CHAPTER IV

LINEAR OPERATOR THEORY OVER THE QUATERNIONS

All LNLS (RNLS) and NLS are metric space. Hence if V is a LNLS (RNLS,NLS) and $A \subseteq V$. Then A is compact if and only if A has the BW property.

<u>Definition</u> 4.1 Let V,W be LNLS's (RNLS's) and $T: V \longrightarrow W$ a left (right) linear map. Then T is said to be a completely continuous map if and only if $A \subseteq V$ bounded implies that T(A) is relatively compact. We shall abbreviate "completely continuous" by "c.c." some mathematicians use the termilogy compact operator. Clearly a c.c. map is continuous.

<u>Remarks</u>: (i) If W is finite dimensional left (right) vector space over $\mathbb{H}$ then continuous implies c.c. [5]

(ii) If V, W are ∞-dimensional LNLS (RNLS). Then continuous does not implies c.c.

<u>Example</u> 4.2 Let $V=W=\ell^2_{\mathbb{H}}$ and let I be the identity map which is continuous. Then $\overline{B(0,1)}$ is closed and bounded. However $\overline{B(0,1)}$ is not compact. To prove this we need only show that $\overline{B(0,1)}$ is not BW.

Consider the sequence $(e_n)_{n\in\mathbb{N}}$ where $e_n = (0,0,\ldots,\overset{n^{th}\text{ place}}{1},0,0,\ldots)$ for all $n \in \mathbb{N}$. Then $(e_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{B(0,1)}$. Since $m \neq n$

implies $d(e_m, e_n) = (2)^{1/2}$, $(e_n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence.

Let V be a LNLS (RNLS) and W a NLS then CC(V,W) will denote the set of all c.c. map from V to W which is a RNLS (LNLS). If V, W are LNLS's (RNLS's) then CC(V,W) is a LNLS (RNLS) over $\mathbb{R}$.

Theorem 4.3 CC(V,W) is a (left) right linear subspace of C(V,W).

Proof: Let $T_1, T_2 \in CC(V,W)$ and $\lambda, \beta \in \mathbb{H}$. Let $A \subseteq V$ be bounded. Must show that $(T_1\lambda)(A)$ is relatively compact. Since $x \longmapsto x\alpha$ is a homeomorphism for all $\alpha \in \mathbb{H}\setminus\{0\}$, $(T_1\lambda)(A)$ is relatively compact. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in T(A). Must show that $(w_n)_{n\in\mathbb{N}}$ has a convergent subsequence. For each $n\in\mathbb{N}$ there exists an $a_n \in A$ such that $w_n = T(a_n)$. Since $(T_1(a_n))_{n\in\mathbb{N}}$ is a sequence in $T_1(A)$ which is relative compact, there exists a convergent subsequence $(T_1(a_n^{(1)}))_{n\in\mathbb{N}}$ of $(T_1(a_n))_{n\in\mathbb{N}}$. Since $(T_2(a_n^{(1)}))_{n\in\mathbb{N}}$ is a sequence in $T_2(A)$ which is relative compact, there exists a convergent subsequence $(T_2(a_n^{(2)}))_{n\in\mathbb{N}}$ of $(T_2(a_n^{(1)}))_{n\in\mathbb{N}}$. Therefore $(T(a_n^{(2)}))_{n\in\mathbb{N}}$ converges since it is a sum of convergent sequence.

Remark: If V, W are LNLS's (RNLS's) then CC(V,W) is left (right) $\mathbb{R}$-linear subspace of C(V,W).

Theorem 4.4 CC(V,W) is closed subspace of C(V,W) if W is a Banach space.

<u>Proof</u>: Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in CC(V,W) such that $(F_n)_{n\in\mathbb{N}}$ converges to F. Must show that $F \in CC(V,W)$. Let $A \subseteq V$ be bounded.

Must show that F(A) is relative compact. Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in F(A). For each $n\in\mathbb{N}$ we can write $w_n = F(x_n)$ for some $x_n \in A$. Hence we must show that $(F(x_n))_{n\in\mathbb{N}}$ has a convergent subsequence. Since $(F_1(x_n))_{n\in\mathbb{N}}$ is a sequence in $F_1(A)$ which is relatively compact, there exists a convergent subsequence $(F_1(x_n^{(1)}))_{n\in\mathbb{N}}$. Since $(F_2(x_n^{(1)}))_{n\in\mathbb{N}}$ is a sequence in $F_2(A)$ which is relatively compact, there exists a convergent subsequence $(F_2(x_n^{(2)}))_{n\in\mathbb{N}}$. By induction we get that for each $k\in\mathbb{N}$ there exists a sequence $(x_m^{(k)})_{n\in\mathbb{N}}$ such that $(F_k(x_n^{(k)}))_{n\in\mathbb{N}}$ converges and $(F_k(x_n^{(k)}))_{n\in\mathbb{N}}$ is a subsequence of $(F_k(x_n^{(k-1)}))_{n\in\mathbb{N}}$. Consider the diagonal sequence $(x_1^{(1)}, x_2^{(2)}, \ldots)$. Then for each $m\in\mathbb{N}$ $(F_m(x_n^{(n)}))_{n\in\mathbb{N}}$ converges. Claim that $(F(x_n^{(n)}))_{n\in\mathbb{N}}$ converges. This claim finish the proof. Since W is a Banach space, must show that $(F(x_n^{(n)}))_{n\in\mathbb{N}}$ is a cauchy sequence. Since A is bounded, there exists a $M > 0$ such that $\|a\| \leq M$ for all $a \in A$. Since F_n converges to F, given $\varepsilon > 0$ there exists a N_ε such that $n > N_\varepsilon$ implies that $\|F_n - F\| < \varepsilon/3M$. Fix $k > N_\varepsilon$. Then $(F_k(x_n^{(n)}))_{n\in\mathbb{N}}$ is a convergent sequence. Hence $(F_k(x_n^{(n)}))_{n\in\mathbb{N}}$ is cauchy, so there exists a N_ε' such that $m, n > N_\varepsilon'$ implies $\|F_k(x_m^{(m)}) - F_k(x_n^{(n)})\| < \varepsilon/3$. Let

$N''_\varepsilon = \max\{N_\varepsilon, N'_\varepsilon\}$. Therefore if m, n $> N''_\varepsilon$ then

$$\|F(x_m^{(m)})-F(x_n^{(n)})\| \leqslant \|F(x_m^{(m)})-F_k(x_m^{(m)})\|+\|F_k(x_m^{(m)})-F_k(x_n^{(n)})\|$$
$$+\|F_k(x_n^{(n)})-F(x_n^{(n)})\|$$
$$\leqslant \|F-F_k\|\,\|x_m^{(m)}\|+\|F_k(x_m^{(m)})-F_k(x_n^{(n)})\|+\|F_k-F\|\,\|x_n^{(n)}\|\leqslant\varepsilon.$$

⋇

<u>Theorem</u> 4.5 CC(V,V) is a two side ideal in C(V,V).

<u>Proof</u>: Let $F\in CC(V,V)$ and $G\in C(V,V)$. Let $A\subseteq V$ be bounded. Since G is continuous, G(A) is bounded. Therefore F(G(A)) is relatively compact. Since A is bounded and F is c.c., F(A) is relatively compact. Since G is continuous and F(A) is relatively compact, G(F(A)) is relatively compact. Hence $G\circ F$, $F\circ G$ are c.c. By Theorem 4.3, CC(V,V) is left linear subspace of C(V,V). Hence we have theorem. ⋇

<u>Example</u> 4.6 Let $V=\ell^2_{\mathbb{H}}$ and $T(x_1,x_2,\ldots)=(\sum_{k=1}^{\infty} a_{1k}x_k, \sum_{k=1}^{\infty} a_{2k}x_k,\ldots)$ where $\sum_{j=1}^{\infty}\sum_{j=1}^{\infty}|a_{ji}|^2<\infty$. Then T is c.c.

<u>Proof</u>: Let $\varepsilon>0$. Then there exists a number p_ε such that $\sum_{j=p_{\varepsilon+1}}^{\infty}\sum_{k=1}^{\infty}|a_{jk}|^2<\varepsilon^2$. We define T_ε by $T_\varepsilon(x)=T_\varepsilon(x_1,x_2,\ldots)=$ $(\sum_{k=1}^{\infty} a_{1k}x_k,\ldots,\sum_{k=1}^{\infty} a_{p_\varepsilon k}x_k,0,0,0,\ldots)$. After the p_ε-the term all entries are 0. Then, since the range of each operator T_ε is finite

dimensional, each T_ε is completely continuous. Now

$$\|T_\varepsilon(x)-T(x)\|^2 = \sum_{j=p_\varepsilon+1}^{\infty} \left|\sum_{k=1} a_{jk}x_k\right|^2 \leq \sum_{j=p_\varepsilon+1}^{\infty}\left[\sum_{k=1}^{\infty} |a_{jk}|^2\right]\left[\sum_{k=1}^{\infty}|x_k|^2\right]$$

$$\leq \varepsilon^2\|x\|^2$$

or $\|T_\varepsilon - T\| < \varepsilon$. Thus T is c.c. by Theorem 3.4 ⋇

<u>Theorem</u> 4.7 Let V be a ∞-dimensional LNLS(RNLS) and $T:V\longrightarrow V$ a c.c. map. Then T can not have a continuous inverse.

<u>Proof</u>: Suppose not. Then T has a continuous inverse T^{-1}.

$I = TT^{-1}$ therefore $I: V\longrightarrow V$ is c.c. The closed unit ball $\overline{B(0,1)}$ is bounded in V therefore $I(\overline{B(0,1)}) = \overline{B(0,1)}$ is relative compact. Since V is ∞-dimensional, there exists a sequence $(x_n)_{n\in \mathbb{N}}$ of left linear independent vectors. Let W_n be the left linear subspace generated by $x_1, x_2, \ldots, x_n$. Then $x_{n+1} \notin W_n$ for all $n \in \mathbb{N}$, also W_n is closed for all $n \in \mathbb{N}$. Let $\alpha_n = d(x_{n+1}, W_n)$. Then $\alpha_n > 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ there exists a $x_n^* \in W_n$ such that $0 < d(x_n^*, x_{n+1}) < 2\alpha_n$. Now

$$d(x_{n+1}-x_n^*, W_n) = \inf_{w\in W_n}\left\{\|x_{n+1}-x_n^*-w\|\right\} = \inf_{w\in W_n}\left\{\|x_{n+1}-(x_n^*-w)\|\right\}$$

$$= \inf_{w\in W_n}\left\{\|x_{n+1}-w\|\right\} = d(x_{n+1}, W_n) = \alpha_n.$$

Let $y_1 = \frac{x_1}{\|x_1\|}$ and if $n > 1$ let $y_n = \frac{x_n - x^*_{n-1}}{\|x_n - x^*_{n-1}\|}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ therefore $(y_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{B(0,1)}$.

$$d(y_{n+1},W_n) = \inf_{w\in W_n}\{\|y_{n+1}-w\|\} = \inf_{w\in W_n}\left\{\left\|\frac{x_{n+1}-x^*_n}{\|x_{n+1}-x^*_n\|} - w\right\|\right\}$$

$$= \frac{1}{\|x_{n+1}-x^*_n\|}\inf_{w\in W_n}\{\|x_{n+1}-x^*_n-w\|\} = \frac{1}{\|x_{n+1}-x^*_n\|}\inf_{w\in W_n}\{\|x_{n+1}-w\|\}$$

$$= \frac{1}{\|x_{n+1}-x^*_n\|} d(x_{n+1},W_n) = \frac{\alpha_n}{\|x_{n+1}-x^*_n\|} > \frac{\alpha_n}{2\alpha_n} = \frac{1}{2}$$

Therefore $d(y_{n+1},W_n) > \frac{1}{2}$ for all $n \in \mathbb{N}$. Hence if $m < n$ then $y_m \in W_m \subseteq W_{n-1}$. Therefore $d(y_m,y_n) \geqslant d(y_n,W_{n-1}) > \frac{1}{2}$, so $(y_n)_{n\in\mathbb{N}}$ cannot have a convergent subsequence since it is not cauchy, a contradiction. ⋇

Let V, W be LNLS (RNLS) and $T: V \longrightarrow W$ a continuous left (right) linear map. Then there exists a natural projection map $T^*: W^* \longrightarrow V^*$ called the adjoin map defined as follows: if $\varphi \in W^*$ defined $[T^*(\varphi)](x) = \varphi(T(x))$. Then T^* is continuous right (left) linear map. In fact

$$\|T^*\| = \sup_{\|\varphi\|=1}\{\|T^*(\varphi)\|\} = \sup_{\|\varphi\|=1}\{\|\varphi\circ T\|\} \leqslant \sup_{\|\varphi\|=1}\{\|\varphi\|\,\|T\|\} = \|T\|$$

<u>Remark</u>: $\|T^*\| = \|T\|$

<u>Proof</u>: Suppose that $T \equiv 0$, then $T^* \equiv 0$. Hence $\|T^*\| = \|T\|$.

