CHAPTER III

HILBERT SPACE OVER THE QUATERNIONS

Definition 3.1 Let V be a left vector space over $\mathbb{H}$. Then a map $\circ : V \times V \longrightarrow \mathbb{H}$ is said to be a left sympletic product (LSP) on V if and only if

(i) $x \circ y = \overline{y \circ x}$ for all $x, y \in V$.

(ii) $(\alpha x + \beta y) \circ z = \alpha(x \circ z) + \beta(y \circ z)$ for all $x, y, z \in V$ and for all $\alpha, \beta \in \mathbb{H}$.

(iii) For each $x \in V$ $x \circ x \geqslant 0$ and $x \circ x = 0$ if and only if $x = 0$.

If $\circ$ is a left sympletic product space on V then the pair $(V, \circ)$ is called a left sympletic product space.

Definition 3.2 Let V be a right vector space over $\mathbb{H}$. Then a map $\circ : V \times V \longrightarrow \mathbb{H}$ is said to be right sympletic product (RSP) if and only if

(i) $x \circ y = \overline{y \circ x}$ for all $x, y \in V$.

(ii) $x \circ (y\alpha + z\beta) = (x \circ y)\alpha + (x \circ z)\beta$ for all $x, y, z \in V$ and for all $\alpha, \beta \in \mathbb{H}$.

(iii) For each $x \in V$ $x \circ x \geqslant 0$ and $x \circ x = 0$ if and only if $x = 0$.

<u>Remark</u>: From now on we shall use the abbreviations "LSPS" and "RSPS" for left sympletic product space and right sympletic product space.

<u>Definition</u> 3.3 Let V be a vector space over $\mathbb{H}$. Then a LSP on V is said to be a bi-LSP if and only if $x\cdot(y\alpha+z\beta) = (x\bar{\alpha})\cdot y+(x\bar{\beta})\cdot z$ for all $x,y,z \in V$ and for all α, $\beta \in \mathbb{H}$.

If $\cdot$ is a bi-LSP on V then the pair $(V,\cdot)$ is called a bi-LSPS.

<u>Definition</u> 3.4 Let V be a vector space over $\mathbb{H}$. Then a RSP on V is said to be a bi-RSP if and only if

$(\alpha x+\beta y)\cdot z = x\cdot(\bar{\alpha}z)+y\cdot(\bar{\beta}z)$ for all $x,y,z \in V$ and for all $\alpha, \beta \in \mathbb{H}$.

If $\cdot$ is a bi-RSP on V then the pair $(V,\cdot)$ is called a bi-RSPS.

<u>Example</u> 3.5 (i) $(\mathbb{H}^n,\cdot)$ is a bi-LSPS where

$$(x_1,x_2,\ldots,x_n)\cdot(y_1,y_2,\ldots,y_n) = \sum_{k=1}^{n} x_k\bar{y}_k .$$

(ii) $(\mathbb{H}^n,\cdot)$ is a bi-RSPS where $(x_1,x_2,\ldots,x_n)\cdot(y_1,y_2,\ldots,y_n)= \sum_{k=1}^{n} \bar{x}_k y_k$.

(iii) $(\ell^2_{\mathbb{H}},\cdot)$ is a bi-LSPS where $(x_1,x_2,\ldots)\cdot(y_1,y_2,\ldots) = \sum_{k=1}^{\infty} x_k\bar{y}_k$.

(iv) $(\ell^2_{\mathbb{H}},\cdot)$ is a bi-RSPS where $(x_1,x_2,\ldots)\cdot(y_1,y_2,\ldots) = \sum_{k=1}^{\infty} \bar{x}_k y_k$.

<u>Remark</u>: All theorems true for LSPS's are true for RSPS's and the proof is same. So we shall only prove theorems for the LSPS case.

Proposition 3.6 Let V be a LSPS and let $x,y,z \in V$ and $\alpha, \beta \in \mathbb{H}$. Then $x.(\alpha y+\beta z) = (x.y)\bar{\alpha}+(x,z)\bar{\beta}$.

Proof: $x.(\alpha y+\beta z) = \overline{(\alpha y+\beta z).x} = \overline{\alpha(y.x)}+\overline{\beta(z.x)}$
$= (x.y)\bar{\alpha}+(x.z)\bar{\beta}$. ⋇

Remark: If V is a bi-LSPS (bi-RSPS) and $\alpha \in \mathbb{R}$ then $x.(y\alpha) = (x.y)\alpha$.

Proposition 3.7 Let V be a bi-LSPS and let $x,y,z \in V$ and $\alpha, \beta \in \mathbb{H}$. Then $(x\alpha+y\beta).z = x.(z\bar{\alpha})+y.(z\bar{\beta})$

Proof: $(x\alpha+y\beta).z = \overline{z.(x\alpha+y\beta)} = \overline{(z\bar{\alpha}).x+(z\bar{\beta}).y}$
$= x.(z\bar{\alpha})+y.(z\bar{\beta})$. ⋇

Remark: (i) In the RSPS case we get that $(y\alpha+z\beta).x = \bar{\alpha}(y.x)+\bar{\beta}(z.x)$ for all $x,y,z \in V$ and for all $\alpha, \beta \in \mathbb{H}$.

(ii) In the bi-RSPS case we get that $z.(\alpha x+\beta y) = (\bar{\alpha}z).x+(\bar{\beta}z).y$ for all $x,y,z \in V$ and for all $\alpha, \beta \in \mathbb{H}$.

Definition 3.8 Let V be a LSPS(RSPS) and let $x \in V$ define
$\|x\| = (x.x)^{1/2}$

Proposition 3.9 (SCHWARZ INEQUALITY) Let V be a LSPS (RSPS). Then $|x.y| \leqslant \|x\| \, \|y\|$ for all $x, y \in V$

Proof: Let $x,y \in V$. Let $A = \|x\|^2$, $B = |x.y|$ and $C = \|y\|^2$. If $x.y = 0$, then we are done. Assume that $x.y \neq 0$ and let $\alpha = \frac{|x.y|}{y.x}$. Therefore $|\alpha| = 1$ and $\alpha(y.x) = |x.y| = B$. Hence for each $r \in \mathbb{R}$ we have that

$$0 \leqslant (x-r\alpha y)\circ(x-r\alpha y) = x\circ x - r(x\circ\alpha y) - r(\alpha y\circ x) + r^2\alpha(y\circ\alpha y)$$

$$= x\circ x - r(x\circ y)\bar{\alpha} - r\alpha(y\circ x) + r^2\alpha(y\circ y)\bar{\alpha}$$

$$= x\circ x - r(x\circ y)\bar{\alpha} - r\alpha(y\circ x) + r^2(y\circ y).$$

Since $(x\circ y)\bar{\alpha}+\alpha(y\circ x) = \overline{(y\circ x)}\bar{\alpha} + \alpha(y\circ x)$

$$= \alpha(y\circ x) + \overline{\alpha(y\circ x)} = B+\bar{B} = 2B,$$

$x\circ x - r(x\circ y)\bar{\alpha} - r\alpha(y\circ x) + r^2(y\circ y) = A-2rB-r^2C \geqslant 0$, hence discriminant $\leqslant 0$. Therefore $(-2B)^2-4AC \leqslant 0$ i.e., $B \leqslant \sqrt{A}\sqrt{C} = \|x\|\,\|y\|$. Hence $|x\circ y| \leqslant \|x\|\,\|y\|$. ※

<u>Corollary</u> 3.10 Let V be a LSPS(RSPS). Then $\|x+y\| \leqslant \|x\|+\|y\|$ for all x, $y \in V$.

<u>Proof</u>: $\|x+y\|^2 = (x+y)\circ(x+y) = x\circ x + x\circ y + y\circ x + y\circ y$

$= x\circ x + x\circ y + \overline{x\circ y} + y\circ y = \|x\|^2 + 2\,\mathrm{Re}(x\circ y) + \|y\|^2 \leqslant \|x\|^2 + 2|\mathrm{Re}(x\circ y)| + \|y\|^2$

$\leqslant \|x\|^2 + 2|x\circ y| + \|y\|^2 \leqslant \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 = (\|x\|+\|y\|)^2.$

Therefore $\|x+y\| \leqslant \|x\|+\|y\|$. ※

<u>Remark</u>: Let V be a LSPS(RSPS). Then $\|\ \|$ is a left (right) norm on V. Therefore every LSPS(RSPS) can be made into a metric space by defining $d(x,y) = \|x-y\|$. A metric determines a topology. Therefore every LSPS(RSPS) is a topological space.

<u>Definition</u> 3.11 Let V be a LSPS(RSPS). Then V is called a Hilbert space if and only if V is complete metric space.

Examples of Hilbert Spaces

(i) $\mathbb{H}^n$. (ii) $\ell^2_{\mathbb{H}}$.

Proposition 3.12 Let V be a NLS. Then there exists a bi-LSP (bi-RSP). on V such that $\|x\| = (x.x)^{1/2}$ if and only if $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+ 2\|y\|^2$ for all x, $y \in V$.

Proof: Suppose that there exists a bi-LSP. on V such that $x = (x.x)^{1/2}$ then $\|x+y\|^2+\|x-y\|^2 = (x+y).(x+y)+(x-y)(x-y) =$ $x.x+x.y+y.x+y.y+x.x+y.y-y.x-x.y = 2\|x\|^2+2\|y\|^2$ for all x, $y\in V$.

Conversely, suppose $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$ for all $x,y\in V$. Define $x.y = \frac{1}{4}\Big[\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2+j\|x+jy\|^2$ $-j\|x-jy\|^2+k\|x+ky\|^2-k\|x-ky\|^2\Big]$ for all x, $y \in V$.

Let $x\in V$. Then $x.x = \frac{1}{4}\Big[\|x+x\|^2-\|x-x\|^2+i\|x+ix\|^2-i\|x-ix\|^2+j\|x+jx\|^2$

$$-j\|x-jx\|^2+k\|x+kx\|^2-k\|x-kx\|^2\Big]$$

$$= \frac{1}{4}\Big[\|2x\|^2+i\|x\|^2(|1+i|^2-|1-i|^2)+j\|x\|^2(|1+j|^2-|1-j|^2)$$

$$+k\|x\|^2(|1+k|^2-|1-k|^2)\Big]$$

$$= \frac{1}{4}\Big[4\|x\|^2+i\|x\|^2(0)+j\|x\|^2(0)+k\|x\|^2(0)\Big] = \|x\|^2$$

Hence we see that $x.x = 0$ if and only if $x = 0$.

