## CHAPTER VI

## EXISTENCE OF PAIRS OF ORTHOGONAL LATIN SQUARES

### 6.1 Existence of Pairs of Orthogonal Latin Squares of Order $n$, $n = 4t+2$, $10 \leq n < 100$.

We shall show that $N(n) \geq 2$ for all $n = 4t+2$ in the range $10 \leq 4t+2 < 100$. By Corollary 4.2.3, we have $N(n) \geq 2$ for all $n$ of the form $n = 6t+4$. It follows that $N(n) \geq 2$ for $n = 10,22, 34,46,58,70,82,94$. Further, we have shown in Chapter IV that $N(18) \geq 2$. Hence it is left to be shown that $N(n) \geq 2$ for $n = 14,26,30,38,42,50,54,62,66,74,78,86,90,98$. The cases $n = 14, 26,38$, need special consideration. First we shall show the existence of pairs of orthogonal Latin squares of order 14. Consider the matrix

$$P_0 = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ 1 & 0 & 0 & 0 \\ 4 & 4 & 6 & 9 \\ 6 & 1 & 2 & 8 \end{bmatrix}$$

whose elements are residues modulo 11 and the three indeterminates $x_1$, $x_2$, $x_3$. Let $P_1$, $P_2$, $P_3$ be obtained from $P_0$ by cyclic permutation of the rows. Put $A_0 = (P_0, P_1, P_2, P_3)$ and let $A_i$ be obtained from $A_0$ by adding $i$ to every residue modulo 11 in $A_0$. Let

$$D = (E, A_0, A_1, \ldots, A_{10}, A^*),$$

where $A^*$ is an OA(3,4) on $x_1$, $x_2$, $x_3$ and E is the $4 \times 11$ matrix whose $i^{th}$ column contains i in every place. It can be verified that D is OA(14,4). Hence $N(14) \geq 2$.

(2) Existence of a pair of orthogonal Latin squares of order 26 can be shown by the same construction starting with the matrix

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & x_1 & x_2 & x_3 \\ 3 & 6 & 2 & 1 & 0 & 0 & 0 \\ 8 & 20 & 12 & 16 & 20 & 17 & 8 \\ 12 & 16 & 7 & 2 & 19 & 6 & 21 \end{pmatrix}$$

with objects taken from the residues modulo 23 and the indeterminates $x_1$, $x_2$, $x_3$ .

Next, we shall show the existence of a pair of orthogonal Latin squares of order 38. Observe that if a pairwise balanced design BIB(41,5,1) exists, then by Theorem 5.1.9, we have

$$N(41-3) = N(38) \geq \min \{ N(3), N(4)-1, N(5)-1 \} = 2 .$$

Hence, it suffices to show that a pairwise balanced design BIB(41,5,1) exists. We construct a pairwise balanced design BIB(41,5,1) as follows :

Take the elements of GF(41) as objects and our blocks are

$$A_t = \{ t, t+1, t+4, t+11, t+29 \} ,$$

$$B_t = \{ t+1, t+10, t+16, t+18, t+37 \} ,$$

where $t = 0,1,\ldots,40$. Observe that the two sets $\{0,1,4,11,29\}$ and $\{1,10,16,18,37\}$ together have the property that every nonzero element d of GF(41) is expressible in exactly one way as

$$d = x_i - x_j$$

where both $x_i$ and $x_j$ are either in the first set or both in the second set. Now, we shall verify that every pair of distinct objects occurs together in exactly one block of our design. Let u and v be distinct elements of GF(41). Since $u - v \neq 0$ hence there exist $x_i$, $x_j$ from the same set such that

$$u - v = x_i - x_j .$$

Put
$$t = u - x_i .$$

Then
$$u = x_i + t \quad \text{and} \quad v = x_j + t .$$

Hence both u and v are in $A_t$ or $B_t$ .

Thus every pair of distinct objects occurs together in exactly one block. Hence the design is a pairwise balanced design BIB(41,5,1).

Now, we consider the cases $n = 30,42,50,66,78,98$.

By Theorem 4.2.1, we have

$$N(30) \geqslant \min\{N(3),\ N(10)\} .$$

Since $N(3) \geqslant 2$ and $N(10) \geqslant 2$, hence $N(30) \geqslant 2$.

By the same argument we see that

$$N(42) = N(3.14) \geqslant \min\left\{N(3),\ N(14)\right\} \geqslant 2$$

$$N(50) = N(5.10) \geqslant \min\left\{N(5),\ N(10)\right\} \geqslant 2$$

$$N(66) = N(3.22) \geqslant \min\left\{N(3),\ N(22)\right\} \geqslant 2$$

$$N(78) = N(3.26) \geqslant \min\left\{N(3),\ N(26)\right\} \geqslant 2$$

$$N(98) = N(7.14) \geqslant \min\left\{N(7),\ N(14)\right\} \geqslant 2.$$

Finally, we shall show that $N(m) \geqslant 2$ for $n = 54,62,74,86,90$. By Theorem 5.2.6, we have

$$N(54) = N(4.11+10) \geqslant \min\left\{N(11), N(10), N(4)-1, N(4+1)-1\right\} .$$

