^n}\right) = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) g$$

where each of the partials is evaluated at $\vec{P}_0$.

<u>Definition 1.6</u> Let f be an infinitely differentiable function on open subset U of R^n into R. Let $\vec{Q}_0$ be a fixed point in U. Then the Taylor series expansion of f at the point $\vec{Q}_0$ is the following power series :

$$f(\vec{P}) = f(\vec{Q}_0)+[(\vec{P}-\vec{Q}_0)\nabla]f(\vec{Q})\Big|_{\vec{Q}=\vec{Q}_0} + \frac{1}{2!}[(\vec{P}-\vec{Q}_0)\nabla]^2 f(\vec{Q})\Big|_{\vec{Q}=\vec{Q}_0} + \cdots$$

$$+ \frac{1}{(n-1)!}[(\vec{P}-\vec{Q}_0)\nabla]^{n-1} f(\vec{Q})\Big|_{\vec{Q}=\vec{Q}_0} + \frac{1}{n!}[(\vec{P}-\vec{Q}_0)\nabla]^n f(\vec{Q})\Big|_{\vec{Q}=\vec{Q}_0}$$

$$+ \cdots\cdots\cdots$$

In order to fully understand the meaning of the formula, write it in three dimensions by setting $\vec{Q}_0 = (a,b,c)$ and $\vec{P} = (x,y,z)$, we get

$$f(x,y,z) = f(a,b,c)+[(x-a)\frac{\partial}{\partial x} +(y-b)\frac{\partial}{\partial y} +(z-c)\frac{\partial}{\partial z}]\ f(a,b,c)$$

$$+ \frac{1}{2!}[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} +(z-c)\frac{\partial}{\partial z}]^2\ f(a,b,c)+ \cdots$$

$$+ \frac{1}{n!}[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y} +(z-c)\frac{\partial}{\partial z}]^n\ f(a,b,c)+ \cdots$$

Expanding we get,

$$f(x,y,z) = f(a,b,c)+(x-a)\frac{\partial}{\partial x}\ f(a,b,c)+(y-b)\frac{\partial}{\partial y}\ f(a,b,c)+(z-c)\frac{\partial}{\partial z}\ f(a,b,c)$$

$$+ \frac{1}{2!}[(x-a)^2\ \frac{\partial^2}{\partial x^2}f(a,b,c)+(y-b)^2\ \frac{\partial^2}{\partial y^2}f(a,b,c)+(z-c)^2\ \frac{\partial^2}{\partial z^2}f(a,b,c)$$

$$+ 2(x-a)(y-b)\ \frac{\partial^2}{\partial x\partial y}\ f(a,b,c)+2(x-a)(z-c)\frac{\partial^2}{\partial x\partial z}f(a,b,c)$$

$$+2(y-b)(z-c)\frac{\partial^2}{\partial y\partial z}f(a,b,c)]+\dots+\frac{1}{n!}[(x-a)^n\frac{\partial^n}{\partial x^n}f(a,b,c)$$

$$+{}^nc_{n-1}(x-a)^{n-1}\frac{\partial^{n-1}}{\partial x^{n-1}}f(a,b,c)((y-b)\frac{\partial f}{\partial y}(a,b,c)$$

$$+(z-c)\frac{\partial}{\partial z}f(a,b,c))+\dots+{}^nc_1(x-a)\frac{\partial}{\partial x}f(a,b,c)((y-b)^{n-1}\frac{\partial^{n-1}}{\partial y^{n-1}}f(a,b,c)$$

$$+{}^{n-1}c_{n-2}(y-b)^{n-2}(z-c)\frac{\partial^{n-1}}{\partial y^{n-2}\partial z}f(a,b,c)$$

$$+\dots+{}^{n-1}c_1(y-b)(z-c)^{n-2}\frac{\partial^{n-1}}{\partial y\partial z^{n-2}}f(a,b,c)+(z-c)^{n-1}\frac{\partial^{n-1}}{\partial z^{n-1}}f(a,b,c))$$

$$+(y-b)^n\frac{\partial^n}{\partial y^n}f(a,b,c)+{}^nc_{n-1}(y-b)^{n-1}(z-c)\frac{\partial^n}{\partial y^{n-1}\partial z}f(a,b,c)$$

$$+\dots+{}^nc_1(y-b)(z-c)^{n-1}\frac{\partial^n}{\partial y\partial z^{n-1}}f(a,b,c)+(z-c)^n\frac{\partial^n}{\partial z^n}f(a,b,c)]+\dots$$

Definition 1.7 A real-valued infinitely differentiable function f defined on an open connected set D of R^n is said to be analytic in D, if, for any point $\vec{Y}_0 \in D$, the Taylor series expansion of the function at the point $\vec{Y}_0$ converges to the function f in some neighbourhood of $\vec{Y}_0$.

Definition 1.8 Let $\vec{f}$ be a vector-valued function defined on open subset D of R^m into R^n, $\vec{f}$ is said to be an analytic function on D if each of its component function is analytic on D.

Theorem 1.9 If the n functions $f^i(\vec{Y},t)$, $i = 1,\ldots n$ are continuous in a closed and bounded region $\bar{G}$ of R^{n+1}, then given any interior point $(\vec{Y}_0,t_0)$ of this region there exists at least one continuously differentiable curve $\vec{Y} = \vec{Y}(t)$ which is defined in an interval $|t-t_0| \leqq a$ and satisfies the system of differentiable equations with initial condition

$$\frac{d\vec{Y}}{dt} = \vec{f}\,(\vec{Y},t) \quad ; \quad \vec{Y}(t_0) = \vec{Y}_0$$

The proof is given in [2] pages 13-18.

Theorem 1.10 Let Ω be an open connected subset of R^{n+1}. For each $i = 1,2,\ldots,n$, let $f^i(\vec{Y},t)$ be c^k (analytic) $\infty \geqq k \geqq 1$ on Ω. Then for any point $(\vec{Y}_0,t_0)\varepsilon\ \Omega$, there exists neighbourhoods U of $\vec{Y}_0$ in R^n and I to t_0 in R such that for any $\vec{Y}_1\ \varepsilon$ U and all t ε I there are unique functions defined on I, $\psi^1_{\vec{Y}_1}(t),\ldots,\psi^n_{\vec{Y}_1}(t)$ such that

$$\frac{d}{dt}\psi^i_{\vec{Y}_1} = f^i(\vec{\psi}_{\vec{Y}_1}, t) \quad \text{and} \quad \psi^i_{\vec{Y}_1}(t_0) = y^i_1 .$$

writing $\psi^i_{\vec{Y}_1}(t) = \psi^i(\vec{Y}_1,t)$ we can also conclude that the functions ψ^i are of class c^{k+1} (analytic) in t and of class c^k (analytic) in $\vec{Y}_1$.

This theorem is called The Fundamental Theorem of Ordinary Differential Equation and the proof for the c^k case is shown in references [2] pages 18-22 and [7] pages 372-373. The case where $f^i(\vec{Y},t)$ are analytic for all $i = 1,\ldots, n$ is given in reference [3] pages 210-215.

