CHAPTER V

CONSTRUCTION OF SETS OF ORTHOGONAL LATIN SQUARES FROM BLOCK DESIGNS

5.1 Pairwise Balanced Design

5.1.1 Definition. Let $K = \{k_1, \ldots, k_m\}$ be a set of m distinct positive integers. Let v, λ be positive integers such that $k \leq v$ for all $k \in K$. A pairwise balanced design $BIB(v, k_1, \ldots, k_m, \lambda)$ is an arrangement of v objects into b blocks such that each block contains k distinct objects for some k belongs to K and every pair of distinct objects occurs in exactly λ blocks .

5.1.2 Remark. If b_i is the number of blocks with k_i elements then $\sum_{i=1}^{m} b_i = b$, $\lambda v(v-1) = \sum_{i=1}^{m} b_i k_i (k_i - 1)$.

Proof Clearly $\sum_{i=1}^{m} b_i = b$. To prove the other observe that there are $\binom{v}{2}$ distinct pairs and each pair of objects occurs in exactly λ blocks. Therefore $\lambda\binom{v}{2}$ is the number of pairs of objects that occurred altogether. On the other hand, we have b_i blocks of k_i distinct elements. This implies that each block contains $\binom{k_i}{2}$ distinct pairs, hence $b_i\binom{k_i}{2}$ is the number of pairs of objects that occurred in these b_i blocks. Therefore the total number of pairs of objects that can be occurred altogether is $\sum_{i=1}^{m} b_i\binom{k_i}{2}$.

Hence

$$\lambda\binom{v}{2} = \sum_{i=1}^{m} b_i \binom{k_i}{2} ,$$

which is equivalent to the required identity.

Q.E.D.

5.1.3 Definition. The b_i blocks with k_i elements of a pairwise balanced design BIB$(v,k_1,\ldots, k_m,\lambda)$ will be called the i^{th} equiblock component. A clear set is a set of equiblock components in which no two blocks have an element in common.

We shall write BIB$(v,k_1,\ldots, k_r; k_{r+1},\ldots, k_m,\lambda)$ to indicate that the first r equiblock components of BIB$(v,k_1,\ldots, k_m,\lambda)$ form a clear set.

5.1.4 Examples. The following are examples of pairwise balanced design BIB$(v,k_1,\ldots, k_m,\lambda)$ and pairwise balanced design BIB$(v,k_1,\ldots, k_r; k_{r+1},\ldots, k_m,\lambda)$. In these examples, we use $1,2,\ldots, v$ to denote objects and $B_1,\ldots, B_b$ to denote blocks. In (1) and (2) we give examples of designs without and with clear set respectively. Example (1) is constructed by the method of trial and error. Example (2) is constructed by the method in the proof of Theorem 5.1.9.

(1) Example of a pairwise balanced design BIB(6,2,3,1).

$B_1 = \{1, 2\}$, $B_2 = \{1, 3\}$

$B_3 = \{1, 4\}$, $B_4 = \{1, 5\}$

$B_5 = \{1, 6\}$, $B_6 = \{3, 5\}$

$B_7 = \{2, 5\}$, $B_8 = \{3, 6\}$

$B_9 = \{2, 6\}$, $B_{10} = \{2, 3, 4\}$

$B_{11} = \{4, 5, 6\}$.

(2) Example of a pairwise balanced design BIB(18,3;4,5,1).

$B_1 = \{9,12,13\}$, $B_2 = \{15,3,5\}$

$B_3 = \{16,4,6\}$, $B_4 = \{7,10,11,16\}$

$B_5 = \{8,11,12,17\}$, $B_6 = \{10,13,14,0\}$

$B_7 = \{11,14,15,1\}$, $B_8 = \{14,17,2,4\}$

$B_9 = \{17,0,5,7\}$, $B_{10} = \{0,1,6,8\}$

$B_{11} = \{1,2,7,9\}$, $B_{12} = \{2,3,8,10\}$

$B_{13} = \{3,6,7,12,14\}$, $B_{14} = \{4,7,8,13,15\}$

$B_{15} = \{5,8,9,14,16\}$, $B_{16} = \{6,9,10,15,17\}$

$B_{17} = \{12,15,16,0,2\}$, $B_{18} = \{13,16,17,1,3\}$

$B_{19} = \{0,3,4,9,11\}$, $B_{20} = \{1,4,5,10,12\}$

$B_{21} = \{2,5,6,11,13\}$.

5.1.5 Theorem. A finite projective plane of order n is a pairwise balanced design BIB(v,k,1) where $v = n^2 + n + 1$, $k = n + 1$.

Proof Take the points as objects and the lines as blocks. Since there are exactly $n^2 + n + 1$ points in plane. Then $v = n^2 + n + 1$. Every line contains exactly $n + 1$ points, then each block contains n + 1 points. Since each pair of points determines a unique line, hence $\lambda = 1$.

Q.E.D.

5.1.6 Theorem. A finite affine plane of order n is a pairwise balanced design BIB(v,k,1) where $v = n^2$, $k = n$.

Proof Take the points as objects and the lines as blocks. Since there are n^2 points in the plane and each line contains n points, hence $v = n^2$ and $k = n$. By axiom 1 of affine plane, any two distinct objects are contained in a unique line, therefore $\lambda = 1$.

Q.E.D.

5.1.7 Theorem. If there is a pairwise balanced design BIB($v,k_1,\ldots, k_r; k_{r+1},\ldots, k_m,1$), then

$$N(v) \geqslant \min\left\{N(k_1),\ldots,N(k_r),\ N(k_{r+1})-1,\ldots,N(k_m)-1\right\}.$$

Proof Let $c = \min\left\{N(k_1)+2,\ldots, N(k_r)+2,\ N(k_{r+1})+1,\ldots, N(k_m)+1\right\}$. Since for $i = 1,\ldots,r$, OA(k_1,c) exist. Let us denote OA(k_i,c) by A_i. Let the object of A_i be $1,\ldots,k_i$. For $i = r + 1,\ldots, m$, OA(k_i,c+1) exist. Let us denote OA(k_i,c+1) by D_i. Let the objects of D_i be $1,\ldots, k_i$. We may permute the columns of D_i so that the first k_i columns of D_i are of the form

$$\begin{matrix} 1 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & k_i \\ \cdot & \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 1 & 2 & \ldots & k_i \end{matrix}$$

From D_i, $i = r+1,\ldots,m$, we form matrices A_i by deleting the first row and first k_i columns of D_i. Note that A_i, $i = r+1,\ldots,m$ are $c \times (k_i^2 - k_i)$ matrix which have in every two rows all columns $\binom{u}{w}$ with $u \neq w$, $u, w = 1,\ldots, k_i$ as submatrices. Let the b blocks of the given pairwise balanced design be $B_1, B_2, \ldots, B_b$.