Assume that $T \not\equiv 0$. Fix $x_0 \in V \setminus \ker T$ and let $y_0 = \frac{1}{\|T(x_0)\|} \cdot T(x_0)$.

Hence $\|y_0\| = 1$. Let U be the left linear subspace of W generated by y_0. Define $F: U \longrightarrow \mathbb{H}$ by $F(\lambda y_0) = \lambda$. Clearly F is continuous a and left linear. In fact $\|F\| = 1$. By the Hahn-Banach Theorem there exists a continuous left linear map $\varphi: W \longrightarrow \mathbb{H}$ such that $\|\varphi\| = \|F\| = 1$ and $\varphi(x) = F(x)$ for all $x \in U$. Since $\varphi(T(x_0)) = (\|T(x_0)\| y_0) = \|T(x_0)\|$, we have that $\|T(x_0)\| = \|\varphi(T(x_0))\| = |[T^*(\varphi)](x_0)| \leq \|T^*(\varphi)\| \, \|x_0\| \leq \|T^*\| \, \|x_0\|$. Hence $\frac{\|T(x_0)\|}{\|x_0\|} \leq \|T^*\|$. If $x_0 \in \ker T \setminus \{0\}$, then $T(x) = 0$. Therefore $\frac{\|T(x)\|}{\|x\|} = 0 \leq \|T^*\|$. Hence $\|T\| = \|T^*\|$.

⋕

<u>Theorem</u> 4.8 Let V, W be LNLS's (RNLS's) and $T: V \longrightarrow W$ is a c.c. map. Then $T^*: W^* \longrightarrow V^*$ is a c.c. map.

<u>Proof</u>: We must show that if $A \subseteq W^*$ is bounded, then $T^*(A)$ is relatively compact in V^*. Since every bounded set is contained in a closed ball center at 0, it is sufficient to show that T^* of every closed ball center at 0 in W^* is relatively compact in V^*. Let $\overline{B^*(0,r)}$ be the closed ball center at 0 radius $r > 0$ in W^*. Then $\overline{B^*(0,r)} = \overline{B^*(0,1)}r$ therefore $T^*(\overline{B^*(0,r)}) = T^*((\overline{B^*(0,1)})r) = T^*(\overline{B^*(0,1)})r$. Since the map $x \longmapsto xr$ is a homeomorphism then if we show that $T^*(\overline{B^*(0,1)})$ is relatively compact in V^* we get that $T^*(\overline{B^*(0,r)})$ is relatively compact. Let $\overline{B(0,1)}$ be the closed ball center at 0 in V therefore $T(\overline{B(0,1)})$ is relatively compact in W,

hence $T(\overline{B(0,1)})$ is compact in W. Define a metric ρ on $\overline{T(B(0,1))}^{*}$ as follows: if $\varphi_1, \varphi_2 \in W^{*}$, then $\rho(\varphi_1,\varphi_2) = \sup_{y \in \overline{T(\overline{B(0,1)})}} \{|\varphi_1(y)-\varphi_2(y)|\}$.

Claim that $\overline{B^{*}(0,1)}$ is relatively compact with respect to the metric ρ. In order to prove the claim, we need only show that $\overline{B^{*}(0,1)}$ is uniformly bounded and equicontinuous by the Arzelá - Asocili Theorem.

1. To show that $\overline{B^{*}(0,1)}$ is uniformly bounded. Note that if $\varphi \in \overline{B^{*}(0,1)}$ then $\|\varphi\| \leq 1$. Let $\varphi \in \overline{B^{*}(0,1)}$. Claim that

$$\sup_{x \in \overline{T(\overline{B(0,1)})}} \{|\varphi(x)|\} = \sup_{x \in T(\overline{B(0,1)})} \{|\varphi(x)|\} \leq \sup_{x \in T(\overline{B(0,1)})} \{\|\varphi\| \|x\|\}.$$

Since φ is continuous on compact set $\overline{T(\overline{B(0,1)})}$, there exists an $z \in \overline{T(\overline{B(0,1)})}$ such that $|\varphi(z)| = \sup_{x \in \overline{T(\overline{B(0,1)})}} \{|\varphi(x)|\}$. If $z \in T(\overline{B(0,1)})$, then done. So assume that $z \notin T(\overline{B(0,1)})$. Then there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $T(\overline{B(0,1)})$ such that $(y_n)_{n\in\mathbb{N}}$ convergent to z. Since $|\varphi|$ is continuous function, $(|\varphi(y_n)|)_{n\in\mathbb{N}}$ converges to $|\varphi(z)|$, hence $\sup_{x \in T(\overline{B(0,1)})} \{|\varphi(x)|\} \geq |\varphi(z)|$. Clearly $\sup_{x \in T(\overline{B(0,1)})} \{|\varphi(x)|\} \leq \sup_{x \in \overline{T(\overline{B(0,1)})}} \{|\varphi(x)|\}$.

Hence we have the claim. Thus

$$\sup_{x \in \overline{T(\overline{B(0,1)})}} \{|\varphi(x)|\} = \sup_{x \in T(\overline{B(0,1)})} \{|\varphi(x)|\} \leq \sup_{x \in T(\overline{B(0,1)})} \{\|x\|\} =$$

$$\sup_{y \in \overline{B(0,1)}} \{\|T(y)\|\} = \sup_{\|y\| < 1} \{\|T(y)\|\} = \|T\|.$$

Since $\varphi \in \overline{B^{*}(0,1)}$ is arbitrary, we have that $\sup_{x \in \overline{T(\overline{B(0,1)})}} \{|\varphi(x)|\} \leq \|T\|$

for all $\varphi \in B^*(0,1)$. Hence $\overline{B^*(0,1)}$ is uniformly bounded with respect to ρ.

2. To show $B^*(0,1)$ is equicontinuous. Let $\varepsilon > 0$ be given and let $\delta_\varepsilon = \varepsilon$. Then for all $\varphi \in \overline{B^*(0,1)}$ and for all $x_1, x_2 \in \overline{T(B(0,1))}$ $\| x_1 - x_2 \| < \delta_\varepsilon$ implies that $|\varphi(x_1) - \varphi(x_2)| = |\varphi(x_1 - x_2)| \leq \|\varphi\| \|x_1 - x_2\| \leq \|x_1 - x_2\| < \varepsilon$. Hence $\overline{B^*(0,1)}$ is equicontinuous. Hence $\overline{B^*(0,1)}$ is relatively compact with respect to the metric ρ. Claim that the map $T^* : B^*(0,1) \longrightarrow V^*$ is an isometry where $B^*(0,1)$ has the metric ρ and V^* has the norm metric. To prove this, let $\varphi_1, \varphi_2 \in \overline{B^*(0,1)}$. Must show that $\|T^*(\varphi_1) - T^*(\varphi_2)\| = \rho(\varphi_1, \varphi_2)$

$$\|T^*(\varphi_1) - T^*(\varphi_2)\| = \sup_{x \in \overline{B(0,1)}} \left\{ \left| [T^*(\varphi_1)](x) - [T^*(\varphi_2)](x) \right| \right\}$$

$$= \sup_{x \in \overline{B(0,1)}} \left\{ \left| \varphi_1(T(x)) - \varphi_2(T(x)) \right| \right\}$$

$$= \sup_{x \in T\,\overline{B(0,1)}} \left\{ \left| (\varphi_1 - \varphi_2)(x) \right| \right\}$$

$$= \sup_{x \in \overline{T(\overline{B(0,1)})}} \left\{ \left| (\varphi_1 - \varphi_2)(x) \right| \right\}$$

$$= \sup_{z \in \overline{T(\overline{B(0,1)})}} \left\{ \left| \varphi_1(z) - \varphi_2(z) \right| \right\}$$

$$= \rho(\varphi_1, \varphi_2).$$

Therefore T^* is an isometry. Hence T^* is a homeomorphism of $\overline{B^*(0,1)}$ with the metric ρ onto $T^*(\overline{B^*(0,1)})$ with respect the norm metric. But

$B^*(0,1)$ is relatively compact with respect to the metric ρ . Hence $T^*(B^*(0,1))$ is relatively compact with respect to the norm metric. Hence T^* is c.c. ⋇

Theorem 4.9 Let V be a LNLS (RNLS) which is also a Banach space $T: V \longrightarrow V$ a c.c. map. If I-T is onto then I-T is 1-1.

Proof: Suppose not. Therefore I-T = A is onto but A is not 1-1. Hence there exists an $x_1 \in V \setminus \{0\}$ such that $A(x_1) = 0$. Given $n \in \mathbb{N}$ let $W_n = \ker A^n$. Then $W_n \subseteq W_{n+1}$ for all $n \in \mathbb{N}$. Claim that $W_n \subset W_{n+1}$ for all $n \in \mathbb{N}$. To prove this, note that since A is onto, there exists an x_2 such that $x_1 = A(x_2)$ and there exists an x_3 such that $A(x_3) = x_2$. By induction there exists an x_n such that $A^n(x_n) = x_{n-1}$ for all $n \in \mathbb{N} \setminus \{1\}$. Since $A^n(x_n) = A^{n-1}(A(x_n))$ $= A^{n-1}(x_{n-1}) = \dots = A(x) = 0$, $x_n \in W_n$. But $x_n \notin W_{n-1}$ since $A^{n-1}(x_n)$ $= A^{n-2}(A(x_n)) = \dots = A(x_2) = x_1 \neq 0$. Hence $W_{n-1} \subset W_n$ for all $n \in \mathbb{N}$. By the same argument as in Theorem 4.7 we can find in each W_n an element y_n such that $\| y_n \| = 1$ and $\| y_n - y \| > \frac{1}{2}$ for all $y \in W_{n-1}$. Then for all $k < n$ $\quad \| T(y_n) - T(y_k) \| = \| y_n - (I-T)(y_n) - y_k + (I-T(y_k) \| > \frac{1}{2}$ because $-(I-T)(y_n) - y_k + (I-T)(y_k)$ lies in W_{n-1}. This show that the sequence $(T(y_n))_{n \in \mathbb{N}}$ can not have a convergent subsequence contra dicting the fact that T is c.c. ⋇

Definition 4.10 Let V be a LNLS (RNLS) and $T: V \longrightarrow V$ a continuous left (right) linear map. Then $\lambda \in \mathbb{H}$ is said to be an eigenvalue of

T if and only if there exists an $v \in V \setminus \{0\}$ such that $T(v) = \lambda v$ $(T(v) = v\lambda)$.

<u>Definition</u> 4.11 Let V be a NLS and $T: V \longrightarrow V$ a continuous left (right) linear map. Then $\lambda \in \mathbb{H}$ is said to be a right (left) eigenvalue of T if and only if there exists an $v \in V \setminus \{0\}$ such that $T(v) = v\lambda$ $(T(v) = \lambda v)$.

<u>Definition</u> 4.12 Let V be a NLS and $T: V \longrightarrow V$ a continuous left linear map. Then $\lambda \in \mathbb{H}$ is said to be a right characteristic value of T if and only if the continuous left linear map $T - I\lambda$ has no continuous inverse.

Left characteristic value is defined dually.

<u>Remark</u>: Let V be a LNLS (RNLS) and $T: V \longrightarrow V$ a continuous left (right) linear map. Then $\lambda \in \mathbb{R}$ is said to be a characteristic value of T if and only if the left (right) linear map $T - \lambda I$ has no continuous inverse.

<u>Remark</u>: If V is a NLS which is also an ∞-left(right) dimensional and $\lambda \in \mathbb{H}$ is a left (right) eigenvalue of T, then λ is a(right) left characteristic value of T. The converse is not true.

<u>Proof</u>: Suppose that λ is a left eigenvalue of T. Then there exists an $v \in V \setminus \{0\}$ such that $T(v) = \lambda v$ therefore $(T-\lambda I)(v) = 0$, hence $T-\lambda I$ is not 1-1. Hence $T-\lambda I$ has no continuous inverse.