We shall use notations $\frac{d}{dt}\vec{\psi} = \mathring{\vec{\psi}}$, $\frac{d^2\vec{\psi}}{dt^2} = \ddot{\vec{\psi}}$, and

$$\frac{d^3\vec{\psi}}{dt^3} = \dddot{\vec{\psi}}$$

Now we want to extend Theorem 1.10 to ordinary differential equations of the third order. We want to show that whenever $\vec{H}$ is c^k (analytic) $k \geq 1$ in $3n+1$ real variables then the differential equations $\dddot{\psi}^i = H^i(\vec{\psi},\mathring{\vec{\psi}},\ddot{\vec{\psi}},t)$, $i = 1,2,\ldots,n$ with initial conditions $\psi^i(t_0) = p^i_0$, $\mathring{\psi}^i(t_0) = u^i_0$, $\ddot{\psi}^i(t_0) = v^i_0$ has a unique solution.

<u>Theorem 1.11</u> Let $\vec{H}$ be c^k (analytic) $k \geq 1$ on an open connected subset Ω of R^{3n+1} then for all $(\vec{p}_0,\vec{u}_0,\vec{v}_0,t_0) \in \Omega$, thereexists a neighbourhood U of $\vec{p}_0$, W of $\vec{u}_0$, V of $\vec{v}_0$ in R^n and an interval I of t_0 in R such that for any $\vec{p}_1 \in U$, $\vec{u}_1 \in W$, $\vec{v}_1 \in V$ there are unique functions $\psi^1_{\vec{p}_1,\vec{u}_1,\vec{v}_1}(t),\ldots, \psi^n_{\vec{p}_1,\vec{u}_1,\vec{v}_1}(t)$ defined on I such that

$$\dddot{\vec{\psi}}_{\vec{p}_1,\vec{u}_1,\vec{v}_1} = \vec{H}(\vec{\psi}_{\vec{p}_1,\vec{u}_1,\vec{v}_1}, \mathring{\vec{\psi}}_{\vec{p}_1,\vec{u}_1,\vec{v}_1}, \ddot{\vec{\psi}}_{\vec{p}_1,\vec{u}_1,\vec{v}_1}, t)$$

and

$$\vec{\psi}_{\vec{p}_1,\vec{u}_1,\vec{v}_1}(t_0) = \vec{p}_1, \mathring{\vec{\psi}}_{\vec{p}_1,\vec{u}_1,\vec{v}_1}(t_0) = \vec{u}_1, \ddot{\vec{\psi}}_{\vec{p}_1,\vec{u}_1,\vec{v}_1}(t_0) = \vec{v}_1$$

writing

$\vec{\psi}_{\vec{p}_1,\vec{u}_1,\vec{v}_1}(t) = \vec{\psi}(\vec{p}_1,\vec{u}_1,\vec{v}_1,t)$ we can also conclude that the function $\vec{\psi}$ is of class c^{k+3} (analytic) in t and of class c^k (analytic) with respect to $(\vec{p}_1, \vec{u}_1, \vec{v}_1)$.

Proof Let $\vec{H}$ be c^k (analytic) $k \geqq 1$ on a subset Ω of R^{3n+1}.

Let $(\vec{p}_0, \vec{u}_0, \vec{v}_0, t_0)$ be any point in Ω.

We shall prove that there exists a unique solution $\vec{\phi} = (\phi^1, \ldots, \phi^n)$ which satisfies the system of third order ordinary differential equations with initial conditions.

$$\dddot{\vec{\psi}} = \vec{H}(\vec{\psi}, \dot{\vec{\psi}}, \ddot{\vec{\psi}}, t)$$

(1) $\vec{\psi}(\vec{p}, \vec{u}, \vec{v}, t_0) = \vec{p}$ (where $\vec{p}$ is a point in some neighbourhood of $\vec{p}_0$)

$\dot{\vec{\psi}}(\vec{p}, \vec{u}, \vec{v}, t_0) = \vec{u}$ (where $\vec{u}$ is a point in some neighbourhood of $\vec{u}_0$)

$\ddot{\vec{\psi}}(\vec{p}, \vec{u}, \vec{v}, t_0) = \vec{v}$ (where $\vec{v}$ is a point in some neighbourhood of $\vec{v}_0$)

Let

$$\psi^1 = y^1, \quad \psi^2 = y^2, \ldots, \psi^n = y^n$$

$$\dot{\psi}^1 = y^{n+1}, \quad \dot{\psi}^2 = y^{n+2}, \ldots, \dot{\psi}^n = y^{2n}$$

$$\ddot{\psi}^1 = y^{2n+1}, \quad \ddot{\psi}^2 = y^{2n+2}, \ldots, \ddot{\psi}^n = y^{3n}$$

Hence

$$\frac{dy^1}{dt} = \dot{\psi}^1 = y^{n+1}, \quad \frac{d}{dt} y^{n+1} = \ddot{\psi}^1 = y^{2n+1}$$

$$\frac{dy^2}{dt} = \dot{\psi}^2 = y^{n+2}, \quad \frac{d}{dt} y^{n+2} = \ddot{\psi}^2 = y^{2n+2}$$

.
.
.

$$\frac{dy^n}{dt} = \dot{\psi}^n = y^{2n}, \quad \frac{d}{dt} y^{2n} = \ddot{\psi}^n = y^{3n}$$

$$\frac{d}{dt} y^{2n+1} = \dddot{\psi}^1 = H^1(y^1,\dots,y^n,y^{n+1},\dots,y^{2n},y^{2n+1},\dots,y^{3n}, t)$$

$$\frac{d}{dt} y^{2n+2} = \dddot{\psi}^2 = H^2(y^1,\dots, y^{3n}, t)$$

$$\vdots$$

$$\frac{d}{dt} y^{3n} = \dddot{\psi}^n = H^n(y^1,\dots, y^{3n}, t)$$

This is a system of ordinary differential equation of the first order which we shall denote by

$$\frac{dy^i}{dt} = f^i(y^1,\dots,y^{3n},t) \qquad i = 1,\dots, 3n$$

(2) where for each $i = 1,\dots,n$, $f^i(y^1,\dots,y^{3n},t) = y^{n+i}$

and for each $i = n+1,\dots,2n$, $f^i(y^1,\dots,y^{3n},t) = y^{2n+i}$

and for each $i = 2n+1,\dots,3n$, $f^i(y^1,\dots,y^{3n},t) = H^{i-2n}(y^1,\dots y^{3n},t)$.

This system of differential equations has initial conditions

$$\vec{Y}(t_0) = (p^1,\dots,p^n, u^1,\dots,u^n, v^1,\dots,v^n) .$$

The n-vector valued function $\vec{f}$ defined above is continuous, hence theorem 1.10 implies that there exists at least one solution of (2).