If B_j has k_i elements, then we form an array S_j from A_i as follows : replacing the numbers $1,\ldots,k_i$ of A_i by the objects in B_j. Form the array

$$C = (S_1, S_2, \ldots, S_b, E)$$

where E is an extra set of columns, each column of E consisting of the same objects repeated c times and E has one column for each object that did not appear in the blocks of the clear set.

We claim that the array C is an OA(v,c). Let any two rows be chosen. We shall show that for any objects u,w, the pair $\binom{u}{w}$ occurs as a submatrix of this two rows. First, let us consider the case $u \neq w$. There exists exactly one block B_j containing both elements and in the corresponding S_j, there will be a column $\binom{u}{w}$ occuring as a submatrix of the two rows. We shall show that u and w do not both occur in any other S, nor can both of them occur in a column of E. If u and w belong to some S's say S_k, $j \neq k$, then u, w belong to B_k which contradicts to the fact that u, w belong to exactly one block. Clearly u and w can not occur in the same column of E.

Next, consider the case u = w. If u is an element of the block B_j in the clear set, then the array S_j contains a pair $\binom{u}{u}$ occuring as a submatrix of the two chosen rows. If u is not the element of the block in the clear set, then the pair $\binom{u}{u}$ occurs as a submatrix of the two rows in E.

Hence any two rows are orthogonal and C is an OA(v,c). Therefore, there are at least c - 2 mutually orthogonal Latin squares of order v and $N(v) \geqslant c - 2$

$$= \min \left\{ N(k_1), \ldots, N(k_r), N(k_{r+1})-1, \ldots, N(k_m)-1 \right\}.$$

Q.E.D.

We shall illustrate the construction given in the proof of the above Theorem by using the pairwise balanced design BIB(18,3;4,5,1) of example (2) in Section 5.1.4. From this Theorem and the existence of the pairwise balanced design BIB(18,3;4,5,1), it follows that

$$N(18) \geqslant \min \left\{ N(3), N(4)-1, N(5)-1 \right\}.$$

By Theorem 2.1.1 and Corollary 2.1.4 we see that N(3) = 2, N(4) = 3, N(5) = 4. Hence

$$N(18) \geqslant \min \left\{ 2, 3-1, 4-1 \right\} = 2.$$

Hence the method of construction described in the above proof will give us a set of 2 mutually orthogonal Latin squares. To construct this pair of orthogonal Latin squares, let

$$\begin{aligned} c &= \min \left\{ N(3)+2, N(4)+1, N(5)+1 \right\} \\ &= \min \left\{ 4, 4, 5 \right\} \\ &= 4 . \end{aligned}$$

So that we need to construct OA(3,4), OA(4,4+1) and OA(5,4+1). By Theorem 4.1.3, we may construct these arrays from 2 orthogonal Latin squares of order 3, 3 orthogonal Latin squares of order 4 and 3 orthogonal Latin squares of order 5. These sets of orthogonal Latin squares are already given in Theorem 2.1.1 and Corollary 2.1.4. They are as shown below :

$$L_1 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$L_1' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad L_2' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix} \qquad L_3' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

$$L_1'' = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix} \qquad L_2'' = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix} \qquad L_3'' = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

Now, apply the method of constructing orthogonal arrays from orthogoanl Latin squares as decribed in the proof of Theorem 4.1.3 to the above sets of orthogonal Latin squares, we obtain the following arrays :

$$OA(3,4) = \begin{bmatrix} 1\,1\,1 & 2\,2\,2 & 3\,3\,3 \\ 1\,2\,3 & 1\,2\,3 & 1\,2\,3 \\ 1\,2\,3 & 3\,1\,2 & 2\,3\,1 \\ 1\,2\,3 & 2\,3\,1 & 3\,1\,2 \end{bmatrix}$$

$$OA(4,4+1) = \left[\begin{array}{cccccccccccccccccccc}
1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4\\
1&2&3&4&1&2&3&4&1&2&3&4&1&2&3&4\\
1&2&3&4&2&1&4&3&3&4&1&2&4&3&2&1\\
1&2&3&4&3&4&1&2&4&3&2&1&2&1&4&3\\
1&2&3&4&4&3&2&1&2&1&4&3&3&4&1&2
\end{array}\right]$$

$$OA(5,4+1) = \left[\begin{array}{ccccccccccccccccccccccccc}
1&1&1&1&1&2&2&2&2&2&3&3&3&3&3&4&4&4&4&4&5&5&5&5&5\\
1&2&3&4&5&1&2&3&4&5&1&2&3&4&5&1&2&3&4&5&1&2&3&4&5\\
1&2&3&4&5&2&3&4&5&1&3&4&5&1&2&4&5&1&2&3&5&1&2&3&4\\
1&2&3&4&5&3&4&5&1&2&5&1&2&3&4&2&3&4&5&1&4&5&1&2&3\\
1&2&3&4&5&4&5&1&2&3&2&3&4&5&1&5&1&2&3&4&3&4&5&1&2
\end{array}\right]$$

Hence

$$A_1 = \left[\begin{array}{ccccccccc}
1&1&1&2&2&2&3&3&3\\
1&2&3&1&2&3&1&2&3\\
1&2&3&3&1&2&2&3&1\\
1&2&3&2&3&1&3&1&2
\end{array}\right]$$

$$D_2 = \left[\begin{array}{cccccccccccccccc}
1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4\\
1&2&3&4&1&2&3&4&1&2&3&4&1&2&3&4\\
1&2&3&4&2&1&4&3&3&4&1&2&4&3&2&1\\
1&2&3&4&3&4&1&2&4&3&2&1&2&1&4&3\\
1&2&3&4&4&3&2&1&2&1&4&3&3&4&1&2
\end{array}\right]$$

$$D_3 = \left[\begin{array}{ccccccccccccccccccccccccc}
1&1&1&1&1&2&2&2&2&2&3&3&3&3&3&4&4&4&4&4&5&5&5&5&5\\
1&2&3&4&5&1&2&3&4&5&1&2&3&4&5&1&2&3&4&5&1&2&3&4&5\\
1&2&3&4&5&2&3&4&5&1&3&4&5&1&2&4&5&1&2&3&5&1&2&3&4\\
1&2&3&4&5&3&4&5&1&2&5&1&2&3&4&2&3&4&5&1&4&5&1&2&3\\
1&2&3&4&5&4&5&1&2&3&2&3&4&5&1&5&1&2&3&4&3&4&5&1&2
\end{array}\right]$$

