INTRODUCTION

Let S be a semigroup. For $T \subseteq S$, T is a *subsemigroup* of S if T forms a semigroup under the same operation on S. For a nonempty subset A of S, let

$$\langle A\rangle = \{a_1a_2\ldots a_n \mid a_i \in A,\ n \in \{1,2,3,\ldots\}\},$$

then $\langle A\rangle$ is a subsemigroup of S and it is called *the subsemigroup of S generated by A*. For $a \in S$, let $\langle a\rangle$ denote $\langle\{a\}\rangle$ and it is called *the cyclic subsemigroup of S generated by a*.

Let S be a semigroup. An element a of S is called an *idempotent* of S if $a^2 = a$. For a semigroup S, let E(S) denote the set of all idempotents of S, that is,

$$E(S) = \{a \in S \mid a^2 = a\}.$$

A semigroup S is a *band* if each element of S is an idempotent of S. Hence, a semigroup S is a band if and only if S = E(S). A commutative band is a *semilattice*.

An element z of a semigroup S is called a *left* [right] *zero* of S if $zx = z$ [$xz = z$] for every $x \in S$. An element z of S is called a *zero* of S if it is both a left zero and a right zero of S. An element e of a semigroup S is called a *left* [right] *identity* of S if $ex = x$ [$xe = x$] for all x in S. An element e of S is called an *identity* of S if it is both a left identity and a right identity of S.

A zero and an identity of a semigroup are unique if exist and they are usually denoted by 0 and 1, respectively.

A nonempty subset G of a semigroup S is a <u>subgroup</u> of S if it is a group under the same operation of S.

Let S be a semigroup with identity 1. An element a of S is called a <u>unit</u> of S if there exists $a' \in S$ such that $aa' = a'a = 1$. Let G be the set of all units of S, that is,

$$G = \{a \in S \mid aa' = a'a = 1 \text{ for some } a' \in S\}.$$

Then G is the greatest subgroup of S having 1 as its identity, and it is called <u>the group of units</u> or <u>the unit group</u> of the semigroup S.

Let S be a semigroup. The <u>order</u> of S is the number of elements in S if S is finite, otherwise S is of <u>infinite order</u>. For $a \in S$, the <u>order</u> of a is the order of <a>.

A semigroup all of whose elements are of finite order is <u>periodic</u>. Hence, a semigroup S is periodic if and only if for all $a \in S$, $a^m = a^n$ for some $m, n \in \{1,2,3...\}$ such that $m \neq n$. Then a group G is a periodic group if and only if for each element a in G, there exists $k \in \{1,2,3...\}$ such that $a^k = 1$ where 1 is the identity of G.

An element a of a semigroup S is <u>regular</u> if $a = axa$ for some $x \in S$. A semigroup S is <u>regular</u> if every element of S is regular.

Let a be an element of a semigroup S. An element x of S is an <u>inverse</u> of a if $a = axa$ and $x = xax$. A semigroup S is an <u>inverse semigroup</u> if every element of S has a unique inverse, and the unique inverse of the element a in S is denoted by a^{-1}. A semigroup S is an inverse semigroup if and only if S is regular and any two idempotents of S commute [2, Theorem 1.17]. Hence, if S is an inverse semigroup, then E(S) is a semilattice.

Let S be a semigroup, A a nonempty subset of S. Then A is called a <u>left</u> [right] <u>ideal</u> of S if $SA \subseteq A$ [$AS \subseteq A$]. We call A an <u>ideal</u> of S if A is both a left ideal and a right ideal of S.

A semigroup S is <u>left simple</u> [right simple, simple] if S is the only left ideal [right ideal, ideal] of S. Hence, a semigroup S is left simple [right simple, simple] if and only if Sa = S [aS = S, SaS = S] for all $a \in S$.

A semigroup S is <u>left cancellative</u> if for a, b, $x \in S$, xa = xb implies a = b. A <u>right cancellative semigroup</u> is defined dually. A cancellative semigroup is a semigroup which is both left cancellative and right cancellative.

Let S and T be semigroups and ψ a map from S into T. The map ψ is <u>homomorphism</u> from S into T if

$$(ab)\psi = (a\psi)(b\psi)$$

for all a, $b \in S$. A semigroup T is a <u>homomorphic image</u> of a semigroup S if there exists a homomorphism from S onto T. A homomorphism ψ from S into T is an <u>isomorphism</u> if ψ is one-to-one. If there is an isomorphism from S onto T, we say that the semigroup S and T are <u>isomorphic</u>, and we write $S \cong T$.