<u>Example</u> 4.13 Let $V = C_{\mathbb{H}}[a,b]$ with respect the supnorm. Fix a non constant function $f_0 \in C_{\mathbb{H}}[a,b]$. Define $T: C_{\mathbb{H}}[a,b] \longrightarrow C_{\mathbb{H}}[a,b]$ by $T(f)$

$= f \circ f_0$ therefore T is left linear. Let $M > 0$ be such that $|f_0(x)| \leq M$ for all $x \in [a,b]$. Hence if $f \in C_{\mathbb{H}}[a,b] \setminus \{0\}$, then $|(f \circ f_0)|(x) = |f(x)|\,|f_0(x)| \leq \|f\| M$ therefore $\|T\| \leq M$. Hence T is continuous. Let $\lambda \in \text{Im } f_0$. Then $(T - I\lambda)f = (f \circ f_0 - f\lambda) = f(f_0 - \lambda)$.

Let W be a nonzero LNLS (RNLS) and $F: W \longrightarrow W$ a continuous left (right) linear map. Claim that if F has a continuous inverse then there exists a $m > 0$ such that $m\|x\| \leq \|F(x)\|$ for all $x \in W$. To prove this, let $x \in W$. Therefore $\|x\| = \|I(x)\| = \|F^{-1}(F(x))\| \leq \|F^{-1}\|\|F(x)\|$. Let $m = \dfrac{1}{\|F^{-1}\|}$ therefore $m > 0$. Then $m\|x\| \leq \|F(x)\|$. Thus we have the claim. To show that $T - I\lambda$ has no continuous inverse i.e. there does not exist $m > 0$ such that $m\|f\| \leq \|(T - I\lambda)(f)\| = \|f(f_0 - \lambda)\|$ for all $f \in C_{\mathbb{H}}[a,b]$. To prove this, suppose not. Then there exists $m > 0$ such that $m\|f\| \leq \|f(f_0 - \lambda)\|$ for all $f \in C_{\mathbb{H}}[a,b]$. Since $\lambda \in \text{Im } f_0$, there exists an $t_0 \in [a,b]$ such that $f_0(t_0) = \lambda$. Choose $\varepsilon > 0$ such that $\varepsilon < m$. Then there exists a $\delta_\varepsilon > 0$ such that $|t - t_0| < \delta_\varepsilon$ implies $|f_0(t) - f_0(t_0)| = |f_0(t) - \lambda| < \varepsilon/2$. Let $x = \sup\{a \leq t < t_0 \,/\, |t - t_0| \geq \delta_\varepsilon\}$ and $y = \inf\{t_0 < t \leq b \,/\, |t - t_0| \geq \delta_\varepsilon\}$. Choose a continuous map $g \in C_{\mathbb{H}}[a,b]$ such that

$$g(t) = \begin{cases} 0 & \text{if } |t - t_0| \geq \delta_\varepsilon \\ \dfrac{x - t}{x - t_0} & \text{if } |t - t_0| < \delta_\varepsilon \text{ and } x < t \leq t_0 \\ \dfrac{y - t}{y - t_0} & \text{if } |t - t_0| < \delta_\varepsilon \text{ and } t_0 \leq t \leq b. \end{cases}$$

Then $\|g\| = 1$ and $\|g(f_0-\lambda)\| = \sup_{t\in[a,b]} \{|g(t)(f_0(t)-\lambda)|\} < \varepsilon$. But $m = m\|g\| \leqslant \|g(f_0-\lambda)\| < \varepsilon$, so a contradiction. Hence we have the claim. Thus $T-I\lambda$ has no continuous inverse. ⋇

<u>Theorem</u> 4.14 Let V be a NLS and $F: V \to V$ a c.c. left (right) linear map. Then if $\lambda \in \mathbb{H}$ is a nonzero right (left) eigenvalue of F the left (right) linear subspace generated by the eigenvectors of λ is finite dimensional.

Let V be a LNLS (RNLS) and $T: V \longrightarrow V$ a c.c. map. We want to show that if I-T is 1-1, then I-T is onto. In order to prove this we'll need some lemmas.

<u>Lemma</u> 4.15 Let V be a LNLS (RNLS) which is also a Banach space and $T: V \longrightarrow V$ a c.c. map. Then Im (I-T) is closed.

<u>Proof</u>: Let $A = I-T$ and dim ker A be n (Use Theorem 3.14) Choose a basis $e_1, e_2, \ldots, e_n$ of ker A and let

$$\varphi_\lambda(e_\beta) = \delta_{\lambda\beta} = \begin{cases} 0 & \text{if } \lambda = \beta \\ 1 & \text{if } \lambda \neq \beta \end{cases}$$

for all $\lambda, \beta \in \{1, 2, \ldots, n\}$. Then $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is a basis of $(\ker A)^*$. Claim that there exists a closed left linear subspace W of V such that there exists a closed left linear subspace W of V such that $V = \ker A \oplus W$. To prove this note that by the Hahn-Banach Theorem we can extend $\varphi_1, \varphi_2, \ldots, \varphi_n$ to continuous left linear maps $\phi_1, \phi_2, \ldots, \phi_n$ from V to $\mathbb{H}$ such that $\|\phi_i\| = \|\varphi_i\|$ for all $i = 1, 2, \ldots, n$. Let $\phi: V \longrightarrow \mathbb{H}^n$ be defined by $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$. Clearly ϕ is continuous and left linear. Let $W = \ker \phi$. Then W is closed. For each $x \in V$ let $U_x = \sum_{i=1}^{n} \phi_i(x)\, e_i \in \ker A$. Choose $j \in \mathbb{N}$ such that $1 \leq j \leq n$. Then

$$\phi_j(x-U_x) = \phi_j(x-\sum_{i=1}^{n} \phi_i(x)e_i) = \phi_j(x)-\sum_{i=1}^{n} \phi_i(x)\varphi_j(e_i)$$

$$= \phi_j(x)-\phi_j(x) \qquad = 0 .$$

Hence $\phi_j(x-U_x) = 0$ for all $j \leq n$. Therefore $\phi(x-U_x) = 0$, so $x-U_x \in \ker \phi$. Now $x = x-U_x+U_x$. Therefore $V = \ker A+W$. To prove that this is a direct sum we must show that $W \cap \ker A = \{0\}$. Let $x \in W \cap \ker A$ therefore $x = \sum_{i=1}^{n} x_i e_i$ for some $x_i \in \mathbb{H}$ and $\phi(x) = 0$ i.e. $\phi_j(x) = 0$ for all $j \leq n$ therefore $0 = \phi_j(x) = \phi_j(\sum_{i=1}^{n} x_i e_i)$ $= \sum_{i=1}^{n} x_i \varphi_j(e_i) = x_j$. Hence $x = 0$. Therefore $W \cap \ker A = \{0\}$. Hence $V = \ker A \oplus W$. Since $\ker A \cap W = \{0\}$, $A/_W$ is 1-1. Claim that $A(W) = \operatorname{Im}(A)$. To prove this note that $W \subseteq V$ therefore $A(W) \subseteq A(V)$. Let $x \in A(V)$ therefore $x \in A(y)$ for some $y \in V$, so $y = u+v$ for some $u \in \ker A$ and $v \in W$, so $x = A(y) = A(u+v) = A(u)+A(v) = 0+A(v)$; hence $x \in A(W)$. Hence $\operatorname{Im} A = A(W)$. To finish the proof we must show that $A(W)$ is closed in V. Now $A/_W : W \longrightarrow A(W)$ and $A/_W$ is a 1-1, onto continuous left linear map. Claim that $A/_W$ is a homeomorphism ie. $(A/_W)^{-1}$ is continuous. Since $(A/_W)^{-1}$ is left linear, we need only show that $(A/_W)^{-1}$ is continuous at 0. To prove this, suppose not. Therefore there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $A(W)$ such that $(v_n)_{n\in\mathbb{N}}$ converges to 0 and $((A/_W)^{-1}(v_n))_{n\in\mathbb{N}}$ does not converge to 0. For each $n\in\mathbb{N}$ there exists a unique $x_n\in W$ such that $v_n = A(x_n)$, so we have a sequence $(x_n)_{n\in\mathbb{N}}$ in W such that $(x_n)_{n\in\mathbb{N}}$ does not converge to 0. But $(A(x_n))_{n\in\mathbb{N}} == (v_n)_{n\in\mathbb{N}}$ converges to 0. There exists a $\varepsilon>0$ for each $m\in\mathbb{N}$ there exists an $n_m\in\mathbb{N}$ such that $n_m > m$ and $\|x_{n_m}\| \geq \varepsilon$. Hence there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that for each $k\in\mathbb{N}$ $\|x_{n_k}\| \geq \varepsilon$ ie. $\frac{1}{\|x_{n_k}\|} < \frac{1}{\varepsilon}$ for all $k\in\mathbb{N}$. Since $\frac{\|x_{n_k}\|}{\|x_{n_k}\|} = 1$ and T is c.c., $\left(\frac{T(x_{n_k})}{\|x_{n_k}\|}\right)_{k\in\mathbb{N}}$ has a convergent subsequence.

Let $(x_{n_{k_l}})_{l\in\mathbb{N}}$ be a subsequence of $(x_{n_k})_{k\in\mathbb{N}}$ such that $\left(\frac{T(x_{n_{k_l}})}{\|x_{n_{k_l}}\|}\right)_{k\in\mathbb{N}}$ converges. Since $\frac{A(x_{n_{k_l}})}{\|x_{n_{k_l}}\|} = \frac{x_{n_{k_l}}}{\|x_{n_{k_l}}\|} - \frac{T(x_{n_{k_l}})}{\|x_{n_{k_l}}\|}$ converges, $\left(\frac{x_{n_{k_l}}}{\|x_{n_{k_l}}\|}\right)_{l\in\mathbb{N}}$ converges. Let $z = \lim_{l\to\infty}\frac{x_{n_{k_l}}}{\|x_{n_{k_l}}\|} \in W$. So $0 = \lim_{l\to\infty}\frac{A(x_{n_{k_l}})}{\|x_{n_{k_l}}\|} =$

$$\lim_{l\to\infty}\frac{(x_{n_{k_l}})}{\|x_{n_{k_l}}\|} - \lim_{l\to\infty}\frac{T(x_{n_{k_l}})}{\|x_{n_{k_l}}\|} = z - T(z) = A(z),$$ hence $z\in\ker A$

therefore $z = 0$, a contradiction since $\|z\| = 1$. Hence $(A/_W)^{-1}$ is continuous. Let $T' = (A/_W)^{-1}$. Claim that $A(W)$ is complete in V.

To prove this let $(y_n)_{n\in\mathbb{N}}$ be a cauchy sequence in A(W). Then $(T'(y_n))_{n\in\mathbb{N}}$ is a cauchy sequence in W. Since W is closed in V and V is a Banach space, W is complete, hence $(T'(y_n))_{n\in\mathbb{N}}$ converges in W. Let $w_0 = \lim_{n\to\infty} T'(y_n)$ therefore $w_0 \in W$. Then $A(w_0) = A(\lim_{n\to\infty} T'(y_n))$ $= \lim_{n\to\infty} A(T'(y_n)) = \lim_{n\to\infty} y_n$. Hence A(W) is complete in V. Since V is a Banach space, A(W) is closed. Hence Im(I-T) is closed. ⋇

Lemma 4.16 Let V be a LNLS (RNLS) which is also a Banach space and $T: V \to V$ a c.c. map. Then $y \in Im(I-T)$ if and only if $\varphi(y) = 0$ for all $\varphi \in \ker(I-T^*)$.

Proof: Let $y \in Im(I-T)$ and $\varphi \in \ker(I-T^*)$. Then there exists an $x \in V$ such that $y = (I-T)(x)$ therefore $\varphi(y) = \varphi((I-T)(x)) =$ $[(I-T^*)(\varphi)](x) = 0$.

Conversely, suppose that $\varphi(y) = 0$ for all $\varphi \in \ker(I-T^*)$. Must Must show that y Im(I-T). Hence we must show that if $y \in \bigcap_{\varphi\in\ker(I-T^*)} \ker\varphi$. Then there exists an $x \in V$ such that $y = x-T(x)$ ie. $y \in Im(I-T)$. We shall prove the contradiction ie. if $y \notin Im(I-T)$, then $y \notin \bigcap_{\varphi\in\ker(I-T^*)} \ker\varphi$.

Let $y \notin Im(I-T)$. Let W be the left linear subspace of V generated by Im(I-T) and y. For each $z \in W$ z has unique representation in the form $z = \lambda y+u$ for some $\lambda \in \mathbb{H}$ and $u \in Im(I-T)$. Define $\varphi(z) = \lambda$ so $\varphi : W \to \mathbb{H}$ is left linear. Claim that φ is continuous. Let

$z \in W \setminus \{0\}$. Therefore $z = \lambda y + u$ for some $\lambda \in \mathbb{H}$ and $u \in \mathrm{Im}(I-T)$.