By deleting the first row and first 4, 5 columns of D_2, D_3 respectively, we get

$$A_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \\ 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 \\ 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 & 3 & 4 & 5 & 1 & 2 & 4 & 5 & 1 & 2 & 3 & 5 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 1 & 2 & 5 & 1 & 2 & 3 & 4 & 2 & 3 & 4 & 5 & 1 & 4 & 5 & 1 & 2 & 3 \\ 4 & 5 & 1 & 2 & 3 & 2 & 3 & 4 & 5 & 1 & 5 & 1 & 2 & 3 & 4 & 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

Next, by using the BIB(18,3; 4,5,1) we form S_j as described in the above proof. We have

$$S_1 = \begin{pmatrix} 9 & 9 & 9 & 12 & 12 & 12 & 13 & 13 & 13 \\ 9 & 12 & 13 & 9 & 12 & 13 & 9 & 12 & 13 \\ 9 & 12 & 13 & 13 & 9 & 12 & 12 & 13 & 9 \\ 9 & 12 & 13 & 12 & 13 & 9 & 13 & 9 & 12 \end{pmatrix}$$

$$S_2 = \begin{pmatrix} 3 & 3 & 3 & 5 & 5 & 5 & 15 & 15 & 15 \\ 3 & 5 & 15 & 3 & 5 & 15 & 3 & 5 & 15 \\ 3 & 5 & 15 & 15 & 3 & 5 & 5 & 15 & 3 \\ 3 & 5 & 15 & 5 & 15 & 3 & 15 & 3 & 5 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 4 & 4 & 4 & 6 & 6 & 6 & 16 & 16 & 16 \\ 4 & 6 & 16 & 4 & 6 & 16 & 4 & 6 & 16 \\ 4 & 6 & 16 & 16 & 4 & 6 & 6 & 16 & 4 \\ 4 & 6 & 16 & 6 & 16 & 4 & 16 & 4 & 6 \end{pmatrix}$$

$$S_4 = \begin{pmatrix} 7 & 10 & 11 & 16 & 7 & 10 & 11 & 16 & 7 & 10 & 11 & 16 \\ 10 & 7 & 16 & 11 & 11 & 16 & 7 & 10 & 16 & 11 & 10 & 7 \\ 11 & 16 & 7 & 10 & 16 & 11 & 10 & 7 & 10 & 7 & 16 & 11 \\ 16 & 11 & 10 & 7 & 10 & 7 & 16 & 11 & 11 & 16 & 7 & 10 \end{pmatrix}$$

$$s_5 = \begin{pmatrix} 8 & 11 & 12 & 17 & 8 & 11 & 12 & 17 & 8 & 11 & 12 & 17 \\ 11 & 8 & 17 & 12 & 12 & 7 & 8 & 11 & 17 & 12 & 11 & 8 \\ 12 & 17 & 8 & 11 & 17 & 12 & 11 & 8 & 11 & 8 & 17 & 12 \\ 17 & 12 & 11 & 8 & 11 & 8 & 17 & 12 & 12 & 17 & 8 & 11 \end{pmatrix}$$

$$s_6 = \begin{pmatrix} 0 & 10 & 13 & 14 & 0 & 10 & 13 & 14 & 0 & 10 & 13 & 14 \\ 10 & 0 & 14 & 13 & 13 & 14 & 0 & 10 & 14 & 13 & 10 & 0 \\ 13 & 14 & 0 & 10 & 14 & 13 & 10 & 0 & 10 & 0 & 14 & 13 \\ 14 & 13 & 10 & 0 & 10 & 0 & 14 & 13 & 13 & 14 & 0 & 10 \end{pmatrix}$$

$$s_7 = \begin{pmatrix} 1 & 11 & 14 & 15 & 1 & 11 & 14 & 15 & 1 & 11 & 14 & 15 \\ 11 & 1 & 15 & 14 & 14 & 15 & 1 & 11 & 15 & 14 & 11 & 1 \\ 14 & 15 & 1 & 11 & 15 & 14 & 11 & 1 & 11 & 1 & 15 & 14 \\ 15 & 14 & 11 & 1 & 11 & 1 & 15 & 14 & 14 & 15 & 1 & 11 \end{pmatrix}$$

$$s_8 = \begin{pmatrix} 2 & 4 & 14 & 17 & 2 & 4 & 14 & 17 & 2 & 4 & 14 & 17 \\ 4 & 2 & 17 & 14 & 14 & 17 & 2 & 4 & 17 & 14 & 4 & 2 \\ 14 & 17 & 2 & 4 & 17 & 14 & 4 & 2 & 4 & 2 & 17 & 14 \\ 17 & 14 & 4 & 2 & 4 & 2 & 17 & 14 & 14 & 17 & 2 & 4 \end{pmatrix}$$

$$s_9 = \begin{pmatrix} 0 & 5 & 7 & 17 & 0 & 5 & 7 & 17 & 0 & 5 & 7 & 17 \\ 5 & 0 & 17 & 7 & 7 & 17 & 0 & 5 & 17 & 7 & 5 & 0 \\ 7 & 17 & 0 & 5 & 17 & 7 & 5 & 0 & 5 & 0 & 17 & 7 \\ 17 & 7 & 5 & 0 & 5 & 0 & 17 & 7 & 7 & 17 & 0 & 5 \end{pmatrix}$$

$$s_{10} = \begin{pmatrix} 0 & 1 & 6 & 8 & 0 & 1 & 6 & 8 & 0 & 1 & 6 & 8 \\ 1 & 0 & 8 & 6 & 6 & 8 & 0 & 1 & 8 & 6 & 1 & 0 \\ 6 & 8 & 0 & 1 & 8 & 6 & 1 & 0 & 1 & 0 & 8 & 6 \\ 8 & 6 & 1 & 0 & 1 & 0 & 8 & 6 & 6 & 8 & 0 & 1 \end{pmatrix}$$

$$s_{11} = \begin{pmatrix} 1 & 2 & 7 & 9 & 1 & 2 & 7 & 9 & 1 & 2 & 7 & 9 \\ 2 & 1 & 9 & 7 & 7 & 9 & 1 & 2 & 9 & 7 & 2 & 1 \\ 7 & 9 & 1 & 2 & 9 & 7 & 2 & 1 & 2 & 1 & 9 & 7 \\ 9 & 7 & 2 & 1 & 2 & 1 & 9 & 7 & 7 & 9 & 1 & 2 \end{pmatrix}$$