Let S be a semigroup. A relation ρ on S is called <u>left compatible</u> if for a, b, c ε S, $a \rho b$ implies $ca \rho cb$. A <u>right compatibility</u> is defined dually. An equivalence relation ρ on S is called a <u>congruence</u> on S if it is both left compatible and right compatible. Then an equivalence relation ρ on S is a congruence on S if and only if for a, b, c ε S, $a \rho b$ imply $ca \rho cb$ and $ac \rho bc$, or equivalently, for a, b, c, d ε S, $a \rho b$ and $c \rho d$ implies $ac \rho bd$. If ρ is a congruence on a semigroup S, then the set

$$S/\rho = \{a\rho \mid a \in S\}$$

with the operation defined by

$$(a\rho)(b\rho) = (ab)\rho \qquad (a, b \in S)$$

is a semigroup, and it is called the <u>quotient semigroup</u> relative to the congruence ρ.

Let P be a nonempty set and $\leqslant$ a relation on P. If the relation $\leqslant$ is reflexive, antisymmetric and transitive, then $\leqslant$ is called a <u>partial order</u> on P, and $(P, \leqslant)$ or P is called a <u>partially ordered set</u>.

A partially ordered set $(P, \leqslant)$ is called a <u>chain</u> if $a \leqslant b$ or $b \leqslant a$ for all a, b ε P.

Let S be a semigroup, and let 0 be a symbol not representing any element of S. The notation $S \cup 0$ denotes the semigroup obtained by extending the binary operation on S to 0 by $0.0 = 0$ and $0.a = a.0 = 0$ for all a ε S, and the notation S^0 denotes the following semigroup :

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero,} \\ S \cup 0 & \text{if } S \text{ has no zero.} \end{cases}$$

Similarly, let S be a semigroup and 1 a symbol not representing any element of S. The notation $S \cup 1$ denotes the semigroup obtained by extending the binary operation on S to 1 by defining $1.1 = 1$ and $1.a = a.$ $a.1 = a$ for all $a \in S$, and the notation S^1 denotes the following semigroup :

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup 1 & \text{if } S \text{ has no identity.} \end{cases}$$

By a group with zero we mean $G \cup 0$ for some group G .

Let S be a semigroup. Define the relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ on S as follow :

$$a \mathcal{L} b \iff S^1 a = S^1 b ,$$
$$a \mathcal{R} b \iff aS^1 = bS^1$$

and

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} .$$

The relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are called <u>Green's</u> <u>relations</u> on S and they are equivalence relations on S. Moreover, $\mathcal{L}$ is right compatible, $\mathcal{R}$ is left compatible, $\mathcal{H} \subseteq \mathcal{L}$ and $\mathcal{H} \subseteq \mathcal{R}$. Equivalent definitions of the Green's relations $\mathcal{L}$ and $\mathcal{R}$ on S are given as follow :

$$a \mathcal{L} b \iff a = xb, \ \ b = ya \ \text{ for some } x,y \in S^1,$$
$$a \mathcal{R} b \iff a = bx, \ \ b = ay \ \text{ for some } x,y \in S^1.$$

If S is a regular semigroup and $a, b \in S$, then

$$a \mathcal{L} b \Longleftrightarrow a = xb,\ b = ya \quad \text{for some } x,y \in S,$$

$$a \mathcal{R} b \Longleftrightarrow a = bx,\ b = ay \quad \text{for some } x,y \in S.$$

For a semigroup S and for $a \in S$, let L_a, R_a and H_a denote the $\mathcal{L}$- class of S containing a, the $\mathcal{R}$- class of S containing a and the $\mathcal{H}$- class of S containing a, respectively.

In any semigroup S, any $\mathcal{H}$- class of S contains at most one idempotent [2, Lemma 2.15], an $\mathcal{H}$- class of S containing an idempotent e of S is a subgroup of S [2, Theorem 2.16], and it is the greatest subgroup of S having e as its identity.

If S is an inverse semigroup, then each $\mathcal{L}$- class and each $\mathcal{R}$- class of S contains exactly one idempotent [2, Theorem 1.17].

For any set A, let $|A|$ denote the cardinality of A.

Let X be a set. A <u>partial transformation</u> of X is a map which its domain and its range are subsets of X. If α is a partial transformation of X, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain and the range of α, respectively. The <u>empty transformation</u> of X is referred as a map with empty domain, and it is denoted by 0. Let T_X denote the set of all partial transformations of X including the empty transformation 0. For $\alpha,\beta \in T_X$, define the product $\alpha\beta$ as follows : If $\nabla\alpha \cap \Delta\beta = \emptyset$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \emptyset$, let $\alpha\beta$ be the composition of $\alpha|_{(\nabla\alpha \cap \Delta\beta)\alpha^{-1}}$ (α restricted to $(\nabla\alpha \cap \Delta\beta)\alpha^{-1}$) and $\beta|_{(\nabla\alpha \cap \Delta\beta)}$. Then for $\alpha,\beta \in T_X$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$ and $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$. Thus T_X is a semigroup under the operation defined above and it is called the <u>partial transformation semigroup</u> on the set X. Hence the empty transformation of X is the zero of T_X and the identity map on X which is denoted by 1_X is the identity of the semigroup T_X. For $\alpha \in T_X$, α is an idempotent of T_X if and only if $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$. Hence

$$E(T_X) = \{\alpha \in T_X \mid \nabla\alpha \subseteq \Delta\alpha \text{ and } x\alpha = x \text{ for all } x \in \nabla\alpha\}.$$