Let $x,y\in V$. Then $\overline{x.y} = \frac{1}{4}\Big[\|y+x\|^2-\|y-x\|^2-i\|ix+i^2y\|^2+i\|ix-i^2y\|^2$

$$-j\|jx+j^2y\|^2+j\|jx-j^2y\|^2-k\|kx+k^2y\|^2+k\|kx-k^2y\|^2\Big]$$

$$= \frac{1}{4}\Big[\|y+x\|^2-\|y-x\|^2-i\|ix-y\|^2+i\|ix+y\|^2-j\|jx-y\|^2$$

$$+j\|jx+y\|^2-k\|kx-y\|^2+k\|kx+y\|^2\Big]$$

$$= \frac{1}{4}\Big[\|y+x\|^2-\|y-x\|^2+i\|y+ix\|^2-i\|y-ix\|^2+j\|y+jx\|^2$$

$$-j\|y-jx\|^2+k\|y+kx\|^2-k\|y-kx\|^2\Big]$$

$$= y \circ x \ .$$

Define $\emptyset : V \times V \times V \longrightarrow \mathbb{H}$ by $\emptyset(x,y,z) = 4\big[(x+y)\circ z - x\circ z - y\circ z\big]$ for all $x, y, z \in V$.

Must show that $\emptyset \equiv 0$. Let $x,y,z \in V$. Then

$$\emptyset(x,y,z) = 4\Big[\frac{1}{4}\big[\|x+y+z\|^2-\|x+y-z\|^2+i\|x+y+iz\|^2-i\|x+y-iz\|^2+j\|x+y+jz\|^2$$

$$-j\|x+y-jz\|^2+k\|x+y+kz\|^2-k\|x+y-kz\|^2\big]-\frac{1}{4}\big[\|x+z\|^2-\|x-z\|^2+$$

$$+i\|x+iz\|^2-i\|x-iz\|^2+j\|x+jz\|^2-j\|x-jz\|^2+k\|x+kz\|^2-k\|x-kz\|^2\big]$$

$$-\frac{1}{4}\big[\|y+z\|^2-\|y-z\|^2+i\|y+iz\|^2-i\|y-iz\|^2+j\|y+jz\|^2+k\|y+kz\|^2$$

$$-k\|y-kz\|^2-j\|y-jz\|^2\big]\Big]$$

$$= \|x+y+z\|^2-\|x+y-z\|^2+i\|x+y+iz\|^2-i\|x+y-iz\|^2+j\|x+y+jz\|^2$$

$$-j\|x+y-jz\|^2+k\|x+y+kz\|^2-k\|x+y-kz\|^2-\|x+z\|^2+\|x-z\|^2-i\|x+iz\|^2$$

$$+i\|x-iz\|^2-j\|x+jz\|^2+j\|x-jz\|^2-k\|x+kz\|^2+k\|x-kz\|^2-\|y+z\|^2$$

$$+\|y-z\|^2-i\|y+iz\|^2+i\|y-iz\|^2-j\|y+jz\|^2+j\|y-jz\|^2-k\|y+kz\|^2$$

$$+k\|y-kz\|^2 \ . \qquad [3.12.1]$$

By hypothesis, $\|x+y+z\|^2 = 2\|x+z\|^2+2\|y\|^2-\|x+z-y\|^2$

$$\|x+y-z\|^2 = 2\|x-z\|^2+2\|y\|^2-\|x-y-z\|^2$$

$$\|x+y+iz\|^2= 2\|x+iz\|^2+2\|y\|^2-\|x+iz-y\|^2$$

$$\|x+y-iz\|^2= 2\|x-iz\|^2+2\|y\|^2-\|x-iz-y\|^2$$

$$\|x+y+jz\|^2= 2\|x+jz\|^2+2\|y\|^2-\|x+jz-y\|^2$$

$$\|x+y-jz\|^2= 2\|x-jz\|^2+2\|y\|^2-\|x-jz-y\|^2$$

$$\|x+y+kz\|^2= 2\|x+kz\|^2+2\|y\|^2-\|x+kz-y\|^2$$

$$\|x+y-kz\|^2= 2\|x-kz\|^2+2\|y\|^2-\|x-kz-y\|^2 .$$

Substitute these equations in [3.12.1] we get that

$$\begin{aligned}\emptyset(x,y,z) = {} & 2\|x+z\|^2+2\|y\|^2-\|x+z-y\|^2-2\|x-z\|^2-2\|y\|^2+\|x-y-z\|^2+2i\|x+iz\|^2 \\ & +2i\|y\|^2-i\|x+iz-y\|^2-2i\|x-iz\|^2-2i\|y\|^2+i\|x-iz-y\|^2+2j\|x+jz\|^2 \\ & +2j\|y\|^2-j\|x+jz-y\|^2+2k\|y\|^2-k\|x+kz-y\|^2-2k\|x-kz\|^2-2k\|y\|^2-2j\|x-jz\|^2 \\ & +k\|x-kz-y\|^2-\|x+z\|^2+\|x-z\|^2-i\|x+iz\|^2+i\|x-iz\|^2-j\|x+jz\|^2-2j\|y\|^2 \\ & +j\|x-jz\|^2-k\|x+kz\|^2+k\|x-kz\|^2-\|y+z\|^2+\|y-z\|^2-i\|y+iz\|^2+j\|x-jz-y\|^2 \\ & +i\|y-iz\|^2-j\|y+jz\|^2+j\|y-jz\|^2-k\|y+kz\|^2+k\|y-kz\|^2. \quad [3.12.2]\end{aligned}$$

Add [3.12.1] and [3.12.2] divide by 2, we get that

$$\phi(x,y,z) = \tfrac{1}{2}(\|x+y+z\|^2+\|x-y-z\|^2)-\tfrac{1}{2}(\|x+y-z\|^2+\|x+z-y\|^2)+\tfrac{1}{2}i(\|x+y+iz\|^2$$

$$+\|x-iz-y\|^2)-\tfrac{1}{2}i(\|x+y-iz\|^2+\|x+iz-y\|^2)+\tfrac{1}{2}j(\|x+y+jz\|^2+\|x-jz-y\|^2)$$

$$-\tfrac{1}{2}j(\|x+y-jz\|^2+\|x+jz-y\|^2)+\tfrac{1}{2}k(\|x+y+kz\|^2+\|x-kz-y\|^2)-\tfrac{1}{2}k(\|x+y-kz\|^2$$

$$-\|x+kz-y\|^2)-\|y+z\|^2+\|y-z\|^2-i\|y+iz\|^2+i\|y-iz\|^2-j\|y+jz\|^2-j\|y-jz\|^2$$

$$-k\|y+kz\|^2+k\|y-kz\|^2.$$

By hypothesis, $\phi(x,y,z) = \|x\|^2+\|y+z\|^2-\|x\|^2-\|y-z\|^2+i\|x\|^2+i\|y+iz\|^2$

$$-i\|x\|^2-i\|y-iz\|^2+j\|x\|^2+j\|y+jz\|^2$$

$$-j\|x\|^2-j\|y-jz\|^2+k\|x\|^2+k\|y+kz\|^2-k\|x\|^2$$

$$-k\|y-kz\|^2-\|y+z\|^2+\|y-z\|^2-i\|y+iz\|^2+i\|y-iz\|^2$$

$$-j\|y+jz\|^2+j\|y-jz\|^2-k\|y+kz\|^2+k\|y-kz\|^2$$

$$= 0$$

Hence $\phi(x,y,z) = 0$. Therefore $(x+y).z = x.z+y.z$ for all $x,y,z \in V$.

Next we must show that $(\alpha x).y = \alpha(x.y)$ for all $x,y \in V$ and for all $\alpha \in \mathbb{H}$. Let $x,y \in V$. Define $\psi: \mathbb{H} \longrightarrow \mathbb{H}$ by $\psi(\alpha) = (\alpha x).y-\alpha(x.y)$ for all $\alpha \in \mathbb{H}$.

Must show that $\psi \equiv 0$. Note that $\psi(0) = 0$, $\psi(1) = 0$ and $\psi(-1) =$ $(-x).y+(x.y)$

$$= \tfrac{1}{4}\Big[\|-x+y\|^2-\|-x-y\|^2+i\|-x+iy\|^2-i\|-x-iy\|^2+j\|-x+jy\|^2-j\|-x-jy\|^2+k\|-x+ky\|^2$$

$$-k\|-x-ky\|^2\Big]+x.y$$

$$= -\tfrac{1}{4}\Big[\|x+y\|^2-\|x-y\|^2-i\|x-iy\|^2+i\|x+iy\|^2-j\|x-jy\|^2+j\|x+jy\|^2-k\|x-ky\|^2$$

$$+k\|x+ky\|^2\Big]+x.y$$

$= -x.y+x.y = 0.$

Let $n \in \mathbb{N}$ then $\varphi(n) = (nx).y-n(x.y) = (\overbrace{x+x+\ldots+x}^{n \text{ times}}).y-n(x.y)$

$= (\overbrace{x.y+\ldots+x.y}^{n \text{ times}})-n(x.y) = n(x.y)-n(x.y) = 0.$ Hence $\varphi(n) = 0$ for all $n \in \mathbb{N}.$

Let $n \in \mathbb{Z}^-$ then $\varphi(n) = (nx).y-n(x.y) = (\overbrace{-x-x-\ldots-x}^{n \text{ times}}).y+ n (x.y)$

$= (-(x.y)-\ldots-(x.y))+ n (x.y) = - n (x.y)+ n (x.y) = 0.$

Hence $\varphi(n) = 0$ for all $n \in \mathbb{Z}^-.$

Let $p \in \mathbb{Z} \setminus \{0\}$ then $\varphi(\frac{1}{p}) = (\frac{1}{p}x).y-\frac{1}{p}(x.y) = \frac{p}{p}\left[(\frac{1}{p}x).y\right]-\frac{1}{p}(x.y)$

$=\left[\frac{1}{p}(x.y)\right]-\frac{1}{p}(x.y) = 0.$ Hence $\varphi(\alpha) = 0$ for all $\alpha \in \mathbb{Q}.$

Let $r \in \mathbb{R}$ there exists a sequence $(\beta_n)_{n \in \mathbb{N}}$ in $\mathbb{Q}$ such that $\lim_{n\to\infty} \beta_n = r.$ Since φ is continuous, $\lim_{n\to\infty} \varphi(\beta_n) = \varphi(r).$ But $\varphi(\beta_n) = 0$ for all $n \in \mathbb{N}.$ Hence $\varphi(r) = 0.$ Therefore $\varphi(r) = 0$ for all $r \in \mathbb{R}.$ Since $\varphi(i) = (ix).y-i(x.y)$