$$S_{12} = \left[\begin{array}{cccccccccccc}
2 & 3 & 8 & 10 & 2 & 3 & 8 & 10 & 2 & 3 & 8 & 10 \\
3 & 2 & 10 & 8 & 8 & 10 & 2 & 3 & 10 & 8 & 3 & 2 \\
8 & 10 & 2 & 3 & 10 & 8 & 3 & 2 & 3 & 2 & 10 & 8 \\
10 & 8 & 3 & 2 & 3 & 2 & 10 & 8 & 8 & 10 & 2 & 3
\end{array}\right]$$

$$S_{13} = \left[\begin{array}{cccccccccccccccccccc}
3 & 6 & 7 & 12 & 14 & 3 & 6 & 7 & 12 & 14 & 3 & 6 & 7 & 12 & 14 & 3 & 6 & 7 & 12 & 14 \\
6 & 7 & 12 & 14 & 3 & 7 & 12 & 14 & 3 & 6 & 12 & 14 & 3 & 6 & 7 & 14 & 3 & 6 & 7 & 12 \\
7 & 12 & 14 & 3 & 6 & 14 & 3 & 6 & 7 & 12 & 6 & 7 & 12 & 14 & 3 & 12 & 14 & 3 & 6 & 7 \\
12 & 14 & 3 & 6 & 7 & 6 & 7 & 12 & 14 & 3 & 14 & 3 & 6 & 7 & 12 & 7 & 12 & 14 & 3 & 6
\end{array}\right]$$

$$S_{14} = \left[\begin{array}{cccccccccccccccccccc}
4 & 7 & 8 & 13 & 15 & 4 & 7 & 8 & 13 & 15 & 4 & 7 & 8 & 13 & 15 & 4 & 7 & 8 & 13 & 15 \\
7 & 8 & 13 & 15 & 4 & 8 & 13 & 15 & 4 & 7 & 13 & 15 & 4 & 7 & 8 & 15 & 4 & 7 & 8 & 13 \\
8 & 13 & 15 & 4 & 7 & 15 & 4 & 7 & 8 & 13 & 7 & 8 & 13 & 15 & 4 & 13 & 15 & 4 & 7 & 8 \\
13 & 15 & 4 & 7 & 8 & 7 & 8 & 13 & 15 & 4 & 15 & 4 & 7 & 8 & 13 & 8 & 13 & 15 & 4 & 7
\end{array}\right]$$

$$S_{15} = \left[\begin{array}{cccccccccccccccccccc}
5 & 8 & 9 & 14 & 16 & 5 & 8 & 9 & 14 & 16 & 5 & 8 & 9 & 14 & 16 & 5 & 8 & 9 & 14 & 16 \\
8 & 9 & 14 & 16 & 5 & 9 & 14 & 16 & 5 & 8 & 14 & 16 & 5 & 8 & 9 & 16 & 5 & 8 & 9 & 14 \\
9 & 14 & 16 & 5 & 8 & 16 & 5 & 8 & 9 & 14 & 8 & 9 & 14 & 16 & 5 & 14 & 16 & 5 & 8 & 9 \\
14 & 16 & 5 & 8 & 9 & 8 & 9 & 14 & 16 & 5 & 16 & 5 & 8 & 9 & 14 & 9 & 14 & 16 & 5 & 8
\end{array}\right]$$

$$S_{16} = \left[\begin{array}{cccccccccccccccccccc}
6 & 9 & 10 & 15 & 17 & 6 & 9 & 10 & 15 & 17 & 6 & 9 & 10 & 15 & 17 & 6 & 9 & 10 & 15 & 17 \\
9 & 10 & 15 & 17 & 6 & 10 & 15 & 17 & 6 & 9 & 15 & 17 & 6 & 9 & 10 & 17 & 6 & 9 & 10 & 15 \\
10 & 15 & 17 & 6 & 9 & 17 & 6 & 9 & 10 & 15 & 9 & 10 & 15 & 17 & 6 & 15 & 17 & 6 & 9 & 10 \\
15 & 17 & 6 & 9 & 10 & 9 & 10 & 15 & 17 & 6 & 17 & 6 & 9 & 10 & 15 & 10 & 15 & 17 & 6 & 9
\end{array}\right]$$

$$S_{17} = \left[\begin{array}{cccccccccccccccccccc}
0 & 2 & 12 & 15 & 16 & 0 & 2 & 12 & 15 & 16 & 0 & 2 & 12 & 15 & 16 & 0 & 2 & 12 & 15 & 16 \\
2 & 12 & 15 & 16 & 0 & 12 & 15 & 16 & 0 & 2 & 15 & 16 & 0 & 2 & 12 & 16 & 0 & 2 & 12 & 15 \\
12 & 15 & 16 & 0 & 2 & 16 & 0 & 2 & 12 & 15 & 2 & 12 & 15 & 16 & 0 & 15 & 16 & 0 & 2 & 12 \\
15 & 16 & 0 & 2 & 12 & 2 & 12 & 15 & 16 & 0 & 16 & 0 & 2 & 12 & 15 & 12 & 15 & 16 & 0 & 2
\end{array}\right]$$

$$S_{18} = \left[\begin{array}{cccccccccccccccccccc}
1 & 3 & 13 & 16 & 17 & 1 & 3 & 13 & 16 & 17 & 1 & 3 & 13 & 16 & 17 & 1 & 3 & 13 & 16 & 17 \\
3 & 13 & 16 & 17 & 1 & 13 & 16 & 17 & 1 & 3 & 16 & 17 & 1 & 3 & 13 & 17 & 1 & 3 & 13 & 16 \\
13 & 16 & 17 & 1 & 3 & 17 & 1 & 3 & 13 & 16 & 3 & 13 & 16 & 17 & 1 & 16 & 17 & 1 & 3 & 13 \\
16 & 17 & 1 & 3 & 13 & 3 & 13 & 16 & 17 & 1 & 17 & 1 & 3 & 13 & 16 & 13 & 16 & 17 & 1 & 3
\end{array}\right]$$

$$S_{19} = \begin{bmatrix} 0 & 3 & 4 & 9 & 11 & 0 & 3 & 4 & 9 & 11 & 0 & 3 & 4 & 9 & 11 & 0 & 3 & 4 & 9 & 11 \\ 3 & 4 & 9 & 11 & 0 & 4 & 9 & 11 & 0 & 3 & 9 & 11 & 0 & 3 & 4 & 11 & 0 & 3 & 4 & 9 \\ 4 & 9 & 11 & 0 & 3 & 11 & 0 & 3 & 4 & 9 & 3 & 4 & 9 & 11 & 0 & 9 & 11 & 0 & 3 & 4 \\ 9 & 11 & 0 & 3 & 4 & 3 & 4 & 9 & 11 & 0 & 11 & 0 & 3 & 4 & 9 & 4 & 9 & 11 & 0 & 3 \end{bmatrix}$$