Let I_X denote the set of all 1-1 partial transformations of X, that is,

$$I_X = \{\alpha \in T_X \mid \alpha \text{ is one-to-one}\}.$$

Then I_X is an inverse subsemigroup of T_X with identity 1_X and zero 0, and it is called the <u>1-1 partial transformation semigroup</u> or the <u>symmetric inverse semigroup</u> on the set X. For $\alpha \in I_X$, the inverse map α^{-1} is the inverse of α in I_X, so $\Delta\alpha^{-1} = \nabla\alpha$, $\nabla\alpha^{-1} = \Delta\alpha$. By a <u>transformation</u> of a set X we mean a mapping of X into itself. Then an element $\alpha \in T_X$ is a transformation of X if and only if $\Delta\alpha = X$. Let $\mathcal{T}_X$ denote the set of all transformations of X, that is,

$$\mathcal{T}_X = \{\alpha \in T_X \mid \Delta\alpha = X\}.$$

Then $\mathcal{T}_X$ is a subsemigroup of T_X with identity 1_X and it is called the <u>full transformation semigroup</u> on the set X. The <u>permutation group</u> on X is denoted by G_X. Then

$$G_X = \{\alpha \in T_X \mid \Delta\alpha = \nabla\alpha = X \text{ and } \alpha \text{ is one-to-one}\}.$$

Observe that $G_X \subseteq I_X \subseteq T_X$ and $G_X \subseteq \mathcal{T}_X \subseteq T_X$. The semigroup of all one-to-one transformations of X and the semigroup of all onto transformations of X are denoted by M_X and E_X, respectively. Hence

$$M_X = \{\alpha : X \to X \mid \alpha \text{ is one-to-one}\}$$

and

$$E_X = \{\alpha : X \to X \mid \alpha \text{ is onto}\}.$$

Let $\alpha,\beta \in T_X$ with $|\nabla\alpha| = |\nabla\beta| = 1$. Assume $\nabla\alpha = \{a\}$ and $\nabla\beta = \{b\}$. Then $\alpha\beta = 0$ if $a \notin \Delta\beta$ and if $a \in \Delta\beta$, then $\alpha\beta = \gamma$ where $\Delta\gamma = \Delta\alpha$ and $\nabla\gamma = \{b\}$. Hence the set of all constant partial transformations of X and the set of all constant transformations of X form subsemigroups of T_X. We denote the semigroup of all constant partial transformations of X and the semigroup of all constant transformations of X by C_X and F_X, respectively. Hence

$$C_X = \{\alpha \in T_X \mid |\nabla\alpha| = 1\} \cup \{0\}$$

and $F_\emptyset = \{0\}$ and if $X \neq \emptyset$, then

$$F_X = \{\alpha \in \mathcal{T}_X \mid |\nabla\alpha| = 1\}.$$

The <u>shift</u> of a partial transformation α of X, $S(\alpha)$, is defined to be the set $\{x \in \Delta\alpha \mid x\alpha \neq x\}$. A partial transformation α of X is said to be <u>almost identical</u> if the shift of α is finite, that is, $|S(\alpha)| < \infty$. Let

$$U_X = \{\alpha \in T_X \mid \alpha \text{ is almost identical}\},$$

$$V_X = \{\alpha \in \mathcal{T}_X \mid \alpha \text{ is almost identical}\}$$

and

$$W_X = \{\alpha \in I_X \mid \alpha \text{ is almost identical}\}.$$

If $\alpha,\beta \in T_X$, then $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$. Hence, U_X, V_X and W_X are subsemigroups of T_X, $\mathcal{T}_X$ and I_X, respectively.

By a <u>transformation semigroup</u> on a set X, we mean a subsemigroup of the partial transformation semigroup on X.

A semigroup S is said to be <u>factorizable</u> if there exists a subgroup G of S such that $S = GE(S)$. A semigroup in which every subsemigroup is factorizable is called a <u>strongly factorizable semigroup</u>.

In the first chapter of this thesis, we study the algebraic structure of strongly factorizable semigroups and characterize strongly factorizable semigroups. To characterize strongly factorizable transformation semigroups on a set X in term of cardinality of X is the purpose of Chapter II. In the last chapter, we characterize maximal strongly factorizable subsemigroup of the symmetric inverse semigroup on a finite set.