$= \frac{1}{4}(\|ix+y\|^2-\|ix-y\|^2+i\|ix+iy\|^2-i\|ix-iy\|^2+j\|ix+jy\|^2-j\|ix-jy\|^2$

$+k\|ix+ky\|^2-k\|ix-ky\|^2)-i(x.y)$

$= \frac{1}{4}(\|ix-i^2y\|^2-\|ix+i^2y\|^2+i\|x+y\|^2-i\|x-y\|^2-ik\|ix-iky\|^2+ik\|ix+iky\|^2$

$+ij\|ix+ijy\|^2-ij\|ix-ijy\|^2)-i(x.y).$

$= \frac{1}{4}(-i^2\|ix-i^2y\|^2+i^2\|ix+i^2y\|^2+i\|x+y\|^2-i\|x-y\|^2-ik\|ix-iky\|^2+ik\|ix+iky\|^2$

$+ij\|ix+ijy\|^2-ij\|ix-ijy\|^2)-i(x\circ y)$

$= \frac{1}{4}i(-i\|x-iy\|^2+i\|x+iy\|^2+\|x+y\|^2-\|x-y\|^2-k\|x-ky\|^2+k\|x+ky\|^2+j\|x+jy\|^2$

$-j\|x-jy\|^2)-i(x\circ y)$

$= i(x\circ y)-i(x\circ y) = 0, \varphi(i) = 0.$ Similarly $\varphi(j) = 0$ and $\varphi(k) = 0.$

Hence $\varphi(\alpha) = 0$ for all $\alpha \in \mathbb{H}$. Thus $(\alpha x)\circ y = \alpha(x\circ y)$ for all $x,$ $y \in V$ and for all $\alpha \in \mathbb{H}$.

Next we must show that $x\circ(y\alpha) = (x\bar{\alpha})\circ y$ for all $x,y \in V$ and for all $\alpha \in \mathbb{H}$. Fix $x,y \in V$. Define $\varphi' : \mathbb{H} \to \mathbb{H}$ by $\varphi'(\alpha) = (x\bar{\alpha})\circ y - x\circ(y\alpha)$ for all $\alpha \in \mathbb{H}$. Must show that $\varphi' \equiv 0$. Since $x\alpha = \alpha x$ and $\alpha y = y\alpha$ for all $\alpha \in \mathbb{R}$, $x\circ(y\alpha) = x\circ(\alpha y) = \overline{(\alpha y)\circ x} = \overline{\alpha(y\circ x)} = \overline{(y\circ x)}\bar{\alpha} = (x\circ y)\alpha = \alpha(x\circ y) = (\alpha x)\circ y = (x\alpha)\circ y = (x\bar{\alpha})\circ y.$

Hence $\varphi'(\alpha) = 0$ for all $\alpha \in \mathbb{R}$. Since

$(x(-i))\circ y = \frac{1}{4}\left[\|-xi+y\|^2-\|-xi-y\|^2+i\|-xi+iy\|^2-i\|-xi-iy\|^2+j\|-xi+jy\|^2\right.$

$\left.-j\|-xi-jy\|^2+k\|-xi+ky\|^2-k\|-xi-ky\|^2\right]$

$= \frac{1}{4}\left[\|-xi-yi^2\|^2-\|-xi+yi^2\|^2+i\|-xi-iyi^2\|^2-i\|-xi+iyi^2\|^2\right.$

$\left.+j\|-xi-jyi^2\|^2-j\|-xi+jyi^2\|^2+k\|-xi-kyi^2\|^2-k\|-xi+kyi^2\|^2\right]$

$= \frac{1}{4}\left[\|-x-yi\|^2-\|-x+yi\|^2+i\|-x-iyi\|^2-i\|-x+iyi\|^2+j\|-x+jyi\|^2\right.$

$\left.-j\|-x+jyi\|^2+k\|-x-kyi\|^2-k\|-x+kyi\|^2\right]$

$$= \frac{1}{4}\Big|\,\|x+yi\|^2-\|x-yi\|^2+i\|x+iyi\|^2-i\|x-iyi\|^2+j\|x-jyi\|$$

$$-j\|x-jyi\|^2+k\|x+kyi\|^2-k\|x-kyi\|^2$$

$= x\circ(yi)$, $\varphi'(i) = 0$. Similarly $\varphi'(j) = 0$ and $\varphi'(k) = 0$. Therefore $\varphi'(\alpha) = 0$ for all $\alpha \in \mathbb{H}$. Hence $(x\bar{\alpha})\circ y = x\circ(y\alpha)$ for all $\alpha \in \mathbb{H}$.

Remarks: (i) The above proof shows that if V is a LNLS(RNLS) then the norm comes from a LSP(RSP) if and only if $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$ for all $x, y \in V$.

(ii) In $C_{\mathbb{H}}[a,b]$ the supnorm does not come from a bi-LSP (bi-RSP)

Proof: Consider $C_{\mathbb{H}}[0,\pi/_2]$. Let $f(t) = \cos t$ and $g(t) = \sin t$ therefore $\|f\| = 1 = \|g\|$, $\|f+g\| = (2)^{1/2}$, $\|f-g\| = 1$, $2\|f\|^2+2\|g\|^2 = 4$ and $\|f+g\|^2+\|f-g\|^2 = 3$. Hence $\|f+g\|^2+\|f-g\|^2 = 3 \neq 4 = 2\|f\|^2+ 2\|g\|^2$.

Definition 3.13 Let V be a LSPS(RSPS). Two nonzero vectors x, $y \in V$ is said to be orthogonal if and only if $x\circ y = 0$. We shall denote this relation by $x \perp y$

Note $x \perp y \implies y \perp x$.

Notation: If V is a LSPS(RSPS) and $x \in V \setminus \{0\}$ let $x^{\perp} = \{y \in V \,/\, x \perp y\}$. Also, if W is a left (right) linear subspace then $W^{\perp} = \{y \in V \,/\, x \perp y \;\; \forall\, x \in W\}$.

<u>Remarks</u>: (i) $x^{\perp}$ and $W^{\perp}$ are closed left (right) linear subspaces of V

<u>Proof</u>: Clearly $x^{\perp}$ is left linear subspace of V. Since $x^{\perp}$ is the inverse image of $\{0\}$ the continuous map $y \longmapsto x \circ y$, $x^{\perp}$ is closed left linear subspace of V. Hence $x^{\perp}$ is closed left linear subspace. Since $W^{\perp} = \bigcap_{x \in W} x^{\perp}$ is closed left linear subspace of V.

(ii) $W \cap W^{\perp} = \{0\}$. Let $x \in W \cap W^{\perp}$ therefore $x \circ y = 0$ for all $y \in W$. Since $x \in W$, $x \circ x = 0$, hence $x = 0$.

We want to show the Riesz Representation the theorem for Hilbert space which says that if V is a LSPS(RSPS) which is also a Hilbert space and $\phi : V \longrightarrow \mathbb{H}$ is a continuous left (right) linear function then there exists a unique $x_0 \in V$ such that $\phi(x) = x \circ x_0$ ($\phi(x) = x_0 \circ x$) for all $x \in V$. In order to prove this we'll need two lemmas.

<u>Lemma</u> 3.14 Every nonempty closed left (right) convex set E in a LSPS(RSPS) V contains a unique element of smallest norm if V is a Hilbert space.

<u>Proof</u>: By proposition 3.12.,

$$\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2 \quad \text{for all } x,y \in V \qquad [3.14.1]$$

Let $\delta = \inf_{x \in E} \{\|x\|\}$ and let $x,y \in E$. Apply [3.14.1] to $\frac{1}{2}x$ and $\frac{1}{2}y$. We get that $\frac{1}{4}\|x-y\|^2 = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \left\|\frac{x+y}{2}\right\|^2$. Since E is convex, $\frac{x+y}{2} \in E$. So

$$\|x-y\|^2 = 2\|x\|^2+2\|y\|^2-4\left\|\frac{x+y}{2}\right\|^2$$

$$\leqslant 2\|x\|^2+2\|y\|^2-4\delta^2 \qquad [3.14.2.]$$

First we'll prove uniquess. Suppose that $x,y \in E$ have that the property that $\|x\| = \|y\| = \delta$. Must show that $x = y$. Since $0 \leqslant \|x-y\|^2 \leqslant 2\|x\|^2+2\|y\|^2-4\delta^2 = 0$, $\|x-y\| = 0$. So $x = y$. Now we'll prove existence. There exists a sequence $(x_n)_{n\in \mathbb{N}}$ in E sucht that $\lim\limits_{n\to\infty} \|x_n\| = \delta$. By 3.14.2 we see that $\forall$ m, $n \in \mathbb{N}$

$\|x_n - x_m\|^2 < 2\|x_n\|^2+2\|x_m\|^2-4\delta^2 \longrightarrow 0$ as m, $n \longrightarrow \infty$ therefore $(x_n)_{n\in\mathbb{N}}$ is a cauchy sequence in $E \subseteq V$. Since V is a Hilbert space, there exists an $x_0 \in V$ such that $\lim\limits_{n\to\infty} x_n = x_0$. Since E is closed in V, $x_0 \in E$. Hence $\|x_0\| = \|\lim\limits_{n\to\infty} x_n\| = \lim\limits_{n\to\infty}\|x_n\| = \delta$.

<u>Lemma</u> 3.15 Let V be a LSPS(RSPS) which is also a Hilbert space and W a closed left (right) linear subspace of V. There exists a unique pair of left linear map $P : V \longrightarrow W$ and $Q : V \longrightarrow W^{\perp}$ such that $x = P(x)+Q(x)$ for all $x \in V$. Furthermore

(i) If $x \in W$ and $y \in W^{\perp}$, then $P(x) = x$, $Q(x) = 0$, $P(y) = 0$ and $Q(y) = y$.

(ii) $\|x-P(x)\| = \inf\limits_{y\in W}\{\|x-y\|\}$ for all $x \in V$.

(iii) $\|x\|^2 = \|P(x)\|^2+\|Q(x)\|^2$ for all $x \in V$.

(iv) If $W \neq V$, then there exists an $y \in V\setminus\{0\}$ such that $y \perp W$ i.e., $y \in W^{\perp}$.