$$S_{20} = \begin{bmatrix} 1 & 4 & 5 & 10 & 12 & 1 & 4 & 5 & 10 & 12 & 1 & 4 & 5 & 10 & 12 & 1 & 4 & 5 & 10 & 12 \\ 4 & 5 & 10 & 12 & 1 & 5 & 10 & 12 & 1 & 4 & 10 & 12 & 1 & 4 & 5 & 12 & 1 & 4 & 5 & 10 \\ 5 & 10 & 12 & 1 & 4 & 12 & 1 & 4 & 5 & 10 & 4 & 5 & 10 & 12 & 1 & 10 & 12 & 1 & 4 & 5 \\ 10 & 12 & 1 & 4 & 5 & 4 & 5 & 10 & 12 & 1 & 12 & 1 & 4 & 5 & 10 & 5 & 10 & 12 & 1 & 4 \end{bmatrix}$$

$$S_{21} = \begin{bmatrix} 2 & 5 & 6 & 11 & 13 & 2 & 5 & 6 & 11 & 13 & 2 & 5 & 6 & 11 & 13 & 2 & 5 & 6 & 11 & 13 \\ 5 & 6 & 11 & 13 & 2 & 6 & 11 & 13 & 2 & 5 & 11 & 13 & 2 & 5 & 6 & 13 & 2 & 5 & 6 & 11 \\ 6 & 11 & 13 & 2 & 5 & 13 & 2 & 5 & 6 & 11 & 5 & 6 & 11 & 13 & 2 & 11 & 13 & 2 & 5 & 6 \\ 11 & 13 & 2 & 5 & 6 & 5 & 6 & 11 & 13 & 2 & 13 & 2 & 5 & 6 & 11 & 6 & 11 & 13 & 2 & 5 \end{bmatrix}$$

To form E, we observe that 0,1,2,7,8,10,11,14,17 do not appear in any blocks of the clean set. Hence

$$E = \begin{bmatrix} 0 & 1 & 2 & 7 & 8 & 10 & 11 & 14 & 17 \\ 0 & 1 & 2 & 7 & 8 & 10 & 11 & 14 & 17 \\ 0 & 1 & 2 & 7 & 8 & 10 & 11 & 14 & 17 \\ 0 & 1 & 2 & 7 & 8 & 10 & 11 & 14 & 17 \end{bmatrix}$$

So that the orthogonal array

$$C = (S_1, \ldots, S_{21}, E)$$

can be obtained. This is an OA(18, 4).

Finally, we can construct 2 orthogonal Latin squares of order 18 from this orthogonal array by the method described in the proof of Theorem 4.1.3. The following table shows the resulting squares, one superimposed on the other.

Table II

0,0	6,8	12,15	4,9	11,3	7,17	8,1	17,5	1,6	3,11	13,14	9,4	16,2	14,10	10,13	2,16	15,12	5,7
8,6	1,1	7,9	13,16	5,10	12,4	0,8	9,2	6,0	2,7	4,12	14,15	10,5	17,3	15,11	11,14	3,17	16,13
16,15	9,7	2,2	8,10	14,17	6,11	13,5	1,9	10,3	7,1	3,8	5,13	15,16	11,6	17,4	0,12	12,0	4,14
11,9	17,16	10,8	3,3	9,11	5,5	7,12	14,6	2,10	0,4	8,2	4,0	6,14	16,17	12,7	15,15	1,13	13,1
9,3	12,10	17,14	0,11	4,4	10,12	6,6	8,13	15,7	11,0	1,5	3,9	5,1	7,15	2,17	13,8	16,16	14,2
17,7	10,4	13,11	15,5	1,12	3,15	11,13	0,17	9,14	16,8	12,1	2,6	4,10	6,2	8,16	5,3	14,9	7,0
1,8	8,0	11,5	14,12	16,6	2,13	4,16	12,14	0,1	10,15	17,9	13,2	3,7	5,11	7,3	9,17	6,4	15,10
5,17	2,9	9,1	12,6	15,13	17,0	3,14	7,7	13,15	1,2	11,16	16,10	14,3	4,8	6,12	8,4	10,11	0,5
6,1	0,6	3,10	10,2	13,7	16,14	1,0	4,15	8,8	14,16	2,3	12,17	17,11	15,4	5,9	7,13	9,5	11,12
4,11	7,2	1,7	11,4	3,0	14,8	17,15	2,1	5,16	9,9	15,17	0,3	12,12	13,13	16,5	6,10	8,14	10,6
14,13	5,12	8,3	2,8	12,5	4,1	15,9	16,11	3,2	6,17	10,10	7,16	1,4	0,14	13,0	17,6	11,7	9,15
3,4	15,14	6,13	9,0	0,9	13,6	5,2	10,16	17,12	4,3	16,7	11,11	8,17	2,5	1,15	14,1	7,10	12,8
15,2	4,5	0,16	7,14	10,1	1,10	14,7	6,3	11,17	13,12	5,4	17,8	9,13	12,9	3,6	16,0	2,15	8,11
10,14	16,3	5,6	1,17	8,15	11,2	2,11	15,8	7,4	12,13	14,0	6,5	13,9	9,12	0,10	4,7	17,1	3,16
13,10	11,15	4,17	6,7	17,2	9,16	12,3	3,12	16,9	8,5	0,13	15,1	7,6	10,0	14,14	1,11	5,8	2,4
12,16	14,11	16,12	5,15	7,8	15,3	10,17	13,4	4,13	17,10	9,6	1,14	2,0	8,7	11,1	3,5	0,2	6,9
2,12	13,17	15,0	17,13	6,16	8,9	16,4	11,10	14,5	5,14	7,11	10,7	0,15	3,1	9,8	12,2	4,6	1,3
7,5	3,13	14,4	16,1	2,14	0,7	9,10	5,0	12,11	15,6	6,15	8,12	11,8	1,16	4,2	10,9	13,3	17,17

5.1.8 <u>Theorem.</u> If there is a pairwise balanced design BIB(v,k,1), then (1) $N(v-1) \geqslant \min\{N(k-1), N(k)-1\}$,

(2) $N(v-x) \geqslant \min\{N(k-x), N(k-1)-1, N(k)-1\}$ for any x such that $1 < x < k$.

<u>Proof</u> First, we show that $N(v-1) \geqslant \min\{N(k-1), N(k)-1\}$. If we delete a single object from the pairwise balanced design BIB(v,k,1), then there are blocks of sizes k and k-1. The set of equiblock components of size k-1 forms a clear set. Hence we get a pairwise balanced design BIB(v-1,k-1;k,1).