<u>Lemma 1.12</u> If $\vec{\phi} = (\phi^1,\dots,\phi^n)$ is a solution of (1), then $\vec{\phi}* = (\phi^1,\dots,\phi^n,\mathring{\phi}^1,\dots,\mathring{\phi}^n,\ddot{\phi}^1,\dots,\ddot{\phi}^n)$ is a solution of (2). Conversely, if $\vec{\phi}* = (\phi^1,\dots,\phi^{3n})$ is a solution of (2), then the first n components $\phi^1,\dots, \phi^n$ are solutions of (1).

<u>Proof</u> Assume that $\vec{\phi}$ is a solution of

(1) $\dddot{\psi}^i = H^i(\psi^1,\ldots,\psi^n,\dot{\psi}^1,\ldots,\dot{\psi}^n,\ddot{\psi}^1,\ldots,\ddot{\psi}^n, t)$ with initial conditions $\psi^i(t_0) = p^i$, $\dot{\psi}^i(t_0) = u^i$, $\ddot{\psi}^i(t_0) = v^i$

Hence for each i = 1,2,..., n, we have

$$\dddot{\phi}^i = H^i(\phi^1,\ldots,\phi^n,\dot{\phi}^1,\ldots,\dot{\phi}^n,\ddot{\phi}^1,\ldots,\ddot{\phi}^n, t)$$

and

$$\phi^i(t_0) = p^i, \quad \dot{\phi}^i(t_0) = u^i, \quad \ddot{\phi}^i(t_0) = v^i$$

we shall prove that $\vec{\phi}^* = (\phi^1,\ldots,\phi^n,\dot{\phi}^1,\ldots,\dot{\phi}^n,\ddot{\phi}^1,\ldots,\ddot{\phi}^n)$ satisfies

$$\frac{dy^i}{dt} = f^i(y^1,\ldots,y^{3n},t) \text{ with initial condition}$$

$$\vec{Y}(t_0) = (p^1,\ldots,p^n, u^1,\ldots,u^n, v^1,\ldots,v^n)$$

Since $\frac{d\phi^1}{dt} = \dot{\phi}^1,\ldots, \frac{d\phi^n}{dt} = \dot{\phi}^n$

by assumption, we get

$$\frac{d^3\phi^1}{dt^3} = \dddot{\phi}^1 = H^1(\phi^1,\ldots\phi^n, \dot{\phi}^1,\ldots,\dot{\phi}^n, \ddot{\phi}^1,\ldots\ddot{\phi}^n, t)$$

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$$\frac{d^3\phi^n}{dt^3} = \dddot{\phi}^n = H^n(\phi^1,\ldots\phi^n, \dot{\phi}^1,\ldots,\dot{\phi}^n, \ddot{\phi}^1,\ldots,\ddot{\phi}^n, t)$$

Since $\phi^i(t_0) = p^i$, $\dot{\phi}^i(t_0) = u^i$, $\ddot{\phi}^i(t_0) = v^i$

Hence $\vec{\phi}^*(t_0) = (\phi^1(t_0),\ldots,\phi^n(t_0), \dot{\phi}^1(t_0),\ldots,\dot{\phi}^n(t_0), \ddot{\phi}^1(t_0),\ldots,\ddot{\phi}^n(t_0))$

$$= (p^1,\ldots,p^n, u^1,\ldots,u^n, v^1,\ldots, v^n)$$

Thus, $\vec{\phi}^* = (\phi^1,\ldots,\phi^n,\dot{\phi}^1,\ldots,\dot{\phi}^n,\ddot{\phi}^1,\ldots,\ddot{\phi}^n)$ is a solution of (2).

<u>Uniqueness</u> We shall prove that $\vec{\phi}^* = (\phi^1,\ldots,\phi^n,\mathring{\phi}^1,\ldots,\mathring{\phi}^n,\ddot{\phi}^1,\ldots,\ddot{\phi}^n)$ is the unique solution of (2).

$\vec{f} = (y^{n+1},\ldots,y^{2n},y^{2n+1},\ldots,y^{3n}, H^1(\vec{Y},t),\ldots H^n(\vec{Y},t))$ is c^k (analytic) $k \geq 1$. By Theorem 1.10, we conclude that

$\vec{\phi}^* = (\phi^1,\ldots,\phi^n,\mathring{\phi}^1,\ldots,\mathring{\phi}^n,\ddot{\phi}^1,\ldots,\ddot{\phi}^n)$ is the unique solution of (2).

Conversely, we assume that $\vec{\phi}^* = (\phi^1,\ldots,\phi^{3n})$ is a solution of (2).
We shall show that $\vec{\phi} = (\phi^1,\ldots,\phi^n)$ is a solution of (1).
By the assumption we have that

$$\mathring{\phi}^1 = \phi^{n+1},\ldots, \mathring{\phi}^n = \phi^{2n}$$

$$\ddot{\phi}^1 = \phi^{2n+1},\ldots,\ddot{\phi}^n = \phi^{3n}$$

Differentiate the above equations with respect to t and using the fact that ϕ^i , $i = 1,\ldots,3n$ satisfy (2), we obtain

$$\dddot{\phi}^1 = H^1(\phi^1,\ldots,\phi^{3n},t) = H^1(\vec{\phi},\mathring{\vec{\phi}},\ddot{\vec{\phi}},t)$$

$$\dddot{\phi}^2 = H^2(\phi^1,\ldots,\phi^{3n},t) = H^2(\vec{\phi},\mathring{\vec{\phi}},\ddot{\vec{\phi}},t)$$

$$\vdots$$

$$\dddot{\phi}^n = H^n(\phi^1,\ldots,\phi^{3n},t) = H^n(\vec{\phi},\mathring{\vec{\phi}},\ddot{\vec{\phi}},t)$$

Therefore, for each $i = 1,\ldots,n$

$$\dddot{\phi}^i = H^i(\vec{\phi},\mathring{\vec{\phi}},\ddot{\vec{\phi}},t)$$

Since $\vec{\phi}^*$ satisfies the initial condition.

$$\vec{\phi}^*(t_0) = (p^1,\ldots,p^n, u^1,\ldots,u^n, v^1,\ldots,v^n) .$$

Hence $\phi^{i}(t_0) = p^{i}$ for $i = 1,\dots,n$

$\phi^{n+i}(t_0) = u^{i}$ for $i = 1,\dots,n$

$\phi^{2n+i}(t_0) = v^{i}$ for $i = 1,\dots,n$

Since $\dot{\phi}^{i} = \phi^{n+i}$ for $i = 1,\dots,n$

Then $\dot{\phi}^{i}(t_0) = \phi^{n+i}(t_0) = u^{i}$ for $i = 1,\dots,n$

$\ddot{\phi}^{i}(t_0) = \phi^{2n+i}(t_0) = v^{i}$ for $i = 1,\dots,n$

Hence $\vec{\phi} = (\phi^{1},\dots,\phi^{n})$ satisfies the equation (1).