<u>Proof</u>: For each $x \in V$ let $x+W = \{x+y / y \in W\}$. Claim that $x+W$ is closed and left convex. Since the map $x \longmapsto x+y$ of V is onto itself is a homeomorphism, we get that $x+W$ is closed. Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha, \beta \geqslant 0$ and $\alpha+\beta = 1$. Let $y, z \in W$. Then $\alpha(x+y)+\beta(x+z) = (\alpha+\beta)x+\alpha y+\beta z = x+\alpha y+\beta z \in x+W$. Hence $x+W$ is left convex. Therefore $x+W$ is closed and left convex. Define $Q(x)$ to be the unique element of smallest norm in $x+W$ [Lemma .14.]. Define $P(x) = x-Q(x) \in x-(x+W) = W$. Hence $P: V \longrightarrow W$. Next we'll show that $Q(x) \in W^{\perp}$ i.e. $Q(x) \cdot y = 0$ for all $y \in W$. If $y = 0$, then done. Assume that $y \in W \setminus \{0\}$ and let $y' = \frac{1}{\|y\|} \cdot y$, hence $\|y'\| = 1$. Let $z = Q(x)$. Since z is the unique element of smallest norm in $x+W$,

$$0 \leqslant z \cdot z \leqslant \|z-\alpha y'\|^2 = (z-\alpha y') \cdot (z-\alpha y')$$

$$= z \cdot z - z(\alpha y') - (\alpha y') \cdot z + (\alpha y') \cdot (\alpha y')$$

$$= \|z\|^2 - (z \cdot y')\bar{\alpha} - \alpha(y' \cdot z) + \alpha\bar{\alpha}(y \cdot y) \quad \text{for all } \alpha \in \mathbb{H}.$$

Hence $0 \leqslant -(z \cdot y')\bar{\alpha} - \alpha(y' \cdot z) + |\alpha|^2$ for all $\alpha \in \mathbb{H}$. Let $\alpha = z \cdot y'$. Then we get that $0 \leqslant -\alpha\bar{\alpha} - \alpha\bar{\alpha} + |\alpha|^2 = -|\alpha|^2 = -|z \cdot y'|^2 \leqslant 0$. Therefore $Q : V \longrightarrow W^{\perp}$. Now we'll prove the uniqueness of Q, suppose that there exists maps $P_1: V \longrightarrow W$ and $Q_1: V \longrightarrow W$ such that $x = P_1(x)+Q_2(x) = P(x)+Q(x)$ for all $x \in W$. Let $x \in V$ therefore $P_1(x)-P(x) = Q(x)-Q_1(x)$. Since $P_1(x)-P(x) \in W$, $Q(x)-Q_1(x) \in W^{\perp}$ and $W \cap W^{\perp} = \{0\}$, $P(x) = P_1(x)$ and $Q(x) = Q_1(x)$. Hence $P_1 = P$ and $Q_1 = Q$. Next we shall show that P and Q are left linear maps. Let $\alpha, \beta \in \mathbb{H}$ and let $x, y \in V$. Then $\alpha x+\beta y = P(\alpha x+\beta y)+Q(\alpha x+\beta y)$. Now $\alpha x+\beta y = \alpha(P(x)+Q(x))+\beta(P(y)+Q(y)) =$

$\alpha P(x)+\alpha Q(x)+\beta P(y)+\beta Q(y)$. Therefore $P(\alpha x+\beta y)+Q(\alpha x+\beta y) = \alpha P(x)+\alpha Q(x) +\beta P(y)+\beta Q(y)$ i.e. $P(\alpha x+\beta y)-\alpha P(x)-\beta P(y) = \alpha Q(x)+\beta Q(y)-Q(\alpha x+\beta y)$. Since $P(\alpha x+\beta y)-\alpha P(x)-\beta P(y) \in W$, $\alpha Q(x)+\beta Q(y)-Q(\alpha x+\beta y) \in W^{\perp}$ and $W \cap W^{\perp} = \{0\}$, we get that $P(\alpha x+\beta y) = \alpha P(x)+\beta P(y)$ and $Q(\alpha x+\beta y) = \alpha Q(x)+\beta Q(y)$.

To prove (i) Note that if $x \in W$, then $x+W = W$. Therefore $Q(x) = \inf_{y \in W} \{\|x+y\|\} = \inf_{y \in W} \{\|y\|\} = 0$. So $x = P(x)+Q(x) = P(x)$. If $y \in W^{\perp}$, then $y = P(y)+Q(y)$. Since $P(y) \in W$, $0 = y \cdot P(y) = (P(y)+Q(y)) \cdot P(y) = P(y) \cdot P(y)+Q(y) \cdot P(y) = \|P(y)\|^2$. Hence $P(y) = 0$, so $Q(y) = y$.

To prove (ii) Let $x \in W$ therefore $\|x-P(x)\| = \|Q(x)\| = \inf_{z \in x+W} \{\|z\|\} = \inf_{y \in W} \{\|x+y\|\} = \inf_{y \in W} \{\|x-y\|\}$.

To prove (iii) Let $x \in V$ therefore $\|x\|^2 = x \cdot x = (P(x) +Q(x)) \cdot (P(x)+Q(x)) = P(x) \cdot P(x)+P(x) \cdot Q(x)+Q(x) \cdot P(x)+Q(x) \cdot Q(x) = \|P(x)\|^2+\|Q(x)\|^2$.

To prove (iv) Let $x \in V \setminus W$ and let $y = Q(x)$. Since $P(x) \in W$ and $x \notin W$, $x \neq P(x)$, hence $y = Q(x) \neq 0$. But $y \in W^{\perp}$ therefore $y \perp W$. #

<u>Corollary</u> 3.16 Let V be a LSPS(RSPS) which is also a Hilbert space and $W \subseteq V$ a closed left (right) linear subspace, then $V = W \oplus W^{\perp}$.

<u>Proof:</u> Let $x \in V$ therefore $x = P(x)+Q(x)$. Hence $V = W+W^{\perp}$. Since $W \cap W^{\perp} = \{0\}$, $V = W \oplus W^{\perp}$. #

<u>Theorem</u> 3.17 (Riesz Representation Theorem For Hilbert space) Let V be a LSPS(RSPS) which is also a Hilbert space and let $\phi : V \longrightarrow \mathbb{H}$ be a continuous left (right) linear function. Then $\exists!\ x_0 \in V$ such that $\phi(x) = x \circ x_0$ ($\phi(x) = x_0 \circ x$) for all $x \in V$.

<u>Proof</u>: If $\phi \equiv 0$, then choose $x_0 = 0$. Assume $\phi \not\equiv 0$. Let $W = \ker\phi$ therefore $W \neq V$ and W closed left linear subspace of V. Since $W \neq V$ $W^{\perp} \neq 0$ by lemma 3.15. Hence there exists an $z \in W^{\perp} \setminus \{0\}$ such that $\phi(z) = 1$. Let $x \in V$. Then let $u_x = x - \phi(x)z$ therefore $\phi(u_x) = \phi(x) - \phi(x)\phi(z) = \phi(x) - \phi(x) = 0$. So $u_x \in \operatorname{Ker}\phi = W$. Hence $0 = u_x \circ z = x \circ z - \phi(x)(z \circ z)$. Therefore $\phi(x)(z \circ z) = x \circ z$. So $\phi(x) = \frac{x \circ z}{\|z\|^2}$. Let $x_0 = \frac{1}{\|z\|^2} \circ z$. Since x is arbitrary, $\phi(x) = x \circ x_0$. To prove uniqueness. Suppose that $\phi(x) = x \circ x_1$ for all $x \in V$ therefore $x \circ x_0 = x \circ x_1$ for all $x \in V$. So $x \circ (x_0 - x_1) = 0$ for all $x \in V$. Hence $(x_0 - x_1) \circ (x_0 - x_1) = 0$. So $x_0 - x_1 = 0$. Hence $x_0 = x_1$. ※

<u>Corollary</u> 3.18 Let V be a LSPS(RSPS) which is also a Hilbert space and let $\phi : V \longrightarrow \mathbb{H}$ be a continuous left (right) conjugate function. Then there exist a unique $x_0 \in V$ such that $\phi(x) = x_0 \circ x$ ($\phi(x) = x \circ x_0$) for all $x \in V$.

<u>Definition</u> 3.19 Let V be a LSPS(RSPS) and $(x_\alpha)_{\alpha \in I} \subsetneq V \setminus \{0\}$. Then $(x_\alpha)_{\alpha \in I}$ is said to be an orthogonal set of vector if and only if $\alpha \neq \beta \Longrightarrow x_\alpha \circ x_\beta = 0$. Also, $(x_\alpha)_{\alpha \in I}$ is said to be orthonormal if and only if $x_\alpha \circ x_\beta = \delta_{\alpha\beta}$ where

$$\delta_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

Proposition 3.20. If V is a LSPS (RSPS) and $(x_\alpha)_{\alpha \in I}$ is orthogonal then they are left (right) linear independent.

Proof: Let $0 = \sum_{n<\infty} \beta_n x_{\alpha_n}$. Must show that $\beta_n = 0$ for all n.

Fix n_0, $0 = 0.x_{\alpha_{n_0}} = (\sum_{n<\infty} \beta_n x_{\alpha_n}).x_{\alpha_{n_0}} = \beta_{n_0} \| x_{\alpha_{n_0}} \|^2$. Since $\| x_{\alpha_{n_0}} \| \neq 0$,

$\beta_{n_0} = 0$. Since n_0 is arbitrary, $\beta_n = 0$ for all n. ⋇

Example 3.21 Let $e_\alpha = (0,0,\ldots,\overset{\alpha^{th}\text{place}}{1},0,0,\ldots,0) \in \mathbb{H}^n$. Then $e_1, e_2, \ldots, e_n$ are orthonormal set of vectors in $\mathbb{H}^n$.

Example 3.22 Let $e_n = (0,0,\ldots,\overset{n^{th}\text{place}}{1},0,0,\ldots) \in \ell^2_{\mathbb{H}}$. Then $(e_n)_{n \in \mathbb{N}}$ is an orthonormal set of vectors in $\ell^2_{\mathbb{H}}$.

Proposition 3.23 Let V be a LSPS (RSPS) which is separable then every orthogonal set in V is countable.