Then $\dfrac{|\varphi(z)|}{\|z\|} = \dfrac{|\varphi(\lambda y+u)|}{\|z\|} = \dfrac{|\lambda|}{\|\lambda y+u\|}$. If $\lambda = 0$, then $\dfrac{|\varphi(z)|}{\|z\|} = 0$.

Assume $\lambda \neq 0$. Then

$$\frac{|\varphi(z)|}{\|z\|} = \frac{|\lambda|}{|\lambda|(\|y+\frac{1}{\lambda}u\|)} = \frac{1}{\|y+\frac{1}{\lambda}u\|} < \frac{1}{d(y,\mathrm{Im}(I-T))} < \infty.$$ Hence

is continuous. By the Hanh-Banach theorem we can extend φ to a continuous left linear map $\emptyset \in V^*$ such that $\|\varphi\| = \|\emptyset\|$. Must show that $\emptyset \in \ker(I-T^*)$. To prove this note that $((I-T^*)\emptyset)(x) = \emptyset((I-T)(x)) = \varphi((I-T)(x)) = 0$, hence $\emptyset \in \ker(I-T^*))$. Also $\emptyset(y) = \varphi(y) = 1 \neq 0$, so $y \notin \ker \emptyset$. Hence $y \notin \bigcap_{\varphi \in \ker(I-T^*)} \ker$. ⋇

<u>Lemma</u> 4.17 Let V, W be LNLS's (RNLS's) and $F: V \longrightarrow W$ a continuous left (right) linear. Then the natural map $\pi: V/_{\ker F} \longrightarrow \mathrm{Im}F$ is a left (right) linear isomorphic if V,W are Banach space and ImF is closed.

<u>Proof</u>: Since π is a 1-1 onto and left linear we shall show that π is continuous at 0. This shall finish the proof. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $V/_{\ker F}$ converging to 0. Must show that $(\pi(\alpha_n))_{n\in\mathbb{N}}$ converges to 0. Since $(\alpha_n)_{n\in\mathbb{N}}$ converges to 0, $(\|\alpha_n\|)_{n\in\mathbb{N}}$ converges to 0. Given $\varepsilon > 0$ and $n \in \mathbb{N}$ there exists an $x_n \in \alpha_n$ such that $\|\pi(\alpha_n)\| = \|F(x_n)\| \leqslant \|F\|\|x_n\| \leqslant \|F\|(\|\alpha_n\| + \varepsilon/_n)$ which converges to 0 therefore $(\|\pi(\alpha_n)\|)_{n\in\mathbb{N}}$ converges to 0, so $(\pi(\alpha_n))_{n\in\mathbb{N}}$ converges to 0. ⋇

<u>Lemma</u> 4.18 Let V be a LNLS (RNLS) which is also a Banach space and $T: V \longrightarrow V$ a c.c.map. Then $\varphi \in \mathrm{Im}(I-T^*)$ if and only if $\varphi(x) = 0$ for all $x \in \ker(I-T)$.

<u>Proof</u>: Let $\varphi \in \mathrm{Im}(I-T^*)$. Then there exists a $\psi \in V^*$ such that $\varphi = (I-T^*)\psi$. Let $x \in \ker(I-T)$ therefore $\varphi(x) = ((I-T^*)\psi)(x) = \psi((I-T)(x)) = \psi(0) = 0$.

Conversely, suppose that $\varphi(x) = 0$ for all $x \in \ker(I-T)$. For each $y \in \mathrm{Im}(I-T)$ there exists an $x \in V$ such that $y = (I-T)(x)$. Define $\eta(y) = \varphi(x)$ this is well-defined since if $y \in (I-T)(x')$ then $x-x' \in \ker(I-T)$ therefore $0 = \varphi(x-x') = \varphi(x)-\varphi(x')$. So $\varphi(x) = \varphi(x')$. Claim that η is left linear. Let $y_1, y_2 \in \mathrm{Im}(I-T)$ and $\alpha \in \mathbb{H}$ there exist $x_1, x_2 \in V$ such that $(I-T)(x_1) = y_1$ and $(I-T)(x_2) = y_2$ hence $\alpha y_1 = \alpha(I-T)(x_1) = (I-T)(\alpha x_1)$. Therefore $\eta(\alpha y_1) = \varphi(\alpha x_1) = \alpha\varphi(x_1) = \alpha\eta(x_1)$ and $\eta(y_1+y_2) = \varphi(x_1+x_2) = \varphi(x_1)+\varphi(x_2) = \eta(x_1)+\eta(x_2)$. Hence we have the claim. Claim that η is continuous To prove this, consider the diagram

$$V \xrightarrow{P} V/_{\ker(I-T)} \xrightarrow{\pi} \mathrm{Im}(I-T) \xrightarrow{\eta} \mathbb{H}$$

$$x \longrightarrow [x] \longrightarrow (I-T)(x) \longrightarrow \eta(I-T)(x) = \varphi(x)$$

Suppose that η is not continuous. Hence there exists an open set $U \subseteq \mathbb{H}$ such that $\eta^{-1}(U)$ is not open in $\mathrm{Im}(I-T)$. Claim that $P^{-1}(\pi^{-1}(\eta^{-1}(U)))$ is open in V. By the open mapping theorem, P is open, so $P(P^{-1}(\pi^{-1}(\eta^{-1}(U))))$ is open in $V/_{\ker(I-T)}$. Since P is onto

$P(P^{-1}(\pi^{-1}(\eta^{-1}(U)))) = \pi^{-1}(\eta^{-1}(U))$ which is open in $V/_{\ker(I-T)}$, a contradiction. Hence $P^{-1}(\pi^{-1}(\eta^{-1}(U)))$ is not open in V. But $P^{-1}(\pi^{-1}(\eta^{-1}(U))) = \varphi^{-1}(U)$ and $\varphi^{-1}(U)$ is open since φ is continuous, a contradiction. Hence η is continuous. By the Hahn-Banach Theorem, there exists a $\psi \in V^*$ such that $\psi/_{Im(I-T)} = \eta$ and $\|\psi\| = \|\eta\|$. Now for all $x \in V$, $[(I-T^*)\psi](x) = \psi((I-T)(x)) = \eta((I-T)(x)) = \varphi(x)$. Hence $(I-T^*)\psi = \varphi$. Thus $\varphi = Im(I-T^*)$. ⋇

<u>Corollary</u> 4.19 Let V be LNLSm(RNLS) which is also a Banach space and T: V⟶V a c.c. map. If $I-T^*$ is a 1-1, then I-T is onto.

<u>Proof</u>: Since $I-T^*$ is 1-1, $\ker(I-T^*) = \{0\}$. Hence for all $y \in V$ $\varphi(y) = 0$ for all $\varphi \in \ker(I-T^*)$. Therefore (I-T) is onto. ⋇

<u>Corollary</u> 4.20 Let V be a LNLS (RNLS) which is also a Banach space and T: V⟶V a c.c. map. If I-T is 1-1, then $I-T^*$ is onto.

<u>Proof</u>: Since I-T is 1-1, $\ker(I-T) = \{0\}$. Hence for all $\varphi \in V^*$ $\varphi(x) = 0$ for all $x \in \ker(I-T)$. Hence $I-T^*$ is onto. ⋇

<u>Theorem</u> 4.21 Let V be a LNLS (RNLS) which is also a Banach space and T : V⟶V a c.c. map. If I-T is 1-1 then I-T is onto.

<u>Proof</u>: Since I-T is 1-1, $I-T^*$ is onto, hence $I-T^*$ is 1-1. Since $I-T^*$ is 1-1, I-T is onto. ⋇

<u>Theorem</u> 4.22 Let V be a NLS which is also a Banach space and T: V⟶V a left (right) linear map which is also c.c. If λ is nonzero right (left) characteristic value of T then λ is a right

(left) eigenvalue of T.

Proof: Since λ is a right characteristic value of T we get that $T-I\lambda$ has no continuous inverse. We must show that λ is a right eigenvalue of T. Suppose not therefore $T-I\lambda$ is 1-1. So $(T-I\ )(-\lambda^{-1})$ is 1-1, hence $I-T\lambda^{-1}$ is 1-1. Since T is c.c., $T\lambda^{-1}$ is c.c. also. Hence $I-T\lambda^{-1}$ is onto, so $I-T\lambda^{-1}$ is 1-1, onto, continuous and left linear map from the Banach space V onto itself. By the open mapping theorem $I-T\lambda^{-1}$ has a continuous left linear inverse. ie. $I-T\lambda^{-1}$ is a homeomorphism. Hence $(I-T\lambda^{-1})(-\lambda)$ is a homeomorphism also. So $T-I\lambda$ is a homeomorphism, hence $T-I\lambda$ has a continuous inverse a contradiction. Hence λ is a right eigenvalue of T. ⋇

Remarks: i) If V is a LNLS (RNLS) which is also a Banach space and $T: V \longrightarrow V$ is a left (right) linear map which is also c.c. and $\lambda \in \mathbb{R}$ is a nonzero characteristic value of T then λ is an eigenvalue of T.

ii) If V is a left (right) finite dimensional vector space over $\mathbb{H}$ which is also a NLS and $T: V \longrightarrow V$ is a NLS and $T: V \rightarrow V$ is a continuous left (right) linear map and $\lambda \in \mathbb{H}$. is a nonzero right (left) characteristic value of T then λ is a right (left) eigenvalue of T.

Theorem 4.23 Let V be a NLS which is also a Banach space and $T: V \longrightarrow V$ a continuous left (right) linear map. Then $(T-I\lambda)^{-1}\left[(T-\lambda I)^{-1}\right]$ exists for all $|\lambda| > \|T\|$ and $\lambda \in \mathbb{R}$.

Proof: Let $\lambda \in \mathbb{R}$ be such that $|\lambda| > \|T\|$. Since $T-I\lambda = (I-T.\frac{1}{\lambda})(-\lambda)$ if $(I-T\lambda)^{-1}$ exists then $(T-I\lambda)^{-1} = (I-T\frac{1}{\lambda})^{-1}(-\frac{1}{\lambda}) = \sum_{k=0}^{\infty} (\frac{T}{\lambda})^{k}(-\frac{1}{\lambda})$. This series converges since $\|\frac{T}{\lambda}\| = \frac{1}{|\lambda|}\|T\| < \frac{1}{|\lambda|}|\lambda| = 1$. Hence $(I-T\lambda)^{-1}$ exists. ⋇

Definition 4.24 Let V, W be LNLS's (RNLS's) which is also a Banach space, an element $T \in C(V,W)$ is called a Fredholm operator from V to W if and only if

i) ker T is finite dimensional

ii) Im T is closed and coker T is finite dimensional where coker $T = W/_{\text{Im } T}$

Theorem 4.25 Let V be a LNLS (RNLS) and $T: V \longrightarrow V$ is c.c. Then I-T is Fredholm.

Proof: By theorem 4.14, ker I-T is finite dimensional. By Lemma 4.15, Im (I-T) is closed. We shall show that coker (I-T) is finite dimensional. This shall finish the proof. If I-T is onto there is nothing to prove. Therefore we may assume that I-T is not onto. Suppose that coker (I-T) is not finite dimensional. Let $(x_\alpha+W)_{\alpha \in I}$ be algebraic an basis of coker (I-T). Choose $\alpha_1 \in I$ and let H_1 be the left linear subspace generated by (I-T) and x_{α_1}. Choose $\alpha_2 \in I \setminus \{\alpha_1\}$ and let H_2 be the left linear subspace generated by H_1 and x_{α_2}. By induction we have H_n is the left linear subspace generated by H_{n-1} and α_n for all $n \in \mathbb{N}$. Hence we have $(I-T)(V) = H_0 \subset H_1 \subset H_2 \ldots$. By the same proof as in Theorem 2.10, H_n is closed for all $n \in \mathbb{N}$ and dim (H_n/H_{n-1}) is 1. By same argument as in Theorem 3.7 we can find in each H_n an element x_n such that $\|x_n\| = 1$ and $\|x_n - y\| > \frac{1}{2}$ for all $y \in H_{n-1}$. Then for all $k < n$ $|T(x_n)-T(x_k)| = |x_n-(I-T)(x_n)-x_k+(I-T)(x_k)| \geq \frac{1}{2}$ because $-(I-T)(x_n) -x_k+ (I-T)(x_k)$ lies in H_{n-1}. This show that the sequence $(T(x_n))_{n\in \mathbb{N}}$ can not have a convergent subsequence sontradicting the fact that T is c.c. Thus proving the lemma. #

Definition 4.26 If T is a Fredholm operator, then we define the index of T to be ind T = dim ker T - dim coker T.