Therefore, $\vec{\phi} = (\phi^{1},\dots,\phi^{n})$ is a solution of (1).

Uniqueness

Suppose $\vec{\Phi}$ is another solution of (1) such that for some $i = 1,\dots, n$, $\phi^{i} \neq \Phi^{i}$.

In the same way, we can prove that $\vec{\Phi}^{*} = (\Phi^{1},\dots,\Phi^{n},\dot{\Phi}^{1},\dots,\dot{\Phi}^{n},\ddot{\Phi}^{1},\dots,\ddot{\Phi}^{n})$ is a solution of (2) which satisfies the given initial condition.

So $\vec{\Phi}^{*} \neq \phi^{*}$ which contradicts Theorem 1.10. Hence for all $i = 1,\dots, n$, we have $\phi^{i} = \Phi^{i}$. That is $\vec{\phi} = \vec{\Phi}$. Thus, $\vec{\phi}$ is the unique solution of (1).

Lemma 1.12 is proved.

Since $(\vec{p}_0,\vec{u}_0,\vec{v}_0,t_0)$ be any point in $\Omega \subseteq R^{3n+1}$. By Theorem 1.10, there exist neighbourhoods U of $\vec{p}_0$, W of $\vec{u}_0$, V of $\vec{v}_0$ and an interval I of t_0 in R such that for all $\vec{p}_1 \in U$, $\vec{u}_1 \in W$, $\vec{v}_1 \in V$, there exists a unique function $\vec{\phi}^{*} = (\phi^1,\ldots,\phi^{3n})$ defined on I such that for each $i = 1,\ldots,3n$, $\phi^i(\vec{p}_1,\vec{u}_1,\vec{v}_1,t)$ satisfies

$$\frac{d\phi^i}{dt} = f^i(\vec{\phi}^{*}, t)$$

$$\phi^i(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = p_1^i \quad \text{for } i = 1,\ldots,n$$

$$\phi^i(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = u_1^i \quad \text{for } i = n+1,\ldots,2n$$

$$\phi^i(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = v_1^i \quad \text{for } i = 2n+1,\ldots,3n$$

The functions ϕ^i are of class c^{k+1} (analytic) in t and of class c^k (analytic) in $(\vec{p},\vec{u},\vec{v})$ for all $i = 1,\ldots,n$.

By Lemma 1.12, $\vec{\phi} = (\phi^1,\ldots,\phi^n)$ is the unique solution of (1) such that for each $i = 1,\ldots,n$, $\dddot{\phi}^i = H^i(\vec{\phi},\mathring{\vec{\phi}},\ddot{\vec{\phi}},t)$ and $\phi^i(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = p_1^i$, $\phi^i(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = u_1^i$,

$$\ddot{\phi}^i \quad (\vec{p}_1,\vec{u}_1,\vec{v}_1,t) = v_1^i$$

Since
$$\phi^{n+i} = \mathring{\phi}^i \quad \text{for } i = 1,\ldots,n$$

$$\phi^{2n+i} = \ddot{\phi}^i \quad \text{for } i = 1,\ldots,n$$

hence

$$\phi^{n+i}(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = \mathring{\phi}^i(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = u_1^i$$

$$\phi^{2n+i}(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = \ddot{\phi}^i(\vec{p}_1,\vec{u}_1,\vec{v}_1,t_0) = v_1^i \quad \text{for } i = 1,\ldots,n$$

For each $i = 1,\ldots,n$, $\mathring{\phi}^i = \phi^{n+i}$ which is of class c^{k+2} (analytic) in t (by Theorem 1.10), $\ddot{\phi}^i = \phi^{2n+i}$, $i = 1,\ldots,n$ which is c^{k+3}

(analytic) in t. Hence $\emptyset^i$ is of class c^{k+3} (analytic) in t for each $i = 1,\ldots,n$

The proof is complete.

Definition 1.13 Let $f^1,\ldots,f^n$ be n c^1 real-valued functions defined on an open set U in R^n and let $\vec{f} = (f^1,\ldots,f^n)$. By the Jacobian of $\vec{f}$ at $\vec{X}_0 \varepsilon$ U we mean the real-valued function $J_{\vec{f}}$ whose values are given by the determinant

$$J_{\vec{f}}(\vec{X}_0) = \begin{vmatrix} \frac{\partial f^1}{\partial x^1}(\vec{X}_0) & \cdots\cdots & \frac{\partial f^1}{\partial x^n}(\vec{X}_0) \\ \vdots & & \vdots \\ \frac{\partial f^n}{\partial x^1}(\vec{X}_0) & \cdots\cdots & \frac{\partial f^n}{\partial x^n}(\vec{X}_0) \end{vmatrix}$$

The notation $\frac{\partial(f^1,\ldots,f^n)}{\partial(x^1,\ldots,x^n)}$ is also used for $J_{\vec{f}}(\vec{X})$ where $\vec{X} = (x^1,\ldots,x^n)$.

Theorem 1.14 (The Inverse Function Theorem)

Let $\vec{f} : U \to R^n$, $\vec{X}_0 \varepsilon$ U where U is an open subset of R^n. If $\vec{f}$ is c^1 function on U and $J_{\vec{f}}(\vec{X}_0) \neq 0$, then there exists open sets V of $\vec{X}_0$ and W of $\vec{f}(\vec{X}_0)$ which are subsets of U and $\vec{f}[U]$ respectively and a unique function $g : W \to V$ such that

(1) $\vec{f}[V] = W$

(2) $\vec{f}$ is a one to one function on V.

(3) $\vec{g}[W] = V$ and $\vec{g}(\vec{f}(\vec{X})) = \vec{X} \quad \forall \vec{X} \in V$

(4) $\vec{g}$ is c^1 function on W.

The proof is in reference [1] pages 144-146.

Theorem 1.15 The set S of all convergent power series in n variables over the field of real numbers is an integral domain.

The proof is in reference [8] pages 129-130.

This theorem says that if the product fg of two real-valued analytic functions f and g is identically zero in some neighbourhood of $\vec{X}_0 \in R^n$, then at least one of the functions f and g is identically zero in a neighbourhood of $\vec{X}_0$.

Theorem 1.16 Let $\vec{V} = (v^1,\dots,v^n)$. Let h be a function of n variables $v^1, v^2,\dots,v^n$ which is analytic. If h is identically zero, then all of the coefficients of the power series are zero.

Proof Since h is analytic in $v^1,v^2,\dots,v^n$. By the definition of analytic function we get,

$$h(v^1,\dots,v^n) = h(\vec{0})+[v^1 \frac{\partial}{\partial v^1}+\dots+ v^n \frac{\partial}{\partial v^n}]h(\vec{0})+\frac{1}{2!}[v^1\frac{\partial}{\partial v^1}+\dots+v^n\frac{\partial}{\partial v^n}]^2 h(\vec{0})$$

$$+\dots+ \frac{1}{k!}[v^1\frac{\partial}{\partial v^1}+\dots+v^n\frac{\partial}{\partial v^n}]^k h(\vec{0})+ \dots$$

For simplicity, we shall use the following notations.