Proof: Let $(x_\alpha)_{\alpha \in I}$ be an orthogonal set in V. Let

$x'_\alpha = \frac{1}{\| x_\alpha \|} . x_\alpha$ for all $\alpha \in I$. Then $(x'_\alpha)_{\alpha \in I}$ is an orthogonal set of vectors. If $\alpha \neq \beta$, then $d(x'_\alpha, x'_\beta) = \| x'_\alpha - x'_\beta \| = (2)^{1/2}$. Let $B(x'_\alpha, 1/2)$ be the open ball center at x'_α radius $\frac{1}{2}$. Claim that if $\alpha \neq \beta$ then

$B(x'_\alpha, 1/2) \cap B(x'_\beta, 1/2) = \emptyset$. To prove this, suppose not i.e. suppose that there exists $\alpha \neq \beta$ such that $B(x'_\alpha, 1/2) \cap B(x'_\beta, 1/2) \neq \emptyset$ therefore there exists a $y \in B(x'_\alpha, 1/2) \cap B(x'_\beta, 1/2)$. So $\|x'_\alpha - y\| < 1/2$ and $\|x'_\beta - y\| < 1/2$. Therefore $(2)^{1/2} = \|x'_\alpha - x'_\beta\| \leqslant \|x'_\alpha - y\| + \|y - x'_\beta\| < 1/2 + 1/2 = 1$, a contradiction. Hence if $\alpha \neq \beta$, then $B(x'_\alpha, 1/2) \cap B(x'_\beta, 1/2) = \emptyset$. Let D be a countable dense subset of V. Then given $\alpha \in I$ there exists an $d'_\alpha \in B(x'_\alpha, 1/2) \cap D$. Hence I is a countable. ※

<u>Definition</u> 3.24 Let $(V, \circ), (V', *)$ be LSPS's (RSPS's). Then $(V, \circ)$ is said to be left (right) isomorphic to $(V', *)$ if and only if there exists a 1-1 onto left (right) linear map $\emptyset : V \longrightarrow V'$ such that $\emptyset(v) * \emptyset(w) = v \circ w$ for all $v, w \in V$.

<u>Definition</u> 3.25 Let V be a LSPS (RSPS). Then $(e_\alpha)_{\alpha \in I}$ is said to be an orthonormal left (right) basis of V if and only if

(i) $(e_\alpha)_{\alpha \in I}$ is an orthonormal set

(ii) The closure of the left (right) linear subspace generated by $(e_\alpha)_{\alpha \in I}$ is V.

<u>Example</u> 3.26 (i) Let $e_\alpha = (0, 0, \ldots, \overset{\alpha^{th}\text{place}}{1}, 0, 0, 0, \ldots)$ for all $\alpha \in \{1, 2, \ldots, n\}$. Then $\{e_\alpha\}_{\alpha \leqslant n}$ is an orthonormal left (right) basis of $\mathbb{H}^n$.

(ii) Let $e_\alpha = (0, 0, \ldots, \overset{\alpha^{th}\text{place}}{1}, 0, 0, \ldots)$ for all $\alpha \in \mathbb{N}$. Then $(e_\alpha)_{\alpha \in \mathbb{N}}$ is an orthonormal left (right) basis of $l^2_{\mathbb{H}}$.

Example of separable space

(i) $\mathbb{H}^n$

(ii) $\ell^2_{\mathbb{H}}$.

Let $D = \{(z_n)_{n\in\mathbb{N}} \in \ell^2_{\mathbb{H}} / z_n \in \mathbb{Q}^4$ and $\exists N \in \mathbb{N} \ni z_n = 0 \ \forall n > N\}$.

Let $\varepsilon > 0$ be given and let $(z_n)_{n\in\mathbb{N}} = z \in \ell^2_{\mathbb{H}}$ therefore $\sum_{\alpha=1}^{\infty} |z_\alpha|^2 < \infty$. Hence there exists an $M \in \mathbb{N}$ such that $\sum_{\alpha=M+1}^{\infty} |z_\alpha|^2 < \varepsilon^2/2$. Since $\mathbb{Q}^4$ is dense in $\mathbb{H}$, for each $n \leqslant M$ there exists an $a_n \in \mathbb{Q}^4$ such that $|z_n - a_n| < \varepsilon/(2M)^{1/2}$ therefore $|z_n - a_n|^2 < \varepsilon^2/2M$ for all $n < M$. Let $a = (a_1, a_2, \ldots, a_M, 0, 0, \ldots)$ therefore $a \in D$ and $\|z-a\| = (\sum_{n=1}^{\infty} |z_n - a_n|^2)^{1/2}$ $= (\sum_{n=1}^{M} |z_n - a_n|^2 + \sum_{n=M+1}^{\infty} |z_n|^2)^{1/2} < (\varepsilon^2/2 + \varepsilon^2/2)^{1/2} = \varepsilon$. Hence $\ell^2_{\mathbb{H}}$ is separable. ⋇

Theorem 3.27 (Extended Gram-Schmidt Orthogonalization Theorem)

Let V be a LSPS (RSPS) and let $(v_n)_{n\in\mathbb{N}}$ be a set of left (right) linear independent vector in V. Then there exists an orthonormal set of vector $(e_n)_{n\in\mathbb{N}}$ in V such that

(i) For each $n \in \mathbb{N}$, $e_1, e_2, \ldots, e_n \in$ the left (right) linear subspace generated by $v_1, v_2, \ldots, v_n$.

(ii) For each $n \in \mathbb{N}$, $v_1, v_2, \ldots, v_n \in$ the left (right) linear subspace generated by $e_1, e_2, \ldots, e_n$.

<u>Proof</u>: Use induction on n. If n = 1 let $e_1 = \frac{1}{\|v_1\|} \cdot v_1$.

So done. Suppose by induction that we have $e_1, e_2, \ldots, e_n$ satisfying the theorem. Let $b_{n,\alpha} = v_n \cdot e_\alpha$ for all $\alpha \in \{1,2,\ldots,n-1\}$. Then let

$h_n = v_n - \sum_{\alpha=1}^{n} b_{n,\alpha} e_\alpha$ therefore $h_n \cdot e_\alpha = -\sum_{\alpha=1}^{n-1} b_{n,\beta}(e_\beta \cdot e_\alpha) + v_n \cdot e_\alpha = 0$

therefore $h_n \perp e_\alpha$ for all $\alpha \leq n-1$. If $h_n = 0$ then $v_n \in$ the left linear subspace generated by $e_1, e_2, \ldots, e_{n-1}$. Hence $v_n \in$ the left linear subspace generate by $v_1, v_2, \ldots v_{n-1}$, a contradiction. Therefore $h_n \neq 0$. So $h_n \cdot h_n > 0$. Let $e_n = \frac{1}{(h_n \cdot h_n)^{1/2}} \cdot h$ therefore

$e_n = \frac{1}{(h_n \cdot h_n)^{1/2}} (v_n - \sum_{\alpha=1}^{n-1} b_{n,\alpha} e_\alpha)$. By induction, for each $\alpha \leq n-1$, e_α is a left linear combination of $v_1, v_2, \ldots, v_\alpha$ therefore $e_n = \sum_{\alpha=1}^{n} \beta_\alpha v_\alpha$ for some $\beta_\alpha \in \mathbb{H}$. Hence $e_n \in$ the left linear generated by $v_1, v_2, \ldots, v_n$.

Also, $v_n = \sum_{\alpha=1}^{n=1} b_{n,\alpha} e_\alpha + h_n = \sum_{\alpha=1}^{n-1} b_{n,\alpha} e_n + (h_n \cdot h_n)^{1/2} e_n$. Hence $v_1, v_2, \ldots, v_n \in$ the left linear subspace generate by $e_1, e_2, \ldots e_{n-1}$, a contradiction. Therefore $h_n \neq 0$. So $h_n \cdot h_n \neq 0$. Let $e_n = \frac{1}{(h_n \cdot h_n)^{1/2}} \cdot h$

therefore $e_n = \frac{1}{(h_n \cdot h_n)^{1/2}} (v_n - \sum_{=1}^{n-1} b_{n,\alpha} e_\alpha)$. By induction, for each $\alpha \leq n-1$ e_α is a left linear combination of $v_1, v_2, \ldots, v_\alpha$ therefore $e_n = \sum_{\alpha=1}^{n} \beta_\alpha v_\alpha$ for some $\beta_\alpha \in \mathbb{H}$. Hence $e_n \in$ the left linear generated by $v_1, v_2, \dots, v_n$. Also, $v_n = \sum_{\alpha=1}^{n-1} b_{n,\alpha} e_\alpha + h_n = \sum_{\alpha=1}^{n-1} b_{n,\alpha} e_n + (h_n \cdot h_n)^{1/2} e_n$

Hence $v_1, v_2, \dots, v_n$ the left linear subspace generated by $e_1, e_2, \dots, e_n$. ⋙

<u>Corollary</u> 3.28 Every separable LSPS (RSPS) $V \neq (0)$ has a countable orthonormal left basis .

<u>Proof</u>: Let $D = (v_n)_{n \in \mathbb{N}}$ be a countable dense set of vectors of V. If $v_1 = 0$ remove it, if $v_1 \neq 0$ do not remove it. Let v_{n_1} be the first nonzero in D. If v_{n_1+1} is a left scalar multiple of v_{n_1} remove it, if not do not remove it. Let v_{n_2} $(n_2 > n_1)$ be the first vector in D which is not a left scalar multiple of v_{n_1} if it exists. If it does not exist let $D' = \{v_{n_1}\}$ and stop the process. If v_{n_2+1} the left linear subspace generated by v_{n_1}, v_{n_2} remove it, if not do not remove it. Let v_{n_3} $(n_3 > n_2)$ be the first vector in D such that $v_{n_3} \notin$ the left linear subspace generated by v_{n_1}, v_{n_2} if it exists. If it does not exist let $D' = \{v_{n_1}, v_{n_2}\}$ and stop process. Continue in this way if the process does not stop. Let D' be the containing the vector $v_{n_1}, v_{n_2}, \dots$ obtained by this process. Therefore $D' \subsetneq D$. D' is a set of left linear independent. By contruction, $D \subseteq$ the left linear subspace generated by D' is dense in V. By the Gram-Schmidt construction applied to D' we get a left orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of V. ⋙

We shall now prove that if $(V, \cdot)$ is a separable infinite dimensional LSPS (RSPS) which is also a Hilbert space then $(V, \cdot)$ is left (right) isomorphic to $\ell^{\infty}_{\mathbb{H}}$. Before we prove this, we'll need some lemmas and propositions.