Let V be a LNLS (RNLS) over $\mathbb{H}$ which is also a Banach space and $T: V \longrightarrow V$ is c.c. We want to show that ind (Id-T) = -ind(Id-T). In order to prove. This we'll need some lemmas.

Lemma 4.27 Let V,W be LNLS's (RNLS's) which are both Banach space and $T: V \longrightarrow W$ a continuous left (right) linear map, then $\ker (T^*) = \{f \in W^* / f/_{T(V)} = 0\}$. If in addition T(V) is closed in W, then $T^*(W^*) = \{f \in V^* / f/_{\ker T} = 0\}$. So, inparticular, $T^*(W^*)$ is closed in V^*.

Proof: $T^*(f) = 0 \Longleftrightarrow [T^*(f)](x) = 0$ for all $x \in V \Longleftrightarrow f(T(x))=0$ for al $x \in V \Longleftrightarrow f \in \{f \in W^*/ f/_{T(V)} = 0\}$. Hence $\ker T^* = \{f \in W^*/f/_{T(V)}=0\}$.

Assume that $T(V)$ is closed in W. For each $l \in \{f \in V^*/f/_{\ker T} = 0\}$ let $\tilde{l}: V/_{\ker T} \to \mathbb{H}$ be defined by $\tilde{l}([x]) = l(x)$. By the same argument as in Lemma 4.17, $\tilde{l} \in (V/_{\ker T})$. Let $P: V \to V/_{\ker T}$ be the natural map. Then $\tilde{l} \circ P = l$ for all $l \in \{f \in V^*/f/_{\ker T} = 0\}$. Let $\tilde{T}: V/_{\ker T} \to (T(V)$ be the natural map. By lemma 4.17, T is a homeomorphism and we let $S: T(V) \to V/_{\ker T}$ be its inverse. Since $\tilde{l} \circ S \in (T(V))^*$, there exists $l' \in W^*$ such that $l'/_{T(V)} = \tilde{l} \circ S$ by Hahn-Banach theorem. Since $\tilde{T} \circ P = T$, $P = S \circ \tilde{T} \circ P = S \circ T$. Hence
$[T^*(l')](x) = l'(T(x)) = \tilde{l} \circ S(T(x)) = \tilde{l}(S(T(x))) = \tilde{l}([x]) = l(x)$
for all $x \in V$ ie. $T^*(l') = l$. Hence $\{f \in V^*/f/_{\ker T} = 0\} \subseteq T^*(W^*)$. Let $l \in T^*(W^*)$ there exists $l' \in W^*$ such that $l = T^*(l')$. Let $x \in \ker T$. Then $l(x) = l'(T(x)) = l'(0) = 0$, so $l \in \{f \in V^*/f/_{\ker T} = 0\}$. Thus $T^*(W^*) \subseteq \{f \in V^*/f/_{\ker T} = 0\}$. Hence $\{f \in V^*/f/_{\ker T} = 0\} = T^*(W^*)$. It follows that $T^*(W^*)$ is closed. ⋕

<u>Corollary</u> 4.28 If W is closed left (right) linear subspace of LNLS (RNLS) V, then P_W^* is a homeomorphism of $(V/_W)^*$ with $\{f \in V^*/f/_W = 0\}$ where $P_W: V \to V/_W$ is the natural map.

<u>Proof</u>: Let f, g $(V/_W)^*$ be such that $P_W^*(f) = P_W^*(g)$ therefore $f(P_W(x)) = g(P_W(x))$ for all $x \in V$. Let $\alpha \in V/_W$. Since P_W is onto, there exists $x \in V$ such that $P_W(x) = \alpha$, so $f(\alpha) = f(P_W(x)) = g(P_W(x)) = g(\alpha)$. Hence P_W^* is 1-1. By lemma 4.27, $P_W^*(V/_W)^* = \{f \in V^*/f/_{\ker PW} = 0\}$

$= \left\{ f \in V^* / f/_W = 0 \right\}$ which is closed in V^*, and hence space. By the open mapping theorem, P_W^* is open. ⋕

Corollary 4.29 If T(V) is closed in W, then

i) $\ker(T^*) \cong (W/_{T(V)})^* = (\operatorname{coker} T)^*$

ii) $(\ker T)^* = V^*/_{T^*(W^*)} = \operatorname{coker}(T^*)$

Proof: By lemma 4.27 and Corollary 4.28, $\ker(T^*) \cong (W/_{T(V)})^* = (\operatorname{coker} T)^*$. Claim $(\ker T)^* \cong V^*/_A$ where $A = \left\{ f \in V^* / f/_{\ker T} = 0 \right\}$. To prove this, let $\varphi : V^*/_A \longrightarrow (\ker T)^*$ be defined by $\varphi([f+A]) = f/_{\ker T}$. Clearly φ is 1-1 and right linear. By the Hahn-Banach Theorem, for each $f \in (\ker T)^*$ there exists an $f' \in V^*$ such that $f'/_{\ker T} = f$. Hence φ is onto. Claim that φ is continuous. Let $(F_n)_{n \in \mathbb{N}}$ be a sequence in $V^*/_A$ such that $(F_n)_{n \in \mathbb{N}}$ converges to 0. We must show that $(\varphi(F_n))_{n \in \mathbb{N}}$ converges to 0. To prove this, given $\varepsilon > 0$ for each $n \in \mathbb{N}$ there exists $f_n \in F_n$ such that $\|f_n\| \leqslant \|F_n\| + \varepsilon/_n$. Since $f_n \in F_n$ for all $n \in \mathbb{N}$, $\varphi(F_n) = f_{n/\ker T}$. Therefore $\|\varphi(F_n)\| = \|f_{n/\ker T}\| \leqslant \|f_n\| \leqslant \|F_n\| + \varepsilon/_n$ which converges to 0, so $(\|\varphi(f_n)\|)_{n \in \mathbb{N}}$ converges to 0. Hence $(\varphi(F_n))_{n \in \mathbb{N}}$ converges to 0. Hence we have the claim. By the open mapping theorem and lemma 4.27, $(\ker T)^* \cong V^*/_A = V^*/_{T^*(W^*)} = \operatorname{coker}(T^*)$. ⋕

Theorem 4.30 Let V be LNLS's (RNLS's) which are also Banach space. Then $\operatorname{ind}(I-T) = -\operatorname{ind}(I-T^*)$.

Proof: By corollary 4.29, $\dim \ker (I-T)^* = \dim (\text{coker } (I-T))^*$ $= \dim \text{coker } (I-T)$ and $\dim \ker (I-T) = \dim (\ker (I-T))^*$ $= \dim \text{coker } (I-T^*)$. Hence

$$\begin{aligned} \text{ind } (I-T) &= \dim \ker (I-T) - \dim \text{coker } (I-T) \\ &= \dim \text{coker } (I-T^*) - \dim \ker (I-T^*) \\ &= -\text{ind } (I-T^*). \end{aligned}$$

Weak Topology Let $(V, \| \ \|)$ be LNLS (RNLS) over $\mathbb{H}$. The norm gives a metric d by defining $d(x,y) = \|x-y\|$ and the metric gives a topology by taking the open balls $B(x_0, \varepsilon) = \{x \in V / d(x,x_0) < \varepsilon\}$ as a base for the topology. We call this the strong topology hence a sequence $(x_n)_{n \in \mathbb{N}}$ converges to x in the strong topology if and only if $\lim_{n \to \infty} \|x_n - x\| = 0$.

Definition 4.31 The weak topology on V is the smallest topology on V which makes every $f \in V^*$ continuous. More precisely, given $\varepsilon > 0$ and $f_1, f_2, \ldots, f_r \in V^*$ then $u = u_{f_1, f_2, \ldots, f_r, \varepsilon} = \{x \in V / |f_\alpha(x)| < \varepsilon \ \ \forall \alpha = 1, 2, \ldots, r\}$ is open and sets of this form are a basis for the neighborhood of 0 (To get a basis for the neighborhood of $x \in V$, take sets of the form $x+U$ where U is a neighborhood of 0 and $x+U = \{x+a / a \in U\}$.

Definition 4.32 The sequence $(x_n)_{n \in \mathbb{N}}$ converges weakly to $x \in V$ if and only if $\lim_{n \to \infty} x_n = x$ in the weakly topology of V.

Theorem 4.33 $\lim_{n\to\infty} x_n = x$ weakly if and only if $\lim_{n\to\infty} f(x_n) = f(x)$ for all $f \in V^*$.

Proof: Obvious.

Remark: Strong convergence $\Longrightarrow$ weak convergence. Since if $\lim_{n\to\infty} x_n = x$ strongly and $f \in V^*$ then $|f(x_n)-f(x)| = |f(x_n-x)| \leqslant \|f\|\|x_n-x\| \longrightarrow 0$ as $n \longrightarrow \infty$. However, it $\dim V = \infty$ then weak convergence $\not\Longrightarrow$ strong convergence.

Example 4.34 Let $V = \ell^2_{\mathbb{H}}$ $e_n = (0,0,\ldots,\overset{n^{th}\text{place}}{1},0,0,\ldots)$ for all $n \in \mathbb{N}$ and $\varphi \in V^*$. Let $\varphi(e_n) = \beta_n$ for all $n \in \mathbb{N}$. By the same proof as in the example 1.24 (i), $(\beta_n)_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}}$, so $\lim_{n\to\infty} \beta_n = 0$. Hence $\lim_{n\to\infty} \varphi(e_n) = \varphi(0)$. Thus $\lim_{n\to\infty} e_n = 0$ weakly. But $(e_n)_{n\in\mathbb{N}}$ is not strongly convergent because if $m \neq n$ $d(e_n,e_n) = \sqrt{2}$ therefore $(e_n)_{n\in\mathbb{N}}$ is not cauchy with respect to the norm metric. #

Remarks: (i) In a finite dimensional left (right) vector space V over $\mathbb{H}$ weak convergence is equivalent to strong convergence.

Proof: Let $\lim_{n\to\infty} x_n = x_0$ weakly. Must show that $\lim_{n\to\infty} x_n = x_0$ with respect to the norm. Let $e_1, e_2, \ldots, e_n$ be a basis of V. Given $\alpha \in \mathbb{N}$ such that $1 \leqslant \alpha \leqslant n$ let $e^\alpha : V \longrightarrow \mathbb{H}$ be the map $e^\alpha(\sum_{\beta=1}^{n} x_\beta e_\beta) = x_\alpha$. Then e^α is left linear. Since V is finite dimensional, e^α is continuous, so $e^{\alpha} \in V^*$. Let $e^{\alpha}(x_n) = x_n^{(\alpha)}$ the α^{th} component of the vector x_n.

Since $\lim_{n \to \infty} x_n = x_0$ weakly, $\lim_{n \to \infty} x_n^{\alpha} = x_0^{\alpha}$ for all $1 \leq \alpha \leq n$.

Since in the finite dimensional case all norms are equivalent without loss of generality let $\| \ \|_{\infty}$ be the norm on V. Given $\varepsilon > 0$ there exists a N_{ε} such that $|x_m^{(\alpha)} - x_0^{(\alpha)}| < \varepsilon$ for all $m > N_{\varepsilon}$ and for all $\alpha \leq n$ therefore $\|x_m - x_0\| = \sup_{1 \leq \alpha \leq n} \{|x_m^{(\alpha)} - x_0^{(\alpha)}|\} \leq \varepsilon$ ie. $\lim_{m \to \infty} x_m = x_0$ with respect to $\| \ \|_{\infty}$ and so $\lim_{m \to \infty} x_m = x_0$ with respect to all norm on V. ⋇

ii) If $(x_n)_{n \in \mathbb{N}}$ is a strongly convergent sequence, then clearly $(x_n)_{n \in \mathbb{N}}$ is bounded.

We want to show that if $(x_n)_{n \in \mathbb{N}}$ a weakly convergent sequence, then $(x_n)_{n \in \mathbb{N}}$ is bounded. In order to prove this we need some lemmas.

<u>Lemma</u> 4.35 Let V be a LNLS (RNLS) and $(x_n)_{n \in \mathbb{N}}$ an unbounded sequence in V. Let $\varphi_0 \in V^*$ and $\overline{B(\varphi_0, r)}$ be the closed ball in V^* center at φ_0 of radius $r > 0$. Then $\{\psi(x_n) / \psi \in \overline{B(\varphi_0, r)}, n \in \mathbb{N}\}$ is unbounded.