1) $c_{j_1} v^{j_1} = \sum_{j_1=1}^{n} c_{j_1} v^{j_1}$, $c_{j_1\dots j_k} v^{j_1}\dots v^{j_k} = \sum_{j_1=1}^{n} \dots \sum_{j_k=1}^{n} c_{j_1\dots j_k} v^{j_1}\dots v^{j_k}$

2) $c_{j_1\dots j_k}$ is the coefficient of $v^{j_1}\dots v^{j_k}$ where $k = 1,2,\dots$

$$D_{j_1} h = \frac{\partial h}{\partial v^{j_1}} \quad , \quad j_1 = 1,\dots, n$$

$$D_{j_1\dots j_k} h = \frac{\partial^k h}{\partial v^{j_1}\dots\partial v^{j_k}} \quad , \quad j_k = 1,\dots, n \; ; \; k = 1,2,\dots$$

Let $\quad c_0 = h(\vec{0})$

Hence

$$h(v^1,\ldots,v^n) = c_0 + c_{j_1} v^{j_1} + c_{j_1 j_2} v^{j_1} v^{j_2} + \ldots + c_{j_1\ldots j_k} v^{j_1} \ldots v^{j_k} + \ldots$$

By the assumption, $h = 0$.

Therefore, we have $h(\vec{0}) = 0$ and $D_{j_1\ldots j_k} h(\vec{0}) = 0$, $k = 1,2,\ldots$

Thus, $c_0 = 0$, $c_{j_1..j_k} = 0$ where $j_k = 1,2,\ldots,n$; $k = 1,2,\ldots$

The proof is complete.

<u>Theorem 1.17</u> Let f and g be two analytic functions in the n variables $v^1,\ldots,v^n$ in the same region such that f and g are identically equal. Let

$$f(v^1,\ldots,v^n) = a_0 + \sum_{j_1=1}^{n} a_{j_1} v^{j_1} + \ldots + \sum_{j_1=1}^{n} \ldots \sum_{j_k=1}^{n} a_{j_1\ldots j_k} v^{j_1} \ldots v^{j_k} + \ldots$$

and $$g(v^1,\ldots,v^n) = b_0 + \sum_{j_1=1}^{n} b_{j_1} v^{j_1} + \ldots + \sum_{j_1=1}^{n} \ldots \sum_{j_k=1}^{n} b_{j_1\ldots j_k} v^{j_1} \ldots v^{j_k} + \ldots$$

Then $a_0 = b_0$, for each $k = 1,2,\ldots, a_{j_1\ldots j_k} = b_{j_1\ldots j_k}$ for $1 \leq j_k \leq n$.

<u>Proof</u> Let $h \equiv f-g$. Hence $h \equiv 0$. Since f, g are analytic functions of $v^1,\ldots v^n$. Hence f-g is analytic in $v^1,\ldots,v^n$, so is h.

Thus, $$h = (a_0-b_0) + \sum_{j_1=1}^{n} (a_{j_1} - b_{j_1}) v^{j_1} + \ldots + \sum_{j_1=1}^{n} .. \sum_{j_k=1}^{n} (a_{j_1..j_k} - b_{j_1..j_k}) v^{j_1} .. v^{j_k} + \ldots$$

By Theorem 1.15, implies that $a_0 - b_0 = 0$, $a_{j_1} - b_{j_1} = 0,\ldots$

$a_{j_1\ldots j_k} - b_{j_1\ldots j_k} = 0$, ...

Hence $a_0 = b_0$, $a_{j_1} = b_{j_1}, \ldots, a_{j_1 \ldots j_k} = b_{j_1 \ldots j_k}$ for $k = 1, \ldots$ and $j_k = 1, \ldots, n$

Then Theorem 1.17 is proved.

Theorem 1.18 Let $b = (b^1, \ldots, b^n) \varepsilon R^n$ be such that $b^i \neq 0$ for $1 \leq i \leq n$ and that the series $\sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} c_{m_1 \ldots m_n} (b^1)^{m_1} \ldots (b^n)^{m_n} < \infty$. Then for any r^i, $0 < r^i < |b^i|$ for $1 \leq i \leq n$, the power series $\sum_{m_1=0}^{\infty} \ldots \sum_{m_n=0}^{\infty} c_{m_1 \ldots m_n} (Z^1)^{m_1} \ldots (Z^n)^{m_n}$ is absolutely convergent for all Z^i, $|Z^i| < r^i$ and it can be rearranged.

The proof is given in reference [4] pages 199-200.

Introduction to Theorem 1.19 which contains the main results in reference [9].

Define
$$\Pi_1(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}, x^{2n+1}) = (x^1, \ldots, x^n)$$
$$\Pi_2(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}, x^{2n+1}) = (x^{n+1}, \ldots, x^{2n})$$
$$\Pi_3(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}, x^{2n+1}) = x^{2n+1} .$$

Let Ω be a connected open set of R^{2n+1} such that if $\vec{p} \varepsilon \Pi_1(\Omega)$, then $(\vec{p}, \vec{v}, 0) \varepsilon \Omega \quad \forall \vec{v} \varepsilon R^n$.

Let $H : \Omega \to R^n$ be analytic. Then H determines a second order differential equation

$$\ddot{\psi}^i = H^i(\psi, \dot{\psi}, t) \qquad i = 1, \ldots, n .$$

By Fundamental Theorem of Ordinary Differential Equation, given initial point $\vec{p} = (p^1, \ldots, p^n) \varepsilon \Pi_1(\Omega)$ and initial vector

$\vec{v} = (v^1,\ldots,v^n) \varepsilon\ R^n$, there exists an open subset $U_{p,v}$ of 0 in $\Pi_3(\Omega)$ and there exists a unique function

$$\psi_{\vec{p},\vec{v}} : U_{\vec{p},\vec{v}} \to \Pi_1(\Omega)$$

satisfying the differential equation and the initial conditions

$$\vec{\psi}_{\vec{p},\vec{v}}(0) = \vec{p}\ ,\quad \dot{\vec{\psi}}_{\vec{p},\vec{v}}(0) = \vec{v}\ .$$

Write $\qquad \Phi(\vec{p},\vec{v},t) = \psi_{\vec{p},\vec{v}}(t)$

This determines an open set $V \subseteq R^{2n+1}$ and a map

$$\Phi \quad : \quad V \to \Pi_1(\Omega)$$

which is analytic by the Fundamental Theorem.