Application of Theorem 5.1.7 gives $N(v-1) \geqslant \min\{N(k-1), N(k)-1\}$.

Next, let $1 < x < k$. We shall show that $N(v-x) \geqslant \min\{N(k-x), N(k-1)-1, N(k)-1\}$. Observe that if we delete x objects belonging to the same block of a pairwise balanced design BIB(v,k,1). Then there are blocks of sizes k-x, k-1 and k. Among these blocks, there is only one block of size k-x. This block alone forms a clear set. The set of equiblock components of size k-1 (alternately k) does not form a clear set. Hence we get a pairwise balanced design BIB(v-x, k-x; k-1,k,1). Application of Theorem 5.1.7 gives

$$N(v-x) \geqslant \min\{N(k-x), N(k-1)-1, N(k)-1\}.$$

Q.E.D.

5.1.9 <u>Theorem.</u> If there is a pairwise balanced design BIB(v,k,1), then $N(v-3) \geqslant \min\{N(k-2), N(k-1)-1, N(k)-1\}$.

<u>Proof</u> If we delete three objects x_1, x_2, x_3 not occuring in the same block from the pairwise balanced design BIB(v,k,1), then

there are blocks of sizes $k-2, k-1$ and k. Since any two distinct objects occur in exactly one block, therefore no two blocks can have more than one object in common. The three blocks of size $k-2$ which have been obtained by deleting $\{x_1,x_2\},\{x_2,x_3\},\{x_1,x_3\}$ have no object in common. Hence they form a clear set. The set of equiblock components of size $k-1$ (alternately k), does not form a clear set. Hence we get a pairwise balanced design BIB($v-3,k-2;\ k-1,k,1$). Application of Theorem 5.1.7 gives

$$N(v-3) \geqslant \min \left\{ N(k-2),\ N(k-1)-1,\ N(k)-1 \right\}.$$

Q.E.D.

5.1.10 <u>Definition</u>. A pairwise balanced design BIB(v,k,λ) is called <u>resolvable</u> if the blocks can be separated into r sets (replications) such that each set contains each object exactly once.

We shall also refer to such a design as a resolvable design with r replications.

<u>Example</u>. The lines of an affine plane of order n can be divided into n+1 sets of parallel lines. So that any affine plane of order n formsa resolvable design with n+1 replications.

5.1.11 <u>Theorem</u>. If there is a resolvable BIB($v,k,1$) with r replications, then

(1) $N(v+x) \geqslant \min \left\{ N(x),\ N(k)-1,\ N(k+1)-1 \right\}$ for $1 \leqslant x \leqslant r-1$.

(2) $N(v+r-1) \geqslant \min \left\{ N(r-1),\ N(k),\ N(k+1)-1 \right\}$.

(3) $N(v+r) \geqslant \min \left\{ N(r),\ N(k+1)-1 \right\}$.

<u>Proof</u> Let x be any integer such that $1 < x \leqslant r$. We form a new design from BIB(v,k,1) by forming the following blocks : To each block of the i^{th} replication add a new object y_i, $i = 1, \ldots, x$. Then add a new block $\{y_1, \ldots, y_x\}$. Clearly any pair of distinct objects occur in exactly one block.

First, let us consider the case $1 < x < r-1$. In this case, at least two replications are added by new objects and at least two replications are left unchanged. Since each replication contains each object exactly once, therefore some two blocks of size k (alternately k+1) have an object in common.

If x,k,k+1 are distinct, then there are blocks of sizes x, k and k+1. Since there is only one block of size x, i.e. $\{y_1, \ldots, y_x\}$. hence this block alone forms a clear set. Therefore we get a pairwise balanced design BIB(v+x,x;k,k+1,1). Application of Theorem 5.1.7 gives $N(v+x) \geqslant \min \{N(x), N(k)-1, N(k+1)-1\}$.

If $x = k$ (alternately k+1), then there are blocks of sizes k and k+1. In this case no block forms a clear set. Hence we get a pairwise balanced design BIB(v+x,k,k+1,1). Application of Theorem 5.1.7 gives $N(v+x) \geqslant \min \{N(k)-1, N(k+1)-1\}$.
Since $x = k$, hence $N(x) = N(k) > N(k) - 1$.
So that we have

$$\min \{N(k)-1, N(k+1)-1\} = \min \{N(k), N(k)-1, N(k+1)-1\}.$$

Hence $\qquad N(v+x) \geqslant \min \{N(x), N(k)-1, N(k+1)-1\}$.

Hence in any case for $1 < x < r-1$, we get $N(v+x) \geqslant \min \{N(x), N(k)-1, N(k+1)-1\}$.

For the case $x = r-1$, there is only one replication left unchanged and at least two replications are added by new objects. Each replication contains each object exactly once, therefore some two blocks of size $k+1$ have an object in common.

If $x,k,k+1$ are distinct, then there are blocks of sizes x, k and $k+1$. Note that there is only one block of size $r-1$. This block does not have any element in common with the replication which is left unchanged. Hence they form a clear set. Hence we get a pairwise balanced design BIB($v+r-1, r-1, k;\ k+1, 1$). Application of Theorem 5.1.7 gives

$$N(v+r-1) \geqslant \min\left\{ N(r-1),\ N(k),\ N(k+1)-1 \right\} .$$

If $x = k$, then there are blocks of sizes k and $k+1$. Since the blocks in the r^{th} replication and the new block $\{y_1, \ldots, y_x\}$ have the same size and have no objects in common, therefore the set of equiblock components of size k forms a clear set. Hence we get a pairwise balanced design BIB($v+r-1, k;\ k+1, 1$). Application of Theorem 5.1.7 gives

$$N(v+r-1) \geqslant \min\left\{ N(k),\ N(k+1)-1 \right\} .$$

Since $k = x = r - 1$, hence we have

$$N(v+r-1) \geqslant \min\left\{ N(r-1),\ N(k),\ N(k+1)-1 \right\} .$$

If $x = k+1$, then there are blocks of sizes k and $k+1$. Again, the set of equiblock components consisting of blocks of size k in the r^{th} replication forms a clear set. Hence we get a pairwise balanced design BIB($v+r-1, k;\ k+1, 1$). Application of Theorem 5.1.7 gives

$$N(v+r-1) \geqslant \min \left\{ N(k),\ N(k+1)-1 \right\} .$$

Since $k+1 = x = r-1$, hence $N(r-1) = N(k+1) > N(k+1)-1$. So that we have

$$\min \left\{ N(k),\ N(k+1)-1 \right\} = \min \left\{ N(r-1),\ N(k),\ N(k+1)-1 \right\} .$$