<u>Proof:</u> Suppose not. Therefore $\{\psi(x_n) / \psi \in \overline{B(\varphi_0, r)}, n \in \mathbb{N}\}$ is bounded. Claim $\{\psi(x_n) / \psi \in \overline{B(0, r)}, n \in \mathbb{N}\}$ is also bounded. To prove this claim, note that if $\psi \in \overline{B(0, r)}$, Then $\psi + \varphi_0 \in \overline{B(\varphi_0, r)}$. Since $\psi(x_n) = \psi(x_n) + \varphi_0(x_n) - \varphi_0(x_n)$, therefore we have the claim. Hence

there exists a K such that $|\psi(x_n)| \leq K$ for all $\psi \in \overline{B(0,r)}$ and for all $n \in \mathbb{N}$. Since the map of $V \longrightarrow V^*$ given by $x \mapsto \psi_x$ is an isometry, $\|x_n\| = \|\psi_{x_n}\| = \sup_{\|\varphi\|=1} \{|\psi_{x_n}(\varphi)|\} = \frac{1}{r} \sup_{\|\varphi\|=r} \{|\psi_{x_n}(\varphi)|\} = \frac{1}{r} \sup_{\|\varphi\|=r} \{|\varphi(x_n)|\} \leq K/r$, so $(x_n)_{n \in \mathbb{N}}$ is bounded, a contradiction.

⋇

<u>Theorem</u> 4.36 Let V be a LNLS (RNLS) and $(x_n)_{n \in \mathbb{N}}$ a weakly convergent sequence, then $(x_n)_{n \in \mathbb{N}}$ is bounded.

<u>Proof</u>: Suppose not. Hence $(x_n)_{n \in \mathbb{N}}$ is unbounded. Let $\overline{B}_0$ be any closed ball in V^*. Then $(\varphi(x_n))_{\varphi \in \overline{B}_0}$ is unbounded by lemma 3.35. Hence there exists a $\varphi_0 \in \overline{B}_0$ and $n_1 \in \mathbb{N}$ such that $|\varphi_0(x_{n_1})| > 1$. Since $\psi_{x_{n_1}}$ is continuous, there exists a neighborhood U such that $\varphi_0 \in U$ and $|\psi_{x_{n_1}}(\varphi)| > 1$ for all $\varphi \in U$. Let $\overline{B}_1$ be a closed ball contained in U. Hence $\overline{B}_1 \subseteq U \cap \overline{B}_0$. So we have that for all $\varphi \in \overline{B}_1$ $|\psi_{x_{n_1}}(\varphi)| > 1$. $(\varphi(x_n))_{\varphi \in \overline{B}_1}$ is unbounded by lemma .35. Hence there exists a $\varphi_1 \in \overline{B}_1$ and there exists an $n_2 \in \mathbb{N}$ such that $|\varphi_1(x_{n_2})| > 2$. The same reasoning as before show that there exists a closed ball $\overline{B}_2 \subseteq \overline{B}_1$ sucht that for all $\varphi \in \overline{B}_2$ $|\varphi(x_{n_2})| > 2$. Continue in this way, we get that for all $k \in \mathbb{N}$ there exist a closed ball $\overline{B}_k \subseteq \overline{B}_{k-1}$ and an $n_k \in \mathbb{N}$ such that $|\varphi(x_{n_k})| > k$ for all $\varphi \in \overline{B}_k$. Then $\bigcap_{k \in \mathbb{N}} \overline{B}_k \neq \emptyset$ [1, P60].

Let $\varphi \in \bigcap_{k \in \mathbb{N}} B_k$ therefore $\varphi \in \overline{B}_k$ for all $k \in \mathbb{N}$, hence $|\varphi(x_{n_k})| > k$.

So $(\varphi(x_{n_k}))_{k\in\mathbb{N}}$ is unbounded. Since $(x_n)_{n\in\mathbb{N}}$ is weakly convergent, there exists an $x_0 \in V$ such that $\lim_{n\to\infty} \varphi(x_n) = \varphi(x_0)$. Since $(\varphi(x_{n_k}))_{k\in\mathbb{N}}$ is a subsequence of convergent sequence, $(\varphi(x_{n_k}))_{k\in\mathbb{N}}$ is convergent, so bounded, a contradiction. ⋇

Weak and Weak $*$ convergence in V^*

Let V be a LNLS (RNLS). Then V^* is a RNLS (LNLS). So it has strong topology which comes from the norm (which is a metric topology). Also, we can give V^* the weak topology. We can put a third topology on V^* called the weak $*$ topology. This topology is the smallest topology making $(\psi_x)_{x\in V}$ continuous. This topology can be described as follows: Let $A \subseteq V$ be a finite set and let $\varepsilon > 0$ be given. Define $U_{A,\varepsilon} = \{f \in V^* / |f(x)| < \varepsilon \;\; \forall x \in A\}$ [if $A = \{x_1, x_2, \ldots, x_m\}$ we shall sometimes write $U_{x_1,x_2,\ldots,x_m,\varepsilon} = \{f \in V^* / |f(x)| < \varepsilon \;\; \forall n = 1,\ldots m\}$] and we take the set of the form $U_{A,\varepsilon}$ to be a neighborhood base of $0 \in V^*$. This generated topology on V^* is called the weak $*$ topology. If $\lim_{n\to\infty} f_n = f$ in this topology we say that $\lim_{n\to\infty} f_n = f$ weakly$*$.

Weak and Weak-convergent in $\bar{V}$ defined similarly.

Theorem 4.37 $\lim_{n\to\infty} f_n = f$ in the weak $*$ (-) topology if and only if $\lim_{n\to\infty} \psi_x(f_n) = \psi_x(f)$ for all $x \in V$.

<u>Proof</u>: Obvious.

<u>Theorem</u> 4.38 Let $(f_n)_{n \in \mathbb{N}} \subseteq V^*(\bar{V})$ where V is LNLS (RNLS) which is also in Banach space such that $\lim_{n \to \infty} f_n = f$ weakly $*(-)$. Then $(f_n)_{n \in \mathbb{N}}$ is bounded.

<u>Proof</u>: Some argument as in Theorem 4.36.

<u>Theorem</u> 4.39 (Generalized Bolzano-Weierstrass Theorem) Let V be any separable LNLS (RNLS). Then every bounded sequence in $V^*(\bar{V})$ has a weakly $*(-)$ convergent subsequence.

<u>Proof</u>: Let $D = (x_n)_{n \in \mathbb{N}}$ be a countable dense subset of V. Let $(f_n)_{n \in \mathbb{N}}$ be a bounded sequence in V^* therefore there exists a M such that $\|f_n\| < M$ for all n N. Since $|f_n(x_1)| \leq \|f_n\| \|x_1\| < \infty$ for all $n \in \mathbb{N}$, $(f_n(x_1))_{n \in \mathbb{N}}$ is a bounded sequence in $\mathbb{H}$, hence there exists a convergent subsequence $(f_n^{(1)}(x_1))_{n \in \mathbb{N}}$ of $(f_n(x_1))_{n \in \mathbb{N}}$. Since $|f_n^{(1)}(x_2)| \leq \|f_n^{(1)}\| \|x_2\| < \infty$ for all $n \in \mathbb{N}$, $(f_n^{(1)}(x_2))_{n \in \mathbb{N}}$ is a bounded sequence in $\mathbb{H}$, hence there exists a convergent subsequence $(f_n^{(2)}(x_2))_{n \in \mathbb{N}}$ of $(f_n^{(1)}(x_2))_{n \in \mathbb{N}}$. Continue in this way there exist a subsequence $(f_n^{(k)})_{n \in \mathbb{N}}$ for all k = 1,2,... such that

1) $(f_n^{(k+1)})_{n \in \mathbb{N}}$ is a subsequence of $(f_n^{(k)})_{n \in \mathbb{N}}$ for all k = 1,2,...

2) $(f_n^{(k)})_{n \in \mathbb{N}}$ converges at the points $x_1, x_2, \ldots, x_k$.

Choose the diagonal $f_1^{(1)}, f_2^{(2)}, \ldots$. This sequence converges for all x_n. Claim that $(f_n^n(x))_{n \in \mathbb{N}}$ converges for all $x \in V$. To prove this, it suffices to show that it is cauchy. Let $x \in V$ and $\varepsilon > 0$ be given. Since D is dense in V, there exists N_ε such that $\|x_{N_\varepsilon} - x\| < \varepsilon/_{3M}$. Since $(f_n^{(n)}(x_{N_\varepsilon}))_{n \in \mathbb{N}}$ converges, there exists and $N \in \mathbb{N}$ such that $\| f_n^{(n)}(x_{N_\varepsilon}) - f_m^m(x_{N_\varepsilon}) \| < \varepsilon/_3$ whenever $m, n > N$. Hence for all $n, m > N$

$$
\begin{aligned}
|f_n^{(n)}(x) - f_m^{(m)}(x)| &\leq |f_n^{(n)}(x) - f_n^{(n)}(x_{N_\varepsilon})| + |f_n^{(n)}(x_{N_\varepsilon}) - f_m^{(m)}(x_{N_\varepsilon})| \\
&\quad + |f_m^{(m)}(x_{N_\varepsilon}) - f_m^{(m)}(x)| \\
&\leq \|f_n^{(n)}\| \, \|x - x_{N_\varepsilon}\| + \varepsilon/_3 \, \|f_m^{(m)}\| \, \|x_{N_\varepsilon} - x\| \\
&\leq M . \varepsilon/_{3M} + \varepsilon/_3 + M . \varepsilon/_{3M} = \varepsilon .
\end{aligned}
$$

Hence $(f_n^{(n)}(x))_{n \in \mathbb{N}}$ is cauchy. Since $\mathbb{H}$ is complete $(f_n^{(n)}(x))_{n \quad N}$ converges. Define $f_0 : V \to \mathbb{H}$ by $f_0(x) = \lim_{n \to \infty} f_n^n(x)$. Then f_0 is left linear. If $x \neq 0$, then $|f_0(x)| = |\lim_{n \to \infty} f_n^{(n)}(x)| = \lim_{n \to \infty} |f_n^{(n)}(x)| \leq \lim_{n \to \infty} \|f_n^{(n)}\| \|x\| \leq M \|x\|$. Hence $\|f_0\| = \sup_{x \neq 0} \left\{ \frac{|f_0(x)|}{\|x\|} \right\} \leq M$. Thus f_0 is continuous. Therefore $f_0 \in V^*$. Also, for all $x \in V$ $\lim_{n \to \infty} f_n^{(n)}(x) = f_0(x)$. Hence $(f_n)_{n \in \mathbb{N}}$ has a subsequence weakly * converging to f_0. ⋇

<u>Theorem</u> 4.40 Let V be a separable LNLS (RNLS), B the closed unit ball in V and $B^* \subset V^*[\bar{V}]$. The closed unit ball in $V^*[\bar{V}]$. The topology on B^* induced by the weak $*$ [-] topology in the same as that induced by the metric $\rho(f,g) = \sum_{n=1}^{\infty} \frac{|f(x_n)-g(x_n)|}{2^n}$ where $\{x_1, x_2, \ldots\}$ is a countable dense subset of B.