<u>Theorem 1.19</u> Suppose there exists a neighbourhood W of (0,0) in R^2 and an analytic function $f : W \to R$ such that

$$\vec{\Phi}(\vec{p},\alpha\vec{v},t) = \vec{\Phi}(\vec{p},\vec{v},f(\alpha,t)) \quad \text{whenever } (\vec{p},\alpha\vec{v},t)\ \varepsilon\ V\ ,$$

$(\alpha,t)\ \varepsilon\ W$. Furthermore, assume that $f(\alpha,0) = 0$, $f(0,t) = 0$ whenever defined. Then the differential equation must have the form

$$\ddot{\psi}^i = \sum_{j=1}^{n}\sum_{k=1}^{n} G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k + c\dot{\psi}^i\ , \qquad c \neq 0 \text{ where}$$

$$f(\alpha,t) = \frac{1}{c}\,\ell n(1-\alpha+\alpha e^{ct})\ , \quad c = \frac{f_{tt}(\beta,0)}{\beta(1-\beta)}$$

$\forall(\beta,0)\ \varepsilon\ W\ ,\ \beta \neq 0,1$

or $$\ddot{\psi}^i = \sum_{j=1}^{n}\sum_{k=1}^{n} G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k \ldots \text{ where } f(\alpha,t) = \alpha t$$

$$\forall i = 1,\ldots,n$$

Theorem 1.20 (Converse to Theorem 1.19)

For each $i = 1,2,\ldots,n$ if ψ^i satisfies the second order ordinary differential equation of the type $\ddot{\psi}^i = \sum_{j=1}^{n} \sum_{k=1}^{n} G^i_{jk}(\vec{\psi})\dot{\psi}^j\dot{\psi}^k + c\dot{\psi}^i$ where the function G^i_{jk} is c' on open subset D of R^n for $j,k = 1,2,\ldots,n$, then for all $\vec{p} \in D$, $\vec{v} \in R^n$, $\alpha \in R$, the solution ψ^i must satisfy the functional equation

$\psi^i(\vec{p},\alpha\vec{v},t) = \psi^i(\vec{p},\vec{v},f(\alpha,t))$ $\forall t \in J(0)$ which is an open interval of zero in R and $f(\alpha,t) = \frac{1}{c}\,\ell n(1-\alpha+\alpha e^{ct})$ when $c \neq 0$ or $\psi^i(\vec{p},\alpha\vec{v},t)$ exists if and only if $\psi^i(\vec{p},\vec{v},\alpha t)$ exists and $\psi^i(\vec{p},\alpha\vec{v},t) = \psi^i(\vec{p},\vec{v},\alpha t)$ when $c = 0$ and G^i_{jk} is analytic on D.

Geometric properties of geodesics, for proof see reference [9] pages 70-76.

Property 1 Given t_0 in R, $\vec{p}_0$ in D. Then there exists a neighbourhood U of the zero vector at $\vec{p}_0$ such that $\forall\vec{v}$ in U, $\vec{\psi}(\vec{p}_0,\vec{v},t_0)$ exists.

Property 2 Given any compact neighbourhood U of the zero vector at $\vec{p}_0$, there exists a neighbourhood V of zero in R such that $\forall t \in V$, $\forall \vec{v} \in U$, $\vec{\psi}(\vec{p}_0,\vec{v},t)$ exists.

Property 3 (Exponential property) Given initial point $\vec{p}_0 \in D$ and $t_0 \in R-\{0\}$, let V be a neighbourhood of $\vec{0}$ such that $\vec{\psi}(\vec{p}_0,\vec{v},t_0)$ exists $\forall\vec{v} \in V$ then the map $\vec{v} \mapsto \vec{\psi}(\vec{p}_0,\vec{v},t_0)$ is a bidifferential map of some open subset of V onto an open set W.

<u>Property 4</u> Given $\vec{p}_0 \varepsilon$ D, $\vec{v}_0$ at $\vec{p}_0$ and any real number α_0, then the solution curve $\vec{\psi}(\vec{p}_0,\vec{v}_0,t)$ with initial condition $\vec{p}_0$, $\vec{v}_0$ agree as points set with the solution curve $\vec{\psi}(\vec{p}_0,\alpha_0\vec{v}_0,t)$ having initial condition $\vec{p}_0,\alpha_0\vec{v}_0$.

In chapter II we shall prove properties 1), 2) and 4) for the solution to hypergeodesic differential equations and shall prove a property similar to property 3).

Let $\dddot{\psi}^i = H^i(\vec{\psi},\mathring{\vec{\psi}},\ddot{\vec{\psi}},t)$ be analytic third order differential equation with initial conditions $\psi^i(t_0) = p_0^i$, $\mathring{\psi}^i(t_0) = u_0^i$, $\ddot{\psi}^i(t_0) = v_0^i$. We are concerned with the first and second derivatives because they are part of the initial conditions. In particular, we are interested in the way that the first and second derivatives transform when we change coordinates.

Suppose $\vec{X}(t) = (x^1(t),\ldots, x^n(t))$ are the equations of a curve with respect to the coordinate system $(x^1,\ldots,x^n)$. Then the components of the first derivative at t_0 are $\left.\frac{dx^i}{dt}\right|_{t=t_0}$ for $i = 1,\ldots,n$ and the components of the second derivative at t_0 are

$$\left.\frac{d^2x^i}{dt^2}\right|_{t=t_0} \quad \text{for } i = 1,\ldots,n \;.$$

Let $\vec{Y} = (y^1,\ldots, y^n)$ be a different coordinate system, let $y^\alpha = f^\alpha(x^1,\ldots,x^n)$ for $\alpha = 1,\ldots,n$ be the equation determining the change of coordinates.

Then $\left.\frac{dy^\alpha}{dt}\right|_{t_0} = \sum_{i=1}^{n} \frac{\partial f^\alpha}{\partial x^i}(\vec{X}(t_0)) \left.\frac{dx^i}{dt}\right|_{t_0}$ which is a linear function of the components of the first derivative with respect to $\vec{X}$ and

$$(1.21)\quad \left.\frac{d^2y^\alpha}{dt^2}\right|_{t_0} = \sum_{i=1}^{n} \frac{\partial f^\alpha}{\partial x^i}(\vec{X}(t_0))\left.\frac{d^2x^i}{dt^2}\right|_{t_0} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}(\vec{X}(t_0))\left.\frac{dx^i}{dt}\right|_{t_0}\left.\frac{dx^j}{dt}\right|_{t_0}$$

which is not a linear function of the components of the second and first derivatives with respect to $\vec{X}$.

This example motivates us to use the following concepts.

Definition 1.22 An action of a semigroup S on a set X is a map $\Phi : S\times X \to X$ such that $\Phi(st,x) = \Phi(s,\Phi(t,x))$ and we shall denote $\Phi(s,x)$ by sx .

Definition 1.23 The triple (S,X,Φ) is called a semigeometry.

Definition 1.24 Let (S,X,Φ) be a semigeometry and $\psi : R\times X \to X$ an action of the semigroup $(R,\times)$ and we shall denote $\psi(\alpha,x)$ by $\alpha.x$. ψ is said to be invariant with respect to the semigeometry (S,X,Φ) if

$$s(\alpha.x) = \alpha.(sx) \qquad \forall s \in S,\ \forall x \in X,\ \forall \alpha \in R.$$

Remark : We call a set with an action of $(R,\times)$ on it a cone.