Hence

$$N(v+r-1) \geqslant \min \left\{ N(r-1),\ N(k),\ N(k+1)-1 \right\} .$$

In any case, we get $N(v+r-1) \geqslant \min \left\{ N(r-1),\ N(k),\ N(k+1)-1 \right\}$ for $x=r-1$

For the case $x = r$, no replication is left unchanged.
If $x \neq k+1$, then there are blocks of sizes $k+1$ and r. There is only one block of size r and this block forms a clear set. Hence we get a pairwise balanced design $BIB(v+r,r;\ k+1,1)$. Application of Theorem 5.1.7 gives

$$N(v+r) \geqslant \min \left\{ N(r),\ N(k+1)-1 \right\} .$$

If $x = k+1$, then all blocks are of size $k+1$. Hence we get a pairwise balanced design $BIB(v+r,k+1,1)$. Application of Theorem 5.1.7 gives

$$N(v+r) \geqslant N(k+1)-1$$

Since $r = k+1$, hence $N(r) = N(k+1) > N(k+1)-1$. Therefore, we have

$$N(v+r) \geqslant \min \left\{ N(r),\ N(k+1)-1 \right\} .$$

Q.E.D.

5.2 Group Divisible Designs

Another kind of designs intimately related to orthogonal arrays are called group divisible designs.

5.2.1 Definition. A group divisible design $GD(v;k,m;\lambda_1, \lambda_2)$ is an arrangement of v objects into blocks such that

(1) each block contains k objects,

(2) the v objects can be divided into ℓ disjoint sets $G_1, G_2, \ldots, G_\ell$ called groups, each containing m elements such that any two objects from the same group occur together in λ_1 blocks while any two objects from different groups occur together in λ_2 blocks.

We shall be concerned with group divisible design in which $\lambda_1 = 0$, $\lambda_2 = 1$ only.

5.2.2 Example. The following is an example of a group divisible design GD(6; 2,3; 0,1). In this example we use 1,...,6 to denote objects and $B_1, \ldots, B_9$ to denote blocks.

$B_1 = \{1, 2\}$, $B_2 = \{3, 4\}$

$B_3 = \{5, 6\}$, $B_4 = \{3, 2\}$

$B_5 = \{5, 4\}$, $B_6 = \{1, 6\}$

$B_7 = \{5, 2\}$, $B_8 = \{1, 4\}$

$B_9 = \{3, 6\}$.

Observe that if we let $G_1 = \{1,3,5\}$, $G_2 = \{2,4,6\}$, then any two objects from the same G_i do not occur together in any block while any pair of objects, one from G_1 and the other from G_2, occur together exactly in one block.

5.2.3 <u>Definition.</u> A group divisible design GD(v;k,m,0,1) is called <u>resolvable</u> if the blocks can be separated into r sets (replications) such that each set contains each objects exactly once.

5.2.4 <u>Theorem.</u> If $k \leq N(m)+1$, then there exists a resolvable GD(km; k,m; 0,1).

<u>Proof</u> We assume that there exists an orthogonal array OA(m, k+1). Let the objects in OA(m, k+1) be 1,..., m. We form a design as follows : Arrange the columns of OA(m, k+1) so that the last row is of the form

$$1,\ldots,1,\ 2,\ldots,2,\ldots,\ m,\ldots,m.$$

Now drop the last row. Replace the number i in the j row by the ordered pair (i,j). These ordered pairs (i,j) will be the km objects of our design. We take as blocks of the design the columns of these ordered pairs.

We claim that this design is resolvable GD(km; k,m; 0,1). First, we show that this design is GD(km; k,m; 0,1). Clearly, there are km objects and each block of the new design contains k objects. Let the km objects (i,j), i = 1,...,m, j = 1,..., k be partitioned into k groups $G_1,\ldots, G_k$ where

$$G_j = \left\{ (i,j) \mid i = 1,\ldots, m \right\}.$$

Thus each group contains m elements. Clearly the objects with the same coordinate never occur in the same block. Claim that the objects whose second coordinates differ occur together in exactly one block. Suppose the contrary, then there exist i, j, i', j' such that $j \neq j'$ and the pairs $(i,j), (i',j')$ occur together in two blocks. Then the corresponding original column $\binom{i}{i'}$ occurs twice, which is a contradiction. Hence any two objects from the same group never occur in the same block, i.e. $\lambda_1 = 0$ while any two objects from different groups occur together in exactly one block, i.e. $\lambda_2 = 1$. Next, we show that this design is resolvable.

Let the blocks of this design be divided into m sets in the same way as the original corresponding columns are divided into m sets of m each, i.e. the i^{th} set being that for which the entry in the last row of OA(m, k+1) is i. Note that the orthogonality of the $k+1^{st}$ rows to the others in OA(m, k+1) assures us that in each of the the m sets, we have each of the km pairs (i,j) exactly once .

Q.E.D.

5.2.5 <u>Theorem.</u> If there is a resolvable GD(v,k,m; 0,1) with r replications, then

(1) $N(v+x) \geq \min\{N(m), N(x), N(k)-1, N(k+1)-1\}$ for $1 < x < r$.

(2) $N(v+r) \geq \min\{N(m), N(r), N(k+1)-1\}$.

<u>Proof</u> Assume that there is a resolvable GD(v; k,m; 0,1) with r replications. Let $w_1, \ldots, w_v$ be distinct objects of the design. Let $B_1, \ldots, B_b$ denote blocks and $G_1, \ldots, G_\ell$ denote groups. Let x

be any integer such that $2 \leqslant x \leqslant r$. We form a new design with $v+x$ objects as follows : Let $\{w_1, \ldots, w_v, y_1, \ldots, y_x\}$ be the set of objects of the new design. Let us take the following as blocks of the new design

1) the set $X = \{y_1, \ldots, y_x\}$,

2) the groups $G_1, \ldots, G_\ell$,

3) the sets $\bar{B}_1, \ldots, \bar{B}_b$, where

$$\bar{B}_j = \begin{cases} B_j \cup \{y_i\} & \text{if } B_j \text{ is in the } i^{th} \text{ replication, } 1 \leqslant i \leqslant x. \\ B_j & \text{if } B_j \text{ is not in any of the first } x \text{ replications.} \end{cases}$$

We shall show that any two distinct objects occur in exactly one block. Let p, q be any two objects of the new design.

<u>Case (i)</u> $\{p, q\} = \{y_i, y_j\}$, $i \neq j$, $i, j = 1, \ldots, x$.

Clearly the pair $\{y_i, y_j\}$ already occurs in the block X.

<u>Case (ii)</u> $\{p, q\} = \{w_i, w_j\}$, $i \neq j$, $i, j = 1, \ldots, v$.