Lemma 3.29 Let V be a LSPS (RSPS), $v_1, v_2, \ldots, v_n$ fixed orthonormal vector in V and $x_0 \in V$ a fix vector. Define $\varphi : \mathbb{H}^n \longrightarrow \mathbb{R}$ by $\varphi(z_1, z_2, \ldots, z_n) = \| x_0 - \sum_{\alpha=1}^{n} z_\alpha v_\alpha \|$. Then φ has a unique minimum where $z_\alpha = x_0 \cdot v_\alpha (z_\alpha = v_\alpha \cdot x_0)$ for all $\alpha = 1, 2, \ldots, n$.

Proof: Let W be the left linear subspace generated by $v_1, v_2, \ldots, v_n$. By lemma 2.10. W is closed. Since $x_0 - W$ is a closed left convex subset of V, $x_0 - W$ has a unique element of minimum norm by lemma 3.14. . By lemma 3.15., there exists an orthogonal projection maps $P: V \longrightarrow W$, $Q: V \longrightarrow W^{\perp}$ such that $\|x_0 - P(x_0)\| = \inf_{y \in W} \{\|x_0 - y\|\}$. Let $y_0 = P(x_0)$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the unique scalar in $\mathbb{H}$ such that $y_0 = \sum_{k=1}^{n} \alpha_k v_k$. Hence $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the unique scalars which minimize φ. Also, $x_0 - y_0 = x_0 - P(x_0) = Q(x_0) \in W^{\perp}$, so $x_0 - y_0 \in W^{\perp}$, hence $(x_0 - y_0) \cdot v_\alpha = 0$ for all $\alpha = 1, 2, \ldots, n$ therefore $x_0 \cdot v_\alpha = y_0 \cdot v_\alpha$ for all $\alpha \leqslant n$. For any fixed $\beta \leqslant n$, $x_0 \cdot v_\beta = y_0 \cdot v_\beta = (\sum_{k=1}^{n} \alpha_k v_k) \cdot v_\beta = \alpha_\beta$ therefore $\alpha_\beta = x_0 \cdot v_\beta$. ⋇

Corollary 3.30 $\sum_{k=1}^{n} |\alpha_k|^2 \leqslant \|x_0\|^2$

<u>Proof</u>: $0 \leqslant \|x_0-y_0\|^2 = (x_0-y_0)\circ(x_0-y_0)$

$$= x_0(x_0-y_0)-y_0\circ(x_0-y_0) = x_0(x_0-y_0)$$

$$= \|x_0\|^2 - x_0\circ(\sum_{k=1}^{n} \alpha_k v_k) = \|x_0\|^2 - \sum_{k=1}^{n} (x_0\circ v_k)\bar{\alpha}_k$$

$$= \|x_0\|^2 - \sum_{k=1}^{n} \alpha_k \bar{\alpha}_k \text{ . Hence } \sum_{k=1}^{n} |\alpha_k|^2 \leqslant \|x_0\|^2 \text{ .}$$

※

<u>Remark</u>: If $(v_\alpha)_{\alpha\in \mathbb{N}}$ is an orthonormal set of vector in the LSPS (RSPS) then for any finite set of indices $n_1, n_2, \ldots, n_k$ $\sum_{\beta=1}^{k} |\alpha_{n_\beta}|^2 \leqslant \|x_0\|^2$ therefore the series $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges and $\sum_{k=1}^{\infty} |\alpha_k|^2 \leqslant \|x_0\|^2$ (Bessel's Inequality).

<u>Theorem</u> 3.31 Let V be a LSPS (RSPS). Then the cardinality of all orthonormal left (right) basis of V is the same, if V is a Hilbert space.

<u>Proof</u>: Let S_1 and S_2 be two orthonormal left bases of V. Let A_1, A_2 be the cardinal numbers of the set S_1, S_2 respectively and let A_0 be the cardinal number of the set of positive integers. If V is finite dimensional then $A_1 = A_2$ so we are done. Assume V is infinite dimensional. For each $x \in S_1$ let $S_2(x) = \{y \in S_2 / y.x \neq 0\}$. Claim that $S_2(x)$ is countable. For each $n \in \mathbb{N}$ let $A_n = \{y \in S_2(x) / |y.x|^2 > \frac{1}{n}\}$. Claim that $\|A_n\| < \infty$. To prove, this, suppose that $\|A_n\| = \infty$. Then there exists a countable $(y_k)_{k\in \mathbb{N}}$ such that $y_k \in S_2(x)$ and $|y_k.x|^2 > 1/_n$ for all $k \in \mathbb{N}$. So

$\sum_{k=1}^{\infty} |y_k \cdot x|^2 > \sum_{k=1}^{\infty} 1/_n = \infty$ a contradiction to Bessel's Inequality.

Hence $\|A_n\| < \infty$ and therefore $\bigcup_{n \in \mathbb{N}} A_n = \{y \in S_2(x) / |y \cdot x|^2 > 0\}$ is countable. Thus $S_2(x)$ is countable. Hence we have the claim.

Claim that $S_2 = \bigcup_{x \in S_1} S_2(x)$. Let $y \in S_2$. If $y \cdot x = 0$ for all $x \in S_1$ then $S_1 \subset S_1 \cup \{y\}$ a contradiction since S_1 is a maximal orthonormal set, so there exists a $x \in S_1$ such that $y \cdot x \neq 0$, hence $y \in S_2(x)$.

Therefore $S_2 = \bigcup_{x \in S_1} S_2(x)$, so $A_2 \leqslant A_0 A_1 = A[\quad]$. Dually $A_1 \leqslant A_0 A_2 = A_2$. [see proposion 3.33 for S_1 is a maximal orthonormal]

※.

<u>Lemma</u> 3.32 Let V be an ∞-dimensional LSPS (RSPS) which is also a Hilbert space and $(e_k)_{k \in \mathbb{N}}$ an orthonormal left (right) basis of V. Thus given $(a_k)_{k \in \mathbb{N}} \in \ell^2_{\mathbb{H}}$ there exists an $v \in V$ such that $v \cdot e_k = a_k$ ($e_k \cdot v = a_k$) and $v = \sum_{k=1}^{\infty} a_k \cdot e_k$ ($v = \sum_{k=1}^{\infty} e_k a_k$).

<u>Proof:</u> Let $v_n = \sum_{k=1}^{n} a_k e_k$ for all $n \in \mathbb{N}$. Claim that $(v_n)_{n \in \mathbb{N}}$ is a cauchy sequence. To prove this, note that $\forall n, p \in \mathbb{N}$

$\|v_{n+p} - v_n\|^2 = \|\sum_{k=n+1}^{n+p} a_k e_k\|^2 = \sum_{k=n+1}^{n+p} |a_k|^2 \longrightarrow 0$ as $n, p \longrightarrow \infty$ since $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. So we get that $(v_n)_{n \in \mathbb{N}}$ is cauchy. Since V is a Hilbert space, there exists an $v \in V$ such that $\lim_{n \to \infty} v_n = v$.

Therefore $v = \lim_{n \to \infty} v_n = \lim_{n \to \infty} \sum_{k=1}^{n} a_k e_k = \sum_{k=1}^{\infty} a_k e_k$. Let $k \in \mathbb{N}$ choose

$n \in \mathbb{N}$ such that $n > k$ therefore $v_n \cdot e_k = (v-v_n+v_n)\cdot e_k = (v-v_n)\cdot e_k + v_n \cdot e_k$ $= (v \cdot v_n)\cdot e_k + (\sum_{\alpha=1}^{n} a_\alpha e_\alpha)\cdot e_k = (v-v_n)\cdot e_k + a_k$. So $v_n \cdot e_k - a_k =$ $(v-v_n)\cdot e_k$ therefore $0 \leqslant |v \cdot e_k - a_k| = |(v-v_n)\cdot e_k| \leqslant \|v-v_n\| \, \|e_k\| =$ $\|v-v_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. So $v \cdot e_k = (\lim_{n\to\infty} v_n)\cdot e_k = \lim_{n\to\infty} (v_n \cdot e_k) = a_k$.

※

Proposition 3.33 Let V be a LSPS (RSPS) which is also a Hilbert space $(e_\alpha)_{\alpha \in \mathbb{N}}$ an orthonormal set in V. Then the following are equivalent:

(i) $(e_\alpha)_{\alpha \in \mathbb{N}}$ is a max orthonormal set in V.

(ii) $(e_\alpha)_{\alpha \in \mathbb{N}}$ is an orthonormal left (right) basis of V.

(iii) For each $x \in V, \|x\|^2 = \sum_{\alpha=1}^{\infty} |x \cdot e_\alpha|^2 \left[\|x\|^2 = \sum_{\alpha=1}^{\infty} |e_\alpha \cdot x_\alpha|^2\right]$

(Pavseval's Identity).

(iv) For each $x, y \in V$ $x \cdot y = \sum_{\alpha=1}^{\infty} (x \cdot e_\alpha)(\overline{y \cdot e_\alpha}) \left[x \cdot y = \sum_{\alpha=1}^{\infty} (\overline{e_\alpha \cdot x})(e_\alpha \cdot y)\right]$.

Proof: (i)$\Longrightarrow$(ii) Let W be the closure of the left linear subspace generated by $(e_\alpha)_{\alpha \in \mathbb{N}}$. Claim that W = V. By corollary 2.9. W is a left linear subspace. Thus W is closed left linear subspace of V. If $W \neq V$, then $W^\perp \neq (0)$. Let $y \in W^\perp \setminus \{0\}$. Then $y \perp e_k$ for all $k \in \mathbb{N}$ therefore $(e_k)_{k \in \mathbb{N}} \cup \left\{\frac{1}{\|y\|} \cdot y\right\} \supset (e_k)_{k \in \mathbb{N}}$ is an orthonormal set in V, a contradiction. Hence W = V.

(ii)$\Longrightarrow$(iii) Let $x \in V$ be arbitrary. First we must show that $x = \sum_{k=1}^{\infty} c_k e_k$ where $c_k = x \circ e_k$ for all $k \in \mathbb{N}$.

By Bessel's Inequality, $\sum_{k=1}^{\infty} |c_k|^2 \leqslant \|x\|^2$, hence by lemma 3.32.,

there exists an $v \in V$ such that $v \circ e_k = c_k$ for all $k \in \mathbb{N}$ and $v = \sum_{k=1}^{\infty} c_k e_k$.

So $v \circ e_k = c_k = x \circ e_k$ for all $k \in \mathbb{N}$ therefore $0 = x \circ e_k - v \circ e_k = (x-v) \circ e_k$ for all $k \in \mathbb{N}$. Claim that $x-v = 0$. To prove this, let $W = \{\alpha(x-v) / \alpha \in \mathbb{H}\}$.