<u>Proof</u>: Since $\rho(f,g) = \sum_{n=1}^{\infty} \frac{|f(x_n)-g(x_n)|}{2^n} \leqslant \sum_{n=1}^{\infty} 2/2^n$ $= \sum_{n=1}^{\infty} 1/2^{n-1} = 1$ for all $f, g \in B^*$, ρ is well-defined. If $\rho(f,g)=0$, then $(f-g)(x_n) = 0$ for all $n \in \mathbb{N}$. Let $x \in B$. Since $(x_n)_{n \in \mathbb{N}}$ is dense in B, there exists $\lim_{k\to\infty} x_{n_k} = x$. Hence $(f-g)(x) = \lim_{k\to\infty} (f-g)(x_{n_k})$ $= 0$. So that $f = g$. The other axioms for a metric space are trivial, ρ is a metric on B^*. To show that the topologies are the same, we must prove that

1. $Q_\varepsilon = \{f \in B^* / \rho(0,f) < \varepsilon\} \supsetneq B^* \cap U_{A,\varepsilon'}$ where $U_{A,\varepsilon'}$ is a weak $*$ neighborhood of $0 \in V^*$ and

2. Every weak $*$ neighborhood $U_{A,\varepsilon} \cap B^* \supset Q_\varepsilon$ a metric neighborhood of $0 \in V^*$.

To prove 1, given $\varepsilon > 0$. Let N be such that $2^{-N} < \varepsilon/2$ and consider the weak $*$ neighborhood of zero $U = U_{x_1, x_2, \ldots, x_N, \varepsilon/2} = \{f \in V^* / |f(x_n)| < \varepsilon/2 \;\; \forall n=1,2,\ldots,N\}$. Then $f \in B^* \cap U$ implies that $\rho(f,0)$

$$= \sum_{n=1}^{N} \frac{|f(x_n)|}{2^n} + \sum_{n=N+1}^{\infty} \frac{|f(x_n)|}{2^n} \leqslant \varepsilon/_2 \left(\sum_{n=1}^{N} 1/_{2^n} \right) + \sum_{n=N+1}^{\infty} 1/_{2^n}$$

$$= \varepsilon/_2 \left(\sum_{n=1}^{N} 1/_{2^n} \right) + \frac{1}{2} N < \varepsilon/_2 + \varepsilon/_2 = \varepsilon \quad \text{and hence } \overset{*}{B} \cap U \subset Q_\varepsilon .$$

This proves 1.

To prove 2, given $\delta > 0$ let $U = U_{y_1, y_2, \ldots, y_m, \delta} = \{ f \in V^* / |f(y_n)| < \delta$ $n = 1,2,\ldots,m\}$ be any weak $*$ neighborhood of $0 \in V^*$.

<u>case</u> 1. $\|y_n\| \leqslant 1$ for all $n = 1,2,\ldots,m$. Since $\{x_1, x_2, \ldots\}$ is dense in B, there exists indices $n_1, n_2, \ldots, n_m$ such that $\|y_k - x_{n_k}\| < \delta/_2$ $k = 1,2,\ldots,m$. Let $N = \max\{n_1, n_2, \ldots, n_m\}$ and $\varepsilon = \delta/_{2^{N+1}}$. Then $f \in Q_\varepsilon$ implies that $|f(y_k)| \leqslant |f(x_{n_k})| + |f(y_k) - f(x_{n_k})| < \delta/_2 + \|f\| \, \|y_k - x_{n_k}\|$ $< \delta/_2 + \delta/_2 = \delta$. Therefore $Q_\varepsilon \subset U \cap \overset{*}{B}$.

<u>case</u> 2. $\|y_k\| > 1$ for some $k \in \{1,2,\ldots,m\}$. Let

$$y'_k = \begin{cases} y_k & \text{if } \|y_k\| \leqslant 1 \\ \dfrac{y_k}{\|y_k\|} & \text{if } \|y_k\| > 1 \end{cases} \qquad \text{and let}$$

$M = \max\{\|y_k\| / k = 1,2,\ldots,m\}$. Then let $U' = U'_{y'_1, y'_2, \ldots, y'_m, \delta/_M}$

Therefore $U' \subsetneq U$. Since $\{x_1, x_2, \ldots\}$ is dense in B, there exists indices $n_1, n_2, \ldots, n_m$ such that $\|y'_k - x_{n_k}\| < \delta/_{2M}$ $k = 1,2,\ldots,m$.

Let $N = \max\{n_1, n_2, \ldots, n_m\}$ and $\varepsilon = \delta/_{2^{N+1}M}$. By the same proof as above $Q_\varepsilon \subset U' \cap B^* \subseteq U \cap B^*$. ⋇

<u>Corollary</u> 4.41 Let V be a separable LNLS (RNLS) which is also a Banach space. Then every bounded subset of $V^*(\bar{V})$ is relatively compact in the weak ✱ (-) topology. The converse is also true.

<u>Proof</u>: Let A be any bounded subset of V^*. Therefore there exists a $r > 0$ such that $A \subseteq \overline{B(0,r)}$. By Theorem 4.40, $\overline{B(0,r)}$ is a metric space. Hence we must show that $\bar{A}$ has the BW property i.e. every sequence $(f_n)_{n\in\mathbb{N}}$ in $\bar{A}$ has a convergent subsequence convering to a point in $\bar{A}$. Let $(f_n)_{n\in\mathbb{N}}$ be any sequence in $\bar{A}$ therefore $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\overline{B(0,r)}$ therefore there exisys a weak ✱ convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ by Theorem 4.41. Hence A is relatively compact in the weak✱ topology. The converse is obvious. ⋇

<u>Theorem</u> 4.42 Let V be a separable LNLS (RNLS) which is also a Banach. Then every closed ball subset of $V^*(\bar{V})$ is compact is compact in the weak ✱ (-) topology (by closed, we mean closed in the strong topology)

<u>Proof</u>: Claim that every closed ball in the strong topology is closed in the weak ✱ topology. Since translation take every closed set into another closed set we need only prove that the closed ball $B_c = \{f \in V^* / \|f\| \leqslant C\}$ is closed in the weak ✱ topology. Suppose that $f_0 \notin B_c$ therefore there exists a $x_0 \in V$ such

that $\|x_0\| = 1$ and $f_0(x) = \alpha > C$. Then the set $U = \left\{ f \in V^* / |f(x_0)| > \frac{1}{2}(\alpha + C) \right\}$ is a weak $*$ neighborhood of f_0 containing no element of B_c therefore B_c is closed in the weak $*$ topology. Since B_c is bounded, it is compact in the weak $*$ topology. ⋇

Definition 4.43 Let V be a LSPS (RSPS) which is also a Hilbert space and $f: V \longrightarrow V$ a continuous left (right) linear map. Then f is said to be self-adjoint if and only if $f(x).y = xf(y)$ for all $x, y \in V$.

Example 4.44 (i) Let $[a_{ij}]_{n \times n}$ be a metric where $a_{ij} \in \mathbb{H}$ and $a_{ij} = \bar{a}_{ji}$ for all $i, j \leqslant n$. Let $f: \mathbb{H}^n \longrightarrow \mathbb{H}^n$ be defined by $(f(x))_i = \sum_{j=1}^{\infty} x_j a_{ij}$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{H}^n$. Then f is continuous left linear map. Claim that f is selt-adjoint. Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{H}^n$. Then $f(x).y$ =

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_j a_{ij} \bar{y}_i = \sum_{i=1}^{n}\sum_{j=1}^{n} x_j \bar{a}_{ji} \bar{y}_i = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \bar{a}_{ij} \bar{y}_j =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \overline{y_j a_{ij}} = x.f(y)$$ ⋇

(ii) Given $l_{\mathbb{H}}^2$ the RSPS structure. Let $f: \ell_{\mathbb{H}}^2 \longrightarrow \ell_{\mathbb{H}}^2$ be defined by $f(x_1, x_2, \ldots,) = (\sum_{j=1}^{\infty} a_{1j} x_j, \sum_{j=1}^{\infty} a_{2j} x_j, \ldots)$ where

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty, \quad \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}| < \infty$$ and $a_{ij} = \bar{a}_{ji}$ for all $i, j \in \mathbb{N}$.

Then f is right linear cand c.c., hence f is continuous. Claim that f is self-adjoint. Let $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in \ell^2_{\mathbb{H}}$.

Then $f(x).y = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overline{a_{ij} x_j} y_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overline{x}_j \overline{a}_{ij} y_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overline{x}_i \overline{a}_{ji} y_j$

$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \overline{x}_i a_{ij} y_j = x.\circ f(y)$. Hence f is self-adjoint. ⋇

Theorem 4.45 Let V be a LSPS (RSPS) and $f: V \longrightarrow V$ be a self-adjoint map. Then all eigenvalues of f are real and two eigenvectors of f corresponding to distinct eigenvalues are orthogonal.

Proof: Let λ be any eigenvalue of f. There exists an $x \in V \setminus \{0\}$ such that $f(x) = \lambda x$. Then $\lambda(x.x) = (\lambda x).x = f(x).x = x.f(x) = x.(\lambda x) = (x.x)\overline{\lambda}$, and hence $\lambda = \overline{\lambda}$ i.e. $\lambda \in \mathbb{R}$. Moreover, if $f(x) = \lambda x$, $f(y) = \mu y$ $(\lambda \neq \mu)$, then $\lambda(x.y) = (\lambda x).y = f(x).y = x.f(y) = x.(\mu y) = (x.y)\overline{\mu} = \mu(x.y)$ and hence $x.y = 0$ i.e., the vectors x and y are orthogonal. ⋇

We now shall prove the Hilbert-Schmidt Theorem. First we shall need some lemmas.

Lemma 4.46 Let V be a LSPS (RSPS) which is also a Hilbert space and $f: V \longrightarrow V$ a c.c self-adjoint map. If $\lim_{n\to\infty} x_n = x$ weakly, then $\lim_{n\to\infty} f(x_n).x_n = f(x).x$.

Proof: $|f(x_n).x_n - f(x).x| \leqslant |f(x_n).x_n - f(x).x_n| + |f(x).x_n - f(x).x|$

$$= |f(x_n - x).x_n| + |x.f(x_n - x)|$$

$$\leqslant (\|x_n\| + \|x\|)\|f(x_n - x)\| .$$

Since $\lim_{n\to\infty} x_n = x$ weakly, there exists a M such that $\|x_n\| \leqslant M$ for all $n \in \mathbb{N}$. Therefore $|f(x_n).x_n - f(x).x| \leqslant (M+\|x\|)\|f(x_n)-f(x)\|$. Since f is c.c. and $(x_n)_{n\in\mathbb{N}}$ is bounded, there exists a strongly subsequence $(f(x_{n_k}))_{k\in\mathbb{N}}$ of $(f(x_n))_{n\in\mathbb{N}}$. Claim that $\lim_{n\to\infty} f(x_n) = f(x)$ weakly. To prove this, let $\psi \in V^*$ therefore there exists a unique $y \in V$ such that $\psi(x) = x.y$ for all $x \in V$. Hence $\psi(f(x_n)) = f(x_n).y = x_n.f(y)$ for all $n \in \mathbb{N}$. Claim $\lim_{n\to\infty} x_n.f(y) = x.f(y)$. Let $\eta(x) = x.f(y)$ for all $x \in V$. Then η is continuous and left linear. Hence $\lim_{n\to\infty} \eta(x_n) = \eta(x)$ i.e., $\lim_{n\to\infty} x_n.f(y) = x.f(y)$. Thus we have the claim. Therefore $\lim_{n\to\infty} \psi(f(x_n)) = \lim_{n\to\infty} x_n.f(y) = x.f(y) = f(x).y = \psi(f(x))$. Hence $\lim_{n\to\infty} f(x_n) = f(x)$ weakly. Let $\lim_{k\to\infty} f(x_{n_k}) = z$ strongly therefore $\lim_{k\to\infty} f(x_{n_k}) = z$ weakly. Since $\lim_{n\to\infty} f(x_n) = x$ weakly and $\lim_{k\to\infty} f(x_{n_k}) = z$ weakly, $\lim_{n\to\infty} \varphi(f(x_n)) = \varphi(f(x))$ and $\lim_{k\to\infty} \varphi(f(x_{n_k})) = \varphi(z)$ for all $\varphi \in V^*$ therefore $\varphi(f(x)) = \varphi(z)$ for all $\varphi \in V^*$. Let $\{e_\alpha\}_{\alpha\in\mathbb{N}}$ be an orthonormal left basis of V. Then if $\varphi_\alpha(x) = x.e_\alpha$ for all $\alpha \in \mathbb{N}$ we get that $\varphi_\alpha(f(x)) = \varphi_\alpha(z)$ for all $\alpha \in \mathbb{N}$ therefore $f(x).e_\alpha = z.e_\alpha$ for all $\alpha \in \mathbb{N}$. i.e., $(f(x)-z).e_\alpha$ for all $\alpha \in \mathbb{N}$. Since $(e_\alpha)_{\alpha\in\mathbb{N}}$ is maximal orthonormal set, $f(x).z = 0$, hence $f(x) = z$.

Hence we have the claim. Thus $\lim_{k\to\infty} f(x_{n_k}) = f(x)$. Claim that $\lim_{n\to\infty} f(x_n) = f(x)$ strongly. Suppose not therefore there exists $\varepsilon_0 > 0$ such that for each $N \in \mathbb{N}$ there exists $n > N$ such that $f(x_n) \notin B(f(x), \varepsilon_0)$. Let $N = 1$ there exists a $m_1 > N$ such that $f(x_{m_1}) \notin B(f(x), \varepsilon_0)$. Let $N = m_1$ there exists $m_2 > m_1$ such that $f(x_{m_2}) \notin B(f(x), \varepsilon_0)$. Continuing we get a sequence $(x_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} f(x_{n_k}) \neq f(x)$. Since $(f(x_{m_k}))_{k\in\mathbb{N}}$ is relatively compact, there exists a strongly convergent subsequence $(f(x_{m_{k_l}}))_{l\in\mathbb{N}}$.