<u>Notations</u> : $a_k b^k = \sum_{k=1}^{n} a_k b^k$, $a_{k_1 k_2 \ldots k_\ell} b^{k_1} b^{k_2} \ldots b^{k_\ell} = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \ldots$

$\ldots \sum_{k_\ell=1}^{n} a_{k_1 k_2 \ldots k_\ell} b^{k_1} b^{k_2} \ldots b^{k_\ell}$ where $\ell = 1,2,\ldots$

$$a_k b^k_{\ell_1 \ell_2} c^{\ell_1} c^{\ell_2} = \sum_{k=1}^{n} \sum_{\ell_1=1}^{n} \sum_{\ell_2=1}^{n} a_k b^k_{\ell_1 \ell_2} c^{\ell_1} c^{\ell_2}$$

$$a_k b^k_\ell c^\ell = \sum_{\ell=1}^{n} \sum_{k=1}^{n} a_k b^k_\ell c^\ell , a_{k_1 k_2} b^{k_1}_{\ell_1} b^{k_2}_{\ell_2} c^{\ell_1} c^{\ell_2}$$

$$= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{\ell_1=1}^{n} \sum_{\ell_2=1}^{n} a_{k_1 \ k_2} b^{k_1}_{\ell_1} b^{k_2}_{\ell_2} c^{\ell_1} c^{\ell_2}$$

Consider $R^{2n} = \{(u^1,\ldots,u^n,v^1,\ldots,v^n)\}$

$$S = \{(a^i_{j_1}, a^i_{j_1 j_2}) | a^i_{j_1 j_2} = a^i_{j_2 j_1} , i,j_1,j_2 = 1,\ldots,n\}$$

S is a semigroup if we define

$(a^i_{j_1}, a^i_{j_1 j_2}).(b^i_{j_1}, b^i_{j_1 j_2}) = (c^i_{j_1}, c^i_{j_1 j_2})$ where $c^i_{j_1} = a^i_k b^k_{j_1}$ and

$$c^i_{j_1 j_2} = a^i_k b^k_{j_1 j_2} + a^i_{k_1 k_2} b^{k_1}_{j_1} b^{k_2}_{j_2}$$

To show that S is semigroup.

$$[(a^i_{j_1}, a^i_{j_1 j_2}).(b^i_{j_1}, b^i_{j_1 j_2})].(c^i_{j_1}, c^i_{j_1 j_2}) = (d^i_{j_1}, d^i_{j_1 j_2}).(c^i_{j_1}, c^i_{j_1 j_2})$$

where $d^i_{j_1} = a^i_k\, b^k_{j_1}$

$$d^i_{j_1j_2} = a^i_k\, b^k_{j_1j_2} + a^i_{k_1k_2}\, b^{k_1}_{j_1}\, b^{k_2}_{j_2}$$

so

$$[(a^i_{j_1}, a^i_{j_1j_2}).(b^i_{j_1}, b^i_{j_1j_2})].(c^i_{j_1}, c^i_{j_1j_2}) = (d^i_k c^k_{j_1}, d^i_k c^k_{j_1j_2} + d^i_{k_1k_2} c^{k_1}_{j_1} c^{k_2}_{j_2})$$

$$= (a^i_\ell b^\ell_k c^k_{j_1}, a^i_\ell b^\ell_k c^k_{j_1j_2} + a^i_\ell b^\ell_{k_1k_2} c^{k_1}_{j_1} c^{k_2}_{j_2} + a^i_{\ell_1\ell_2} b^{\ell_1}_{k_1} b^{\ell_2}_{k_2} c^{k_1}_{j_1} c^{k_2}_{j_2})$$

$$(a^i_{j_1}, a^i_{j_1j_2}).[(b^i_{j_1}, b^i_{j_1j_2}).(c^i_{j_1}, c^i_{j_1j_2})] = (a^i_{j_1}, a^i_{j_1j_2}).(h^i_{j_1}, h^i_{j_1j_2})$$

where $h^i_{j_1} = b^i_\ell c^\ell_{j_1}$

$$h^i_{j_1j_2} = b^i_\ell c^\ell_{j_1j_2} + b^i_{\ell_1\ell_2} c^{\ell_1}_{j_1} c^{\ell_2}_{j_2}$$

then

$$(a^i_{j_1}, a^i_{j_1j_2}).[(b^i_{j_1}, b^i_{j_1j_2}).(c^i_{j_1}, c^i_{j_1j_2})] = (a^i_k h^k_{j_1}, a^i_k h^k_{j_1j_2} + a^i_{k_1k_2} h^{k_1}_{j_1} h^{k_2}_{j_2})$$

$$= (a^i_k b^k_\ell c^\ell_{j_1}, a^i_k b^k_\ell c^\ell_{j_1j_2} + a^i_k b^k_{\ell_1\ell_2} c^{\ell_1}_{j_1} c^{\ell_2}_{j_2} + a^i_{k_1k_2} b^{k_1}_{\ell_1} c^{\ell_1}_{j_1} b^{k_2}_{\ell_2} c^{\ell_2}_{j_2})$$

Hence S is a semigroup.

S acts on R^{2n} as follows :

$$(a^i_{j_1}, a^i_{j_1j_2}).(u^1,\ldots,u^n,v^1,\ldots,v^n) = (a^1_{j_1} u^{j_1},\ldots,a^n_{j_1} u^{j_1},\ a^1_{j_1} v^{j_1} + a^1_{j_1j_2} u^{j_1} u^{j_2},\ldots$$

$$\ldots,\ a^n_{j_1} v^{j_1} + a^n_{j_1j_2} u^{j_1} u^{j_2})$$

To show that S acts on R^{2n}. That is, we want to show that $(st).\alpha = s(t.\alpha)$ for all s, t ε S and $\alpha \varepsilon R^{2n}$.

Let $s = (a^i_{j_1}, a^i_{j_1j_2})$

$t = (b^i_{j_1}, b^i_{j_1j_2})$

$\alpha = (u^1,\ldots,u^n, v^1,\ldots,v^n)$

$$(st).\alpha = [(a^i_{j_1},a^i_{j_1j_2}).(b^i_{j_1},b^i_{j_1j_2})].(u^1,\ldots,u^n,v^1,\ldots,v^n)$$

$$= (c^i_{j_1},c^i_{j_1j_2}).(u^1,\ldots,u^n,v^1,\ldots,v^n) \text{ where } c^i_{j_1} = a^i_k\, b^k_{j_1}$$

$$c^i_{j_1j_2} = a^i_k b^k_{j_1j_2} + a^i_{k_1k_2} b^{k_1}_{j_1} b^{k_2}_{j_2}$$

$$= (c^1_{j_1} u^{j_1},\ldots,c^n_{j_1} u^{j_1},\ c^1_{j_1} v^{j_1}+c^1_{j_1j_2} u^{j_1} u^{j_2},\ldots,c^n_{j_1} v^{j_1}+c^n_{j_1j_2} u^{j_1} u^{j_2})$$