Then $e_k \in W^{\perp}$ for all $k \in \mathbb{N}$. Therefore $W^{\perp} \supseteq$ the left linear subspace generated by $(e_k)_{k \in \mathbb{N}}$. Since $W^{\perp}$ is closed and $(e_k)_{k \in \mathbb{N}}$ is an orthonormal left basis of V, the closure of left linear subspace generated by $(e_k)_{k \in \mathbb{N}}$ is $V \subseteq W^{\perp}$. Hence $V = W^{\perp}$, so $x-v \in W^{\perp}$, hence

$x-v = 0$ i.e. $x = v = \sum_{k=1}^{\infty} c_k e_k$. Hence $\|x\|^2 = \|\sum_{k=1}^{\infty} (x \circ e_k) e_k\|^2 =$

$$\|\lim_{n\to\infty} \sum_{k=1}^{n} (x \circ e_k) e_k\|^2 = \lim_{n\to\infty} \|\sum_{k=1}^{n} (x \circ e_k) e_k\|^2 = \lim_{n\to\infty} \left(\left[\sum_{k=1}^{n} (x \circ e_k) e_k\right]\right.$$

$$\left.\left[\sum_{k=1}^{n} (x \circ e_k) e_k\right]\right) = \lim_{n\to\infty} \sum_{k=1}^{n} (x \circ e_k)(\overline{x \circ e_k}) = \lim_{n\to\infty} \sum_{k=1}^{n} |x \circ e_k|^2 = \sum_{k=1}^{\infty} |x \circ e_k|^2.$$

(iii)$\Longrightarrow$(iv) Fix $x, y \in V$ and let $\lambda \in \mathbb{H}$ be arbitrary. Now

$$x \circ x + \lambda(y \circ x) + (x \circ y)\bar{\lambda} + (\lambda y) \circ (\lambda y) = (x+\lambda y) \circ (x+\lambda y) = \|x+\lambda y\|^2 = \sum_{k=1}^{\infty} |(x+\lambda y) \circ e_k|^2.$$

Since $\sum_{k=1}^{\infty} (x \circ e_k)\overline{[\lambda(y \circ e_k)]} < \infty$ and $\sum_{k=1}^{\infty} [\lambda(y \circ e_k)(\overline{x \circ e_k})] < \infty$. Hence

$$\|x+\lambda y\|^2 = \sum_{k=1}^{\infty} |(x+\lambda y)\cdot e_k|^2 = \lim_{n\to\infty} \sum_{k=1}^{n} |(x+\lambda y)\cdot e_k|^2$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left[((x+\lambda y)\cdot e_k)(\overline{(x+\lambda y)\cdot e_k})\right] = \lim_{n\to\infty} \sum_{k=1}^{n} \left[(x\cdot e_k)(\overline{x\cdot e_k})\right.$$

$$\left. +(x\cdot e_k)(\overline{\lambda(y\cdot e_k)})+\lambda(y\cdot e_k)(\overline{x\cdot e_k})+\lambda(y\cdot e_k)(\overline{\lambda(y\cdot e_k)})\right]$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} (x\cdot e_k)(\overline{x\cdot e_k}) + \lim_{n} \sum_{k=1}^{n} (x\cdot e_k)(\overline{\lambda(y\cdot e_k})+ \lim_{n\to\infty} \sum_{k=1}^{n} \lambda(y\cdot e_k)(\overline{x\cdot e_k})$$

$$+\lim_{n\to\infty} \sum_{k=1}^{n} \lambda(y\cdot e_k)(\overline{\lambda(y\cdot e_k)}) = \sum_{k=1}^{\infty} (x\cdot e_k)(\overline{x\cdot e_k}) + \sum_{k=1}^{\infty} (x\cdot e_k)(\overline{\lambda(y\cdot e_k})$$

$$+ \sum_{k=1}^{\infty} \lambda(y\cdot e_k)(\overline{x\cdot e_k}) + \sum_{k=1}^{\infty} \lambda(y\cdot e_k)(\overline{\lambda(y\cdot e_k)}).$$ Since $\|x\|^2 = \sum_{k=1}^{\infty} |x\cdot e_k|^2$

and $\|\lambda y\|^2 = \sum_{k=1}^{\infty} |(\lambda y)\cdot e_k|^2$ we get that $\lambda(y\cdot x)+(x\cdot y)\bar{\lambda}$

$$= \left[\sum_{k=1}^{\infty} (x\cdot e_k)(y\cdot e_k)\right]\bar{\lambda} + \lambda\left[\sum_{k=1}^{\infty} (y\cdot e_k)(\overline{x\cdot e_k})\right]$$ for all $\lambda \in \mathbb{H}$.

Let $\lambda = i$ therefore

$$i(y\cdot x)-(x\cdot y)i = i\left[\sum_{k=1}^{\infty} (y\cdot e_k)(\overline{x\cdot e_k})\right] - \left[\sum_{k=1}^{\infty} (x\cdot e_k)(y\cdot e_k)\right]i.$$ Hence

$$i\left[y\cdot x- \sum_{k=1}^{\infty} (y\cdot e_k)(\overline{x\cdot e_k})\right] = \left[x\cdot y- \sum_{k=1}^{\infty} (x\cdot e_k)(\overline{y\cdot e_k})\right] i$$ i.e.

$$i\left[y\cdot x- \sum_{k=1} (y\cdot e_k)(\overline{x\cdot e_k})\right] - \left[\overline{y\cdot x- \sum_{k=1}^{\infty} (y\cdot e_k)(\overline{x\cdot e_k})}\right] i = 0 \quad [3.33.1]$$

Similarly letting $\lambda = j$ we get that

$$j\left[y\cdot x- \sum_{k=1}^{\infty} (y\cdot e_k)(\overline{x\cdot e_k})\right] - \left[\overline{y\cdot x- \sum_{k=1}^{\infty} (y\cdot e_k)(\overline{x\cdot e_k})}\right] j = 0 \quad [3.33.2]$$

Similarly letting $\lambda = k$ we get that

$$k\left[y\cdot x-\sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})\right]-\left[\overline{(y\cdot x)-\sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})}\right]k \qquad [3.33.3]$$

Let $\lambda = 1+i$ therefore

$$(1+i)(y\cdot x)+(x\cdot y)(1-i) = (1+i)\left(\sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})\right)+\left(\sum_{k=1}(x\cdot e_k)(\overline{y\cdot e_k})\right)(1-i),$$

so $y\cdot x+i(y\cdot x)+(x\cdot y)-(x\cdot y)i = \sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})+i\left(\sum_{k=1}(y\cdot e_k)(\overline{x\cdot e_k})\right)+$

$\sum_{k=1}^{\infty}(x\cdot e_k)(\overline{y\cdot e_k})-\left(\sum_{k=1}^{\infty}(x\cdot e_k)(\overline{y\cdot e_k})\right)i$. Since $i(y\cdot x)+(x\cdot y)i =$

$i\left(\sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})\right)+\left(\sum_{k=1}^{\infty}(x\cdot e_k)(\overline{y\cdot e_k})\right)\overline{i}$,

$y\cdot x+x\cdot y = \sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})+\sum_{k=1}^{\infty}(x\cdot e_k)(\overline{y\cdot e_k})$. So $y\cdot x+\overline{y\cdot x}-\left[\sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})\right.$

$$\left.-\overline{\sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})}\right] = 0 \qquad [3.33.4]$$

From [3.33.1], [3.33.2], [3.33.3], [3.33.4] and by proposition [1.2] and [1.3], $y\cdot x = \sum_{k=1}^{\infty}(y\cdot e_k)(\overline{x\cdot e_k})$.

(iv)$\Longrightarrow$(i) Suppose not therefore $(e_k)_{k\in\mathbb{N}}$ is not max orthonormal set, so there exists an $e\in V\setminus\{0\}$ such that $e\cdot e_k = 0$ for all $k\in\mathbb{N}$. Let $x = y = e$. By (iv) $0 \neq \|e\|^2 = \sum_{k=1}^{\infty}(e\cdot e_k)(\overline{e\cdot e_k})$

$= \sum_{k=1}^{\infty}|e\cdot e_k|^2 = 0$, a contradiction. #

Remark: We proved that if V is a LSPS (RSPS) which is also a Hilbert space and $(e_k)_{k \in \mathbb{N}}$ is a left (right) basis then any $x \in V$ can be written in the form $x = \sum_{k=1}^{\infty} c_k e_k$. where $c_k = x \cdot e_k$.

Corollary 3.34 If $(e_k)_{k \in \mathbb{N}}$ is an orthonormal left (right) basis of a Hilbert space V and $x \cdot e_k = 0$ $(e_k \cdot x = 0)$ for all $k \in \mathbb{N}$ then $x = 0$.

Proof: By proposition 3.33, $0 = \sum_{k=1}^{\infty} (x \cdot e_k)(\overline{x \cdot e_k}) = x \cdot x$, so $x = 0$. ⋇

Theorem 3.35 Let $(V, \cdot)$ be a separable ∞-dimensional LSPS (RSPS) which is also a Hilbert space. Then V is left (right) isomorphic to $\ell^2_{\mathbb{H}}$.

Proof: Let $(e_k)_{k \in \mathbb{N}}$ be an orthonormal left basis of V [Corollary 3.28]. Define the map $F: V \longrightarrow \ell^2_{\mathbb{H}}$ by $F(x) = (x \cdot e_k)_{k \in \mathbb{N}}$ for all $x \in V$. By Bessel's Inequality, F is well-define. Clearly F is left linear. To show that F is onto. Let $(z_n)_{n \in \mathbb{N}} \in \ell^2_{\mathbb{H}}$ then by lemma 3.32 there exists an $v \in V$ such that $v \cdot e_k = z_k$ for all $k \in \mathbb{N}$ and $v = \sum_{k=1}^{\infty} z_k e_k$. Hence $F(v) = (z_n)_{n \in \mathbb{N}}$. By corollary 3.34. F is 1-1. By proposition 3.33. (iv) $F(x) \cdot F(y) = x \cdot y$. ⋇

Theorem 3.36 Let V be an n-dimensional LSPS (RSPS). Then V is a left (right) isomorphic to $\mathbb{H}^n$.