<u>Case (ii a)</u> w_i, w_j are from the same group, say G_k for some k. Hence the pair $\{w_i, w_j\}$ occurs in the block G_k.

<u>Case (ii b)</u> w_i, w_j are from different groups. Hence the pair $\{w_i, w_j\}$ occurs in exactly one block B_t of GD(v; k,m; 0,1). So that $\{w_i, w_j\}$ occurs in $\bar{B}_t$.

<u>Case (iii)</u> $\{p, q\} = \{w_i, y_j\}$ $i = 1, \ldots, v$, $j = 1, \ldots, x$.

Since w_i occurs once in the j^{th} replication, hence the pair $\{w_i, y_j\}$ occurs in exactly one block of the form $\bar{B}_t = B_t \cup \{y_j\}$, where B_t is a block in the j^{th} replication.

To show (1) in the conclusion of the Theorem, we shall consider the following cases:

<u>Case 1</u> $2 \leq x \leq r - 2$.

<u>Case 2</u> $x = r - 1$.

<u>Case 1</u>. $2 \leq x \leq r-2$. In this case, at least two replications are added by new objects and at least two replications are left unchanged. Since the original design is resolvable, therefore some two blocks of size k (alternately k+1) have an object in common. There are 10 subcases to be considered.

Case 1.1. All x,m,k,k+1 are distinct. In this case there are blocks of sizes x,m,k and k+1. Among these blocks, there is only one block of size x. It is clear that this block together with the set of equiblock components of size m form a clear set. Hence we get a pairwise balanced design BIB(v+x,x,m;k,k+1,1). Application of Theorem 5.1.7 gives (1)

Case 1.2. x = m and x, k, k+1 are distinct. In this case there are blocks of sizes x,k,k+1. Since the block X and blocks $G_1, \ldots, G_\ell$ have the same size and have no object in common, therefore the set of equiblock components of size x forms a clear set, Hence we get a pairwise balanced design BIB(v+x,x; k,k+1,1). Application of Theorem 5.1.7 gives

$$N(v+x) \geq \min \{ N(x), N(k)-1, N(k+1)-1 \} .$$

Since $m = x$, therefore $N(m) = N(x)$. Hence

$$N(v+x) \geq \min \{ N(m), N(x), N(k)-1, N(k+1)-1 \} .$$

The remaining cases can be considered in similar manner. We summarize the result of our consideration in the following table :

Table III

Case	Conditions on x,m,k,k+1	Design Obtained
1	$x = m = k$	BIB(v+x,k,k+1,1)
2	$x = m = k+1$	BIB(v+x,k,k+1,1)
3	$x = m$, $x \neq k$, $x \neq k+1$	BIB(v+x,x; k,k+1,1)
4	$x \neq m$, $x = k$, $m = k+1$	BIB(v+x,k,k+1,1)
5	$x \neq m$, $x = k$, $m \neq k+1$	BIB(v+x,m; k,k+1,1)
6	$x \neq m$, $x = k+1$, $m = k$	BIB(v+x,k,k+1,1)
7	$x \neq m$, $x = k+1$, $m \neq k$	BIB(v+x, m; k,k+1,1)
8	$x \neq m$, $x \neq k$, $m = k+1$	BIB(v+x,x; m,k,1)
9	$x \neq m$, $x \neq k+1$, $m = k$	BIB(v+x,x; k,k+1,1)
10	x,m,k,k+1 are distinct	BIB(v+x,x,m; k,k+1,1)

When Theorem 5.1.7 is applied to each case, we can conclude that

$$N(v+x) \geq \min \{ N(m), N(x), N(k)-1, N(k+1)-1 \} .$$

Case 2 $x = r-1$. In this case, exactly one replication is left unchanged (i.e. the r^{th} replication). No two blocks in the r^{th} replication have an object in common.

To determine the design obtained, there are 10 subcases to be considered. They are summarized in the following table :

Table IV

Case	Conditions on x,m,k,k+1	Design Obtained
1	$x = m = k$	BIB(v+x,k,k+1,1)
2	$x = m = k+1$	BIB(v+x,k; k+1,1)
3	$x = m,\ x \neq k,\ x \neq k+1$	BIB(v+x,x; k,k+1,1)
4	$x \neq m,\ x = k,\ m = k+1$	BIB(v+x,k; k+1,1)
5	$x \neq m,\ x = k,\ m \neq k+1$	BIB(v+x,m; k,k+1,1)
6	$x \neq m,\ x = k+1,\ m = k$	BIB(v+x,k,k+1,1)
7	$x \neq m,\ x = k+1,\ m \neq k$	BIB(v+x,m; k, k+1,1)
8	$x \neq m,\ x \neq k,\ m = k+1$	BIB(v+x,x,k; k+1,1)
9	$x \neq m,\ x \neq k+1,\ m = k$	BIB(v+x,x; k,k+1,1)
10	x,m,k,k+1 are distinct	BIB(v+x,x,m; k,k+1,1)

When Theorem 5.1.7 is applied to each case, we can conclude that

$$N(v+x) \geq \min \left\{ N(m),\ N(x),\ N(k)-1,\ N(k+1)-1 \right\}.$$

Next, we shall show (2). To do this we let $x = r$. In this case, no replication is left unchanged. To determine the design obtained, there are 5 subcases to be considered. They are summarized in the following table :

Table V

Case	Conditions on r,m,k+1	Design Obtained
1	$r = m = k+1$	BIB(v+r, k+1, 1)
2	$r = m,\ m \neq k+1$	BIB(v+r, m; k+1,1)
3	$r \neq m,\ m \neq k+1,\ r = k+1$	BIB(v+r, m; k+1,1)
4	$r \neq m,\ m = k+1$	BIB(v+r, r; k+1,1)
5	r, m, k+1 are distinct	BIB(v+r, r, m; k+1,1)

When Theorem 5.1.7 is applied to each case, we can conclude that

$$N(v+r) \geqslant \min \left\{ N(r),\ N(m),\ N(k+1)-1 \right\}.$$

Q.E.D.

Theorem 5.2.4 and 5.2.5 may be combined to yield the important result.

5.2.6 <u>Theorem.</u> If $k \leqslant N(m)+1$, then for $1 < x < m$,

$$N(km+x) \geqslant \min \left\{ N(m),\ N(x),\ N(k)-1,\ N(k+1)-1 \right\}.$$

<u>Proof</u> By Theorem 5.2.4, we get a resolvable GD(km; k,m; 0,1) with m replications. By (1) of Theorem 5.2.5, we obtain

$$N(km+x) \geqslant \min \left\{ N(m),\ N(x),\ N(k)-1,\ N(k+1)-1 \right\}.$$

Q.E.D.