$$= (a^1_k b^k_{j_1} u^{j_1},\ldots,a^n_k b^k_{j_1} u^{j_1},\ a^1_k b^k_{j_1} v^{j_1}+a^1_k b^k_{j_1j_2} u^{j_1} u^{j_2}+a^1_{k_1k_2} b^{k_1}_{j_1} b^{k_2}_{j_2} u^{j_1} u^{j_2},\ldots,$$

$$\ldots,\ a^n_k b^k_{j_1} v^{j_1}+a^n_k b^k_{j_1j_2} u^{j_1} u^{j_2}+a^n_{k_1k_2} b^{k_1}_{j_1} b^{k_2}_{j_2} u^{j_1} u^{j_2})$$

$$s(t.\alpha) = (a^i_{j_1},a^i_{j_1j_2}).[(b^i_{j_1},b^i_{j_1j_2}).(u^1,\ldots,u^n,v^1,\ldots,v^n)]$$

$$= (a^i_{j_1},a^i_{j_1j_2}).(x^1,\ldots,x^n,y^1,\ldots,y^n) \text{ where } x^m = b^m_{j_1} u^{j_1}$$

$$y^m = b^m_{j_1} v^{j_1}+b^m_{j_1j_2} u^{j_1} u^{j_2}$$

for $m = 1,\ldots,n$

$$s(t.\alpha) = (a^1_{j_1}x^{j_1},\dots,a^n_{j_1}x^{j_1},a^1_{j_1}y^{j_1}+ a^1_{j_1j_2}x^{j_1}x^{j_2},\dots,a^n_{j_1}y^{j_1}+ a^n_{j_1j_2}x^{j_1}x^{j_2})$$

$$= (a^1_{j_1}b^{j_1}_k u^k,\dots,a^n_{j_1}b^{j_1}_k u^k,\ a^1_{j_1}b^{j_1}_k v^k+ a^1_{j_1}b^{j_1}_{k_1k_2}u^{k_1}u^{k_2}$$

$$+ a^1_{j_1j_2}b^{j_1}_{k_1}u^{k_1}b^{j_2}_{k_2}u^{k_2},\dots,a^n_{j_1}b^{j_1}_k v^k+ a^n_{j_1}b^{j_1}_{k_1k_2}u^{k_1}u^{k_2}$$

$$+ a^n_{j_1j_2}b^{j_1}_{k_1}u^{k_1}b^{j_2}_{k_2}u^{k_2})$$

Hence S acts on R^{2n} as defined. Notice that the semigroup action comes from composition of functions and the action comes from the equation on page 20. R acts on R^{2n} as follows :

$$\alpha.(u^1,\dots,u^n,v^1,\dots,v^n) = (\alpha u^1,\dots,\alpha u^n,\alpha^2 v^1,\dots,\alpha^2 v^n)$$

To show this is an action, let α, $\beta \in R$, $(u^1,\dots,u^n,v^1,\dots,v^n) \in R^{2n}$

$$(\alpha.\beta).(u^1,\dots,u^n,v^1,\dots,v^n) = (\alpha\beta u^1,\dots,\alpha\beta u^n,\alpha^2\beta^2 v^1,\dots,\alpha^2\beta^2 v^n)$$

$$\alpha.[\beta.(u^1,\dots,u^n,v^1,\dots,v^n)] = \alpha.(\beta u^1,\dots,\beta u^n,\beta^2 v^1,\dots,\beta^2 v^n)$$

$$= (\alpha\beta u^1,\dots,\alpha\beta u^n,\alpha^2\beta^2 v^1,\dots,\alpha^2\beta^2 v^n)$$

Hence we see the map is an action.

Thus we see that equation (1.20) gives an action of $S = \{(\frac{\partial f^\alpha}{\partial x^i}, \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j})\}_{\alpha,j,i=1,..,n}$ on R^{2n} (denote $\frac{\partial f^\alpha}{\partial x^i}$ by $a^i_{j_1}$ and $\frac{\partial^2 f^\alpha}{\partial x^i\partial x^j}$ by $a^i_{j_1j_2}$) and we define

$\alpha.(\vec{u},\vec{v}) = (\alpha u^1,\dots,\alpha u^n,\alpha^2 v^1,\dots,\alpha^2 v^n)$ where $\alpha \in R$, $\vec{u}$, $\vec{v} \in R^n$ where $\vec{u}$ represents $\left.\frac{d\vec{X}}{dt}\right|_{t_0}$ and $\vec{v}$ represents $\left.\frac{d^2\vec{X}}{dt^2}\right|_{t_0}$.

Claim that this scalar product is invariant under change of coordinates.

That is, we want to show that $s(\alpha.x) = \alpha.(s\ x)$ for all $s \in S$, $x \in R^{2n}$, $\alpha \in R$

$$s(\alpha.x) = (a^i_{j_1}, a^i_{j_1 j_2})[\alpha.(u^1,\ldots,u^n,v^1,\ldots,v^n)]$$

$$= (a^i_{j_1}, a^i_{j_1 j_2})(\alpha u^1,\ldots,\alpha u^n,\alpha^2 v^1,\ldots,\alpha^2 v^n)$$

$$= (\alpha a^1_j u^j,\ldots,\alpha a^n_j u^j,\alpha^2 a^1_j v^j+\alpha^2 a^1_{j_1 j_2} u^{j_1} u^{j_2},\ldots$$

$$\ldots, \alpha^2 a^n_j v^j + \alpha^2 a^n_{j_1 j_2} u^{j_1} u^{j_2})$$

$$\alpha.(sx) = \alpha.[(a^i_{j_1}, a^i_{j_1 j_2})(u^1,\ldots,u^n, v^1,\ldots,v^n)]$$

$$= \alpha.(a^1_j u^j,\ldots,a^n_j u^j, a^1_j v^j + a^1_{j_1 j_2} u^{j_1} u^{j_2},\ldots$$

$$\ldots, a^n_j v^j + a^n_{j_1 j_2} u^{j_1} u^{j_2})$$

$$= (\alpha a^1_j u^j,\ldots, \alpha a^n_j u^j, \alpha^2 a^1_j v^j + \alpha^2 a^1_{j_1 j_2} u^{j_1} u^{j_2},\ldots$$

$$\ldots,\alpha^2 a^n_j v^j + \alpha^2 a^n_{j_1 j_2} u^{j_1} u^{j_2})$$

Thus the scalar multiplication defined above is invariant.

Since the change of coordinates is not a linear change, we cannot define $\alpha.(\vec{u},\vec{v}) = (\alpha u^1,\ldots,\alpha u^n,\alpha v^1,\ldots,\alpha v^n)$ as one would do in linear algebra because this definition would not be invariant with respect to the semigroup action. This explains why we use this definition of $\alpha.(\vec{u},\vec{v})$ which may have seemed strange when first given in the abstract.