Proof: Let $x_1, x_2, \ldots, x_n$ be an orthonormal left basis of V [Theorem .27]. Let $F(x_k) = e_k = (0,0,\ldots,1,0,0,\ldots 0)$ (1 in the k^{th} place). Then we can extended F to a left linear map $F: V \longrightarrow \mathbb{H}^n$. Clearly F is 1-1 and onto. Let v, w $\in$ V. Then therefore $v = \sum_{k=1}^{n} \alpha_k x_k$, $w = \sum_{k=1}^{n} \beta_k x_k$ for some $\alpha_k, \beta_k \in \mathbb{H}$. Then $F(v) \cdot F(w) = (\sum_{k=1}^{n} \alpha_k e_k) \cdot (\sum_{k=1}^{n} \beta_k e_k)$

$= \sum_{k=1}^{n} \alpha_k \beta_k = (\sum_{k=1}^{n} \alpha_k x_k) \cdot (\sum_{k=1}^{n} \beta_k x_k) = v \cdot w$. Hence V is an left isomorphic to $\mathbb{H}^n$. #

Example of a non separable which is also a Hilbert space

Let I be any uncountable set. Let

$\ell^2_{\mathbb{H}}(I) = \{f: I \longrightarrow \mathbb{H} / f(\alpha) = 0$ except countable many $\alpha \in I$ and $\sum_{\alpha \in I} |f(\alpha)|^2 < \infty\}$

and define $f \cdot g = \sum_{\alpha \in I} f(\alpha) \cdot \overline{g(\alpha)}$ for all f, $g \in \ell^2_{\mathbb{H}}(I)$. Then $(\ell^2_{\mathbb{H}}(I), \cdot)$ is a LSPS. By the same proof as in $\ell^2_{\mathbb{H}}$ where $1 \leq p < \infty$ is a Hilbert space. Claim that $\ell^2_{\mathbb{H}}(I)$ is a non-separable.

For each $\beta \in I$ let $e_\beta \in \ell^2_{\mathbb{H}}(I)$ where

$$e_\beta(\alpha) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

Then $(e_\beta)_{\beta \in I}$ is uncountable and if $\beta_1 \neq \beta_2$ then $d(e_{\beta_1}, e_{\beta_2}) =$

$\| e_{\beta_1} - e_{\beta_2} \| = (2)^{1/2}$. By the same proof as in proposition 3.23., $\alpha \neq \beta \Longrightarrow$

$B(e_\alpha, 1/2) \cap B(e_\beta, 1/2) = \emptyset$. If $\ell^2_{\mathbb{H}}(I)$ has a countable dense subset D then there exists an $d_\alpha \in D \cap B(e_\alpha, 1/2)$ for all $\alpha \in I$. Hence $(d_\alpha)_{\alpha \in I}$ is uncountable a contradiction. Therefore $(\ell^2_{\mathbb{H}}(I), \circ)$ is non-separable LSPS. By the same argument as above $(\ell^2_{\mathbb{H}}(I), \circ)$ is non-separable RSPS where $f \circ g = \sum_{\alpha \in I} \overline{f(\alpha)} g(\alpha)$ for all f, g $\ell^2_{\mathbb{H}}(I)$.

Theorem 3.37 Let $(V, \circ)$ be an ∞-dimensional LSPS (RSPS) which is also a Hilbert space. Then there exists a set I such that V is left (right) isomorphic to $\ell^2_{\mathbb{H}}(I)$.

Proof: Let Λ be the collection of all orthonormal sets in V. Let $x \in V \setminus \{0\}$ therefore $\left\{\frac{1}{\|x\|} \circ x\right\} \in \Lambda$. So $\Lambda \neq \emptyset$. Let $\mathcal{A}$ be any chain in Λ. Clearly $\bigcup_{B \in \mathcal{A}} B \in \Lambda$. By zorn lemma, there exists a max orthonormal set in V. Let $(e_k)_{k \in I}$ be a maximal orthonormal set. By the same argument as in proposition 3.33., $(e_k)_{k \in I}$ is an orthonormal left basis of V. If $x \in V$ define $F(x) = f$ where $f(\alpha) = x \circ e_\alpha$ for all $\alpha \in I$. Must show that $F(x) \in \ell^2_{\mathbb{H}}(I)$. Given $n \in \mathbb{N}$. Let $\Lambda_n = \left\{\alpha \in I / |x \circ e_\alpha|^2 > \frac{1}{n}\right\}$. By the same argument as in the theorem 3.31 $\bigcup_{n \in \mathbb{N}} \Lambda_n$ is countable. Also $\sum_{\alpha \in \bigcup_{n \in \mathbb{N}} \Lambda_n} |x \circ e_\alpha|^2 = \sum_{\alpha \in \bigcup_{n \in \mathbb{N}} \Lambda_n} |f(\alpha)|^2 < \infty$ by Bessel's Inequality. Hence $F(x) \in \ell^2_{\mathbb{H}}$. To show that F is left linear, let x, $y \in V$ and $\beta \in \mathbb{H}$. Let $F(x) = f$, $F(y) = g$, $F(x+y) = f'$ and $F(\beta x) = g'$.

Let $\alpha \in I$. Then $f'(\alpha) = (x+y) \circ e_\alpha = x \circ e_\alpha + y e_\alpha = f(\alpha)+g(\alpha) = (f+g)\alpha$ and $g'(\alpha) = (\beta x) \circ e = \beta(x \circ e_\alpha) = \beta f(\alpha)$. Therefore $F(x+y) = f' = f+g = F(x)+F(y)$ and $F(\beta x) = g' = \beta f = \beta F(x)$. To show F is onto, let $f \in \ell^2_{\mathbb{H}}(I)$. Given $n \in \mathbb{N}$ let $A_n = \{\alpha \in I / |f(\alpha)|^2 > 1/n\}$. Since $\sum_{\alpha \in I} |f(\alpha)|^2 < \infty$, A_n is finite and $A_n \subseteq A_m$ for all $m > n$. Let $A_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and let $v_n = \sum_{j=1}^{n} f(\alpha_j) e_{\alpha_j}$. Claim that $(v_n)_{n \in \mathbb{N}}$ is a cauchy. To prove this, note that for each $n, p \in \mathbb{N}$ $\|v_{n+p} - v_n\|^2 = \|\sum_{j=n+1}^{n+p} f(\alpha_j) e_{\alpha_j}\|^2 = \sum_{j=n+1}^{n+p} |f(\alpha_j)|^2$. Since $\sum_{\alpha \in I} |f(\alpha)|^2 < \infty$, for each $\varepsilon > 0$ there exists a N_ε such that $\sum_{\alpha = N+1}^{\infty} |f(\alpha)|^2 < \varepsilon^2$.

Hence $\sum_{j=n+1}^{n+p} |f(\alpha_j)|^2 < \varepsilon^2$ for all $n > N$ and for all $p \in \mathbb{N}$, i.e. $\|v_{n+p} - v_n\| < \varepsilon$ for all $n > N_\varepsilon$ and for all $p \in \mathbb{N}$. Thus $(v_n)_{n \in \mathbb{N}}$ is cauchy. Since V is a Hilbert space there exists an $v \in V$ such that $\lim_{n \to \infty} v_n = v$. Want to show that $F(v) = f$. Fix $\alpha \in I$. Then $v \cdot e_\alpha = \lim_{n \to \infty} v_n \circ e_\alpha = \lim_{n \to \infty} (v_n \circ e_\alpha) = f(\alpha)$ therefore $F(v) = f$.

To show F is 1-1. Let $x \in V \setminus \{0\}$ be such that $F(x) = 0$ therefore $x \cdot e_\alpha = 0$ for all $\alpha \in I$ therefore $(e_\alpha)_{\alpha \in I} \cup \{\frac{1}{\|x\|} \cdot x\} \supset (e_\alpha)_{\alpha \in I}$ is orthonormal contradicting the maximality of the orthonormal set $(e_\alpha)_{\alpha \in I}$. By the same argument as in proposition 3.33, $F(x)\,F(y) = x \cdot y$. #

Theorem 3.38 Let V be a LSPS (RSPS) which is also a Hilbert space. Then $(\bar{V}, \|\ \|)$ is a left (right) isomorphic to $(V, \|\ \|)$

Proof: Let $\varphi \in \bar{V}$ there exists a unique $x_0 \in V$ such that $\varphi(x) = x_0 \circ x$ for all $x \in V$. Define $F: \bar{V} \longrightarrow V$ by $F(\varphi) = x_0$. Clearly F is 1-1 and onto. Claim that F is left linear. Let $\alpha_1, \alpha_2 \in \mathbb{H}$ and $\varphi_1, \varphi_2 \in \bar{V}$ there exists a unique $x_0 \in V$ and a unique $y_0 \in V$ such that $\varphi_1(x) = x_0 \circ x$ and $\varphi_2(x) = y_0 \circ x$ for all $x \in V$. Let $x \in V$ therefore $(\alpha_1\varphi_1 + \alpha_2\varphi_2)(x) = \alpha_1\varphi_1(x) + \alpha_2\varphi_2(x) = \alpha_1(x_0 \circ x) + \alpha_2(y_0 \circ x) = (\alpha_1 x_0) \circ x + (\alpha_2 y_0) \circ x$ $= (\alpha_1 x_0 + \alpha_2 y_0) \circ x$. Therefore $F(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 x_0 + \alpha_2\ y_0 = \alpha_1 F(\varphi_1) + \alpha_2 F(\varphi_2)$. Hence F is left linear. If $\varphi \equiv 0$, then $x_0 = 0$, so $\|\varphi\| = 0$, hence $\|F(\varphi)\| = \|x_0\| = 0$. Therefore $\|\varphi\| = \|x_0\|$. Assume that $\varphi \not\equiv 0$ therefore $x_0 \neq 0$ and $|\varphi(x)| = |x_0 \circ x| \leqslant \|x_0\| \|x\|$, so if $x \neq 0$ then $\frac{|\varphi(x)|}{\|x\|} \leqslant \|x_0\|$. Hence $\|\varphi\| \leqslant \|x_0\| = \|F(\varphi)\|$. Since $|\varphi(x_0)| = |x_0 \circ x_0|$ $= \|x_0\|^2$, $\frac{|\varphi(x_0)|}{\|x_0\|} = \|x_0\|$, so $\|\varphi\| > \|x_0\|$. Therefore $\|\varphi\| \geqslant \|x_0\| = \|F(\varphi)\|$. $\|\varphi\| = \|F(\varphi)\|$. Thus $(\bar{V}, \|\ \|)$ is a left isomorphic to $(V, \|\ \|)$. #

Remark: $\bar{V}$ is homeomorphic to V^* by the map $f \mapsto \bar{f}$.