Let $\lim_{l\to\infty} f(x_{m_{k_l}}) = y$. Since $\lim_{k\to\infty} f(x_{m_k}) \neq f(x)$, $f(x) \neq y$.

Since $\lim_{n\to\infty} f(x_n) = f(x)$ weakly, $\lim_{n\to\infty} \varphi(f(x_n)) = \varphi(f(x))$ for all $\varphi \in V^*$, so $\lim_{l\to\infty} \varphi(f(x_{m_{k_l}})) = \varphi(f(x))$ for all $\varphi \in V^*$. Since $\lim_{l\to\infty} \varphi(f(x_{m_{k_l}})) = \varphi(y)$ for all $\varphi \in V^*$, $\varphi(f(x)) = \varphi(y)$ for all $\varphi \in V^*$. By the same argument as above, $f(x) = y$, a contradiction. Hence we have the claim. Thus $\lim_{n\to\infty} f(x_n) \cdot x_n = f(x) \cdot x$. ⁂

<u>Lemma</u> 4.47 Let V be a separable LSPS (RSPS) which is also a Hilbert space. Let $f: V \to V$ be a nonzero c.c. self-adjoint map. Let $Q(x) = f(x) \cdot x$ for all $x \in V$. Suppose that there exists a $x_0 \in \overline{B(0,1)}$ such that $\sup_{x\in\overline{B(0,1)}} \{|Q(x)|\} = |Q(x_0)|$. Then if $y \perp x_0$ we must have that $f(x_0) \cdot y = x_0 \cdot f(y) = 0$. In particular x_0 is an eigenvector of f.

<u>Proof:</u> Suppose $Q \equiv 0$. Let $x_0 \in V \setminus \{0\}$. Then let $x' = \frac{x_0}{\|x_0\|}$ therefore $\|x'\| = 1$. Let $a \in \mathbb{R} \setminus \{0\}$ be arbitrary and $f(x') = y$.

Then let $x = \frac{x' + ay}{(1+|a|^2\|y\|^2)^{1/2}}$ therefore $\|x\| = 1$.

$$0 = Q(x) = f(x).x = \frac{(f(x')+af(y)).(x'+ay)}{1+|a|^2\|y\|^2}$$

$$= \frac{[f(x').x' + a(f(x').y) + a(f(y).x') + a^2(f(y).y)]}{1+|a|^2\|y\|^2}$$

therefore $0 = a[f(x').y + f(y).x'] = a[f(x').y + \overline{f(x').y}] = 2a\,\mathrm{Re}(f(x').y)$

$= 2a(f(x').y)$, hence $f(x').f(x') = \frac{f(x_0).f(x_0)}{\|x_0\|^2} = 0$, so $f(x_0) = 0$.

Hence $f \equiv 0$, a contradiction. So assume that $Q \not\equiv 0$. Then $x_0 \neq 0$.

Claim that $\|x_0\| = 1$. Suppose not therefore $0 < \|x_0\| < 1$. Let

$x_1 = \frac{x_0}{\|x_0\|}$ then $\|x_1\| = 1$. Hence $Q(x_1) = \frac{1}{\|x_0\|^2} \cdot Q(x_0)$, so

$|Q(x_1)| > |Q(x_0)|$, a contradiction. Thus we have the claim.

Fix $y \perp x_0$. Let $a \in \mathbb{R} \setminus \{0\}$ and let $x = \frac{x_0+ay}{(1+|a|^2\|y\|^2)^{1/2}}$. Then

$\|x\| = 1$. $Q(x) = f(x).x = \frac{1}{1+|a|^2\|y\|^2}(f(x)+af(y)).(x_0+ay)$

$$= \frac{1}{1+|a|^2\|y\|^2}\left[f(x_0).x_0 + a(f(x_0).y) + a(f(y).x_0 + |a|^2(f(y).y)\right]$$

$$= \frac{1}{1+|a|^2\|y\|^2}\left[Q(x_0) + a(f(x_0).y + \overline{f(x_0).y}) + |a|^2(f(y).y)\right].$$

If $|a|$ is sufficiently small we get that $Q(x) \approx Q(x_0)+a(2Re(f(x_0).y))$.

If $f(x_0).y \neq 0$, then a can be chosen to make $|Q(x)| > |Q(x_0)|$, a contradiction. Hence $f(x_0).y = x_0.f(y) = 0$. Since

$V = \{\alpha x_0\}_{\alpha\in \mathbb{H}} \oplus \{\alpha x_0\}^{\perp}_{\alpha\in \mathbb{H}}$, $f(x_0) = y+z$ for some $y \in \{\alpha x_0\}_{\alpha\in \mathbb{H}}$

and $z \in \{\alpha x_0\}^{\perp}_{\alpha\in \mathbb{H}}$ therefore $0 = f(x_0).z = y.z+z.z = 0+z.z = z.z$,

so $z = 0$. Hence x_0 is an eigenvalue of f. ⋇

Theorem 4.48 (Hilbert-Schmidt) Let V be a separable LSPS (RSPS). Let $f: V \longrightarrow V$ ne a nonzero c.c. seft-adjoint map. Then there exists a countable set of orthonormal eigenvectors $(e_n)_{n\in \mathbb{N}}$ of f such that every vector $v\in V$ has unique representation in the form $x = \Sigma\, \beta_n e_n + x'$ where $x' \in \ker f$ also, $f(x) = \Sigma\, \lambda_n \beta_n e_n$ where λ_n is the eigenvalue of e_n and if the number of eigenvectors is ∞ then $\lim_{n\to\infty} \lambda_n = 0$.

Proof: For each $x\in V$, $|f(x).x| \leq \|f(x)\|\,\|x\| \leq \|f\|\,\|x\|\,\|x\| = \|f\|\,\|x\|^2$. If $\|x\| \leq 1$, then $|f(x).x| \leq \|f\| < \infty$, so $\{|f(x).x|\}_{\|x\|\leq 1}$ is bounded. Let $M_1 = \sup_{\|x\|\leq 1} \{|f(x).x|\}$. Let $(x_n)_{n\in \mathbb{N}}$ be a sequence of elements of V such that $\|x_n\| = 1$ and $\lim_{n\to\infty} |f(x_n).x_n| = M_1$. Since V is a Hilbert space, V is left isomorphic to $\bar{V}$ and so by Theorem 4.4 2, the closed unit ball center at 0 is weakly compact, hence there exists weakly convergent subsequence $(x_{n_k})_{k\in \mathbb{N}}$. Let $\lim_{k\to\infty} x_{n_k} = y$ therefore $y \in$ the closed unit ball i.e. $\|y\| < 1$. By lemma 4.46, $\lim_{k\to\infty} f(x_{n_k}).x_{n_k}$

$= f(y).y$, so $\lim_{k\to\infty} |f(x_{n_k}).x_{n_k}| = |f(y).y|$. Hence $|f(y).y| = M_1$.

By lemma 4.48, y is an eigenvector of f ($\therefore y \neq 0$). Claim that $\|y\| = 1$, if not, then $\|y\| < 1$. Let $y' = \frac{y}{\|y\|}$, then $\|y'\| = 1$ and $|f(y').y'| = \frac{1}{\|y\|^2} |f(y).y| > M_1$, a contradiction. Hence we have the claim. Let $e_1 = y$ and let λ_1 be the eigenvalue of e_1 therefore $|f(e_1).e_1| = |\lambda| \, |e_1.e_1| = |\lambda| \, \|e_1\|^2 = |\lambda|$. So $|\lambda| = M_1$.

Let $W_1 = (\alpha e_1)_{\alpha \in \mathbb{N}}$ then W_1 is a closed left linear subspace of V, hence W_1 is a separable LSPS which is also a Hilbert space and so is $W_1^{\perp}$. Let $w_1' \in W_1^{\perp}$ then $f(w_1').e_1 = w_1'.f(e_1) = w_1'.(\lambda_1 e_1) = (w'.e_1)\bar{\lambda} = 0$ therefore $f(w_1') \in W_1^{\perp}$. Hence $f: W_1^{\perp} \to W_1^{\perp}$ is c.c. and self adjoint.

Let $M_2 = \sup_{\substack{\|x\| \leq 1 \\ x \in W^{\perp}}} \{|f(x).x|\}$. Hence $M_2 \leq M_1$. By the same argument as before there exists an $e_2 \in W_1^{\perp}$ which is an eigenvector of f such that $\|e_2\| = 1$. Again let λ_2 be the eigenvalue of e_2 then $|\lambda_2| = M_2$, so $|\lambda_2| \leq |\lambda_1|$. Let $W_2 = \{\alpha_1 e_1 + \alpha_2 e_2\}_{\alpha_1, \alpha_2 \in \mathbb{H}}$. Again $f: W_2^{\perp} \to W_2^{\perp}$ is c.c. and self adjoint. Therefore by the same reasoning there exists an $e_3 \in W_2^{\perp}$ such that $\|e_3\| = 1$ which is an eigenvector of f and the eigenvalue λ_3 of e_3 has the property that $|\lambda_3| \leq |\lambda_2|$.

Continue in the way. There are two possibilities:

<u>case</u> 1. Suppose that there exists an $n_0 \in \mathbb{N}$ such that $f(x).x = 0$ on $W_{n_0}^{\perp}$. By the same proof as in lemma 4.47, $f/_{W_{n_0}^{\perp}} = 0$. Hence $W_{n_0}^{\perp} \subseteq \ker f$. Since $V = W_{n_0} \oplus W_{n_0}^{\perp}$, for each $x \in V$ x can be written uniquely in the form $x = a+a'$ where $a \in W_{n_0}$ and $a' \in W_{n_0}^{\perp} \subseteq \ker f$. Since $a \in W_{n_0}$, $a = \sum_{n=1}^{n_0} \beta_n e_n$ for some $\beta_n \in \mathbb{H}$. Hence for each $x \in V$

$$f(x) = f(a)+f(a') = \sum_{n=1}^{n_0} \lambda_n \beta_n e_n .$$

<u>case</u> 2. For each $n \in \mathbb{N}$ there exists an $x_n \in W_n^{\perp}$ such that $f(x_n).x_n \neq 0$. In this case there exists infinitely many eigenvector $(\lambda_n)_{n\in \mathbb{N}}$ and they are ordered so that $|\lambda_n| \geq |\lambda_{n+1}|$. Claim that $\lim_{n\to\infty} \lambda_n = 0$. Since V is left isomorphic to $\ell^2_{\mathbb{H}}$ and $\lim_{n\to\infty} e'_n = 0$ weakly in $\ell^2_{\mathbb{H}}$ where $e'_n = (0,0,1,0,0,\ldots)$ (1 in the n^{th} place) for all $n \in \mathbb{N}$, $\lim_{n\to\infty} e_n = 0$ weakly in V. Hence $\lim_{n\to\infty} f(e_n) = 0$ strongly (same proof as in lemma 4.46) therefore $\lim_{n\to\infty} \|f(e_n)\| = 0$. But $\|f(e_n)\| = \|\lambda_n e_n\| = |\lambda_n| \|e_n\| = |\lambda_n|$, so $\lim_{n\to\infty} \lambda_n = 0$. Let W_∞ be the closure of the left linear subspace generated by $(e_\alpha)_{\alpha\in \mathbb{N}}$. If $W_\infty^{\perp} = 0$, then $V = W_\infty$ and we are done (since let $x' = 0$). If $W_\infty^{\perp} \neq 0$, then choose $x \in W_\infty^{\perp} \setminus \{0\}$ therefore we have that $\left| \frac{f(x)}{\|x\|} . \frac{x}{\|x\|} \right| \leq |\lambda_n| = M_n$ for all $n \in \mathbb{N}$ therefore $0 \leq |f(x).x| \leq |\lambda_n| \, \|x\|^2 \longrightarrow 0$. Hence $f(x).x = 0$.

By the same proof as in lemma [illegible].47, $f/_{W_\infty^\perp} = 0$. Since $V = W_\infty \oplus W_\infty^\perp$, for each $x \in V$ can be written uniquely in the form $x = a+a'$ where $a \in W_\infty$ and $a' \in W_\infty^\perp$, so $x = \sum_{n=1}^{\infty} \beta_n e_n + a'$. ⋇