Since $N(11) = 10$, $N(10) \geqslant 2$, $N(4) = 3$, $N(5) = 4$, hence

$$\min\left\{N(11),\ N(10),\ N(4)-1,\ N(4+1)-1\right\} \geqslant \min\left\{10, 2, 3-1, 4-1\right\} = 2.$$

Therefore $N(54) \geqslant 2$. By the same argument, we see that

$$N(62) = N(4.13+10) \geqslant \min\left\{N(13), N(10), N(4)-1, N(4+1)-1\right\} \geqslant 2.$$

$$N(74) = N(4.16+10) \geqslant \min\left\{N(16), N(10), N(4)-1, N(4+1)-1\right\} \geqslant 2.$$

$$N(86) = N(4.19+10) \geqslant \min\left\{N(19), N(10), N(4)-1, N(4+1)-1\right\} \geqslant 2.$$

$$N(90) = N(4.19+14) \geqslant \min\left\{N(19), N(14), N(4)-1, N(4+1)-1\right\} \geqslant 2.$$

6.2 <u>Existence of Pairs of Orthogonal Latin Squares of Order n, $n > 6$.</u>

We shall show that $N(n) \geqslant 2$ for all $n = 4t+2$ in the range $100 < 4t+2 \leqslant 726$. Observe that if n can be written in the form $n = 4m+x$, where $N(m) \geqslant 3$ and $N(x) \geqslant 2$, then Theorem 5.2.6 can be applied with $k = 4$ and we have

$$N(n) = N(4m+x) \geq \min\left\{ N(m), N(x), N(4)-1, N(5)-1 \right\} \geq 2.$$

Hence it suffices to demonstrate that each $n = 4t+2$ in the above range can be represented in this form. This can be shown by choosing suitable values of m and x. Note that for each n the choice of m determines the value of x. The following table shows how m should be chosen to guarantee that $N(m) \geq 3$ and $N(x) \geq 2$.

Table VI

| n | m | x |
|---|---|---|
| $102 \leq n \leq 114$ | 23 | $10 \leq x \leq 22$ |
| $118 \leq n \leq 134$ | 27 | $10 \leq x \leq 26$ |
| $138 \leq n \leq 154$ | 31 | $14 \leq x \leq 30$ |
| $158 \leq n \leq 182$ | 37 | $10 \leq x \leq 34$ |
| $186 \leq n \leq 218$ | 44 | $10 \leq x \leq 42$ |
| $222 \leq n \leq 262$ | 53 | $10 \leq x \leq 50$ |
| $266 \leq n \leq 318$ | 64 | $10 \leq x \leq 62$ |
| $322 \leq n \leq 382$ | 77 | $14 \leq x \leq 74$ |
| $386 \leq n \leq 458$ | 92 | $18 \leq x \leq 90$ |
| $462 \leq n \leq 562$ | 113 | $10 \leq x \leq 110$ |
| $556 \leq n \leq 694$ | 139 | $10 \leq x \leq 138$ |
| $698 \leq n \leq 726$ | 172 | $10 \leq x \leq 38$ |

6.2.1 Lemma. If $n = 4t+2$, $n \geq 730$, then there exist positive integers $g$, $u$ such that

$$n = 4(36g) + 4u + 10 ,$$

where $g \geq 5$, $0 \leq u \leq 35$.

Proof Since $n = 4t+2 \geq 730$, hence $t \geq \frac{728}{4} = 182$.

Thus $n = 4t_1 + 10$ where $t_1 = t - 2 \geq 180$.

By division algorithm, there exist integers $g$ and $u$ such that

$$t_1 = 36g + u ,$$

and $$0 \leq u \leq 35 .$$

From $t_1 \geq 180$, we have $g \geq \frac{180}{36} - \frac{u}{36} = 5 - \frac{u}{36}$.

Since $g$ is an integer and $\frac{u}{36} < 1$, hence $g \geq 5$.

Therefore, we have $n = 4(36g+u)+10$ where $g \geq 5$, $0 \leq u \leq 35$.

Q.E.D.

6.2.2 Theorem. For every $n > 6$, there exists a pair of orthogonal Latin squares of order $n$.

Proof It suffices to prove that $N(n) \geq 2$ for $n = 4t+2$, $n \geq 730$.

Since $n \geq 730$, by Lemma 6.2.1, we have

$$n = 4(36g) + 4u + 10 ,$$

where $g \geq 5$, $0 \leq u \leq 35$.

By Corollary 2.3.4, it can be seen that $N(36g) \geq 3$.

Since $0 \leqslant u \leqslant 35$, therefore $10 \leqslant 4u + 10 \leqslant 150$. Hence $N(4u+10) \geqslant 2$. As $g \geqslant 5$, we have $36g \geqslant 180$. Therefore $4u + 10 < 36g$. Apply Theorem 5.2.6 with $k = 4$, $m = 36g$, $x = 4u + 10$, we have

$$N(n) \geqslant \min \{ N(36g), N(4u+10), N(4)-1, N(5)-1 \}$$
$$\geqslant \min \{ 3,2,2,3 \} = 2.$$

Therefore there exists a pair of orthogonal Latin squares of order n, $n > 6$.

Q.E.D.