CHAPTER II

BANACH SPACE OVER THE QUATERNIONS

Definition 2.1 Let V be a left vectorspace over $\mathbb{H}$. A map $\| \,\| : V \longrightarrow \mathbb{R}$ is said to be left norm on V if and only if

(i) $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$.

(ii) $\|\alpha v\| = |\alpha| \, \|v\|$ for all $v \in V$ and for all $\alpha \in \mathbb{H}$.

(iii) $\|v+w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

If $\| \,\|$ is a left norm on V then the pair $(V, \| \,\|)$ is called a left normed linear space. We shall abbreviate it by LNLS.

Definition 2.2 Let V be a right vector space over $\mathbb{H}$. A map $\| \,\| : V \longrightarrow \mathbb{R}$ is said to be right norm on V if and only if

(i) $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$.

(ii) $\|v\alpha\| = \|v\| \, |\alpha|$ for all $v \in V$ and for all $\alpha \in \mathbb{H}$.

(iii) $\|v+w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

If $\| \,\|$ is a right norm then the pair $(V, \| \,\|)$ is called a right normed linear space. We shall abbreviate it by RNLS.

Definition 2.3 Let V be a vector space over H. Then $(V, \| \,\|)$ is called a normed linear space if $\| \,\|$ is both a left normed and right norm. We shall abbreviate it by NLS.

Given a LNLS(RNLS) V, define $d(v,w) = \|v-w\|$ then d is a metric on V hence V is a topological space.

<u>Definition</u> 2.4 If a LNLS(RNLS) is complete with respect to the metric d then we shall call V a Banach space.

<u>Example</u> 2.5 (i) $(\mathbb{H}^n, \| \|_p)$ where $\|x\|_p = (\sum_{\alpha=1}^{n} |x_\alpha|^p)^{1/p}$ for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{H}^n$ and $1 \leq p < \infty$.

<u>Proof</u>: Let $(z_\alpha)_{\alpha \in \mathbb{N}}$ be a cauchy sequence in $\mathbb{H}$. Given $\varepsilon > 0$ there exists a N_ε such that $\|z_m - z_k\|_p < \varepsilon$ for all m, $k > N_\varepsilon$. Hence for each $\alpha \in \{1,2,\ldots,n\}$ $(z_m^{(\alpha)})_{m \in \mathbb{N}}$ is a acauchy sequence in $\mathbb{H}$. Since $\mathbb{H}$ is complete, there exists an $z_0^{(\alpha)} \in \mathbb{H}$ such that $\lim_{m\to\infty} z_m = z_0^{(\alpha)}$. Let $z_0 = (z_0^{(1)}, z_0^{(2)}, \ldots, z_0^{(n)})$. Claim that $\lim_{m\to\infty} z_m = z_0$. Since $\lim_{m\to\infty} z_m^{(\alpha)} = z_0^{(\alpha)}$ for all $\alpha \in \{1,2,\ldots,n\}$, there exists a N'_ε such that $|z_m^{(\alpha)} - z_0^{(\alpha)}| < \frac{\varepsilon}{n^{1/p}}$ for all $m > N'_\varepsilon$ and for all $\alpha \in \{1,2,\ldots,n\}$. Therefore $|z_m^{(\alpha)} - z_0^{(\alpha)}|^p < \varepsilon^p/n$ for all $\alpha \in \{1,2,\ldots,n\}$ so $\sum_{\alpha=1}^{n} |z_m^{(\alpha)} - z_0^{(\alpha)}|^p < \varepsilon^p$. Hence $\lim_{m\to\infty} z_m = z_0$. Thus $\mathbb{H}^n$ is a Banach space. ⋇

(ii) In $\ell^p_{\mathbb{H}} = \{(z_n)_{n \in \mathbb{N}} / z_n \in \mathbb{H} \ \forall n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} |z_n|^p < \infty\}$ and $\|z\|_p = (\sum_{n=1}^{\infty} |z_n|^p)^{1/p}$ where $1 \leq p < \infty$.

Proof: Let $(z_n)_{n\in\mathbb{N}}$ be a cauchy sequence in $\ell^p_{\mathbb{H}}$. Given $\varepsilon > 0$. There exists an $N_\varepsilon \in \mathbb{N}$ such that $\|z_n - z_m\|_p < \frac{\varepsilon}{2}$ for all m, $n > N_\varepsilon$. Therefore for each $\alpha \in \mathbb{N}$, $|z_n^{(\alpha)} - z_m^{(\alpha)}| < \varepsilon/2$ for all m, $n > N_\varepsilon$. Hence for each $\alpha \in \mathbb{N}$ $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathbb{H}$. Since $\mathbb{H}$ is complete, there exists an $z_0^{(\alpha)} \in \mathbb{H}$ such that $\lim\limits_{n} z_n^{(\alpha)} = z_0^{(\alpha)}$. Let $z_0 = (z_0^{(\alpha)})_{\alpha\in\mathbb{N}}$. Claim that $\lim\limits_{n\to\infty} z_n = z_0$. Since $\sum\limits_{\alpha=1}^{\infty} |z_m^{(\alpha)} - z_0^{(\alpha)}|^p < (\varepsilon/2)^p$ for all m, $n > N_\varepsilon$, $\sum\limits_{\alpha=1}^{M} |z_m^{(\alpha)} - z_n^{(\alpha)}|^p < (\varepsilon/2)^p$ for all $M \in \mathbb{N}$ and for all m, $n > N_\varepsilon$. Fix M, m and let $n \to \infty$ in $\sum\limits_{=1}^{M} |z_m^{(\alpha)} - z_n^{(\alpha)}|^p < (\varepsilon/2)^p$ we get that $\sum\limits_{\alpha=1}^{M} |z_m^{(\alpha)} - z_0^{(\alpha)}|^p < (\varepsilon/2)^p$.

Since M is arbitrary, $\sum\limits_{\alpha=1}^{\infty} |z_m^{(\alpha)} - z_0^{(\alpha)}|^p \leqslant (\varepsilon/2)^p$. Hence $\lim\limits_{m\to\infty} z_m = z_0$.

Since $\sum\limits_{\alpha=1}^{\infty} |z_m^{(\alpha)} - z_0^{(\alpha)}|^p \leqslant (\varepsilon/2)^p$ for all $m > N_\varepsilon$, for each $m > N_\varepsilon$ $z_m - z_0 \in \ell^p_{\mathbb{H}}$ which is a vector space over $\mathbb{H}$ so $z_0 \in \ell^p_{\mathbb{H}}$. ⋇

(iii) In $\ell^\infty_{\mathbb{H}} = \left\{(z_n)_{n\in\mathbb{N}} / z_n \in \mathbb{H} \ \ \forall n \in \mathbb{N} \text{ and } \sup\limits_{n\in\mathbb{N}}\{|z_n|\} < \infty\right\}$ and $\|z\|_\infty = \sup\limits_{n\in\mathbb{N}}\{|z_n|\}$.

Proof: Let $(z_n)_{n\in\mathbb{N}}$ be a cauchy sequence in $\ell^\infty_{\mathbb{H}}$. Given $\varepsilon > 0$ there exists a N_ε such that $\|z_m - z_n\|_\infty < \varepsilon/2$ for all m, $n > N_\varepsilon$. Hence for each $\alpha \in \mathbb{N}$ $|z_m^{(\alpha)} - z_n^{(\alpha)}| < \varepsilon/2$ for all m, $n > N_\varepsilon$. Therefore for each $\alpha \in \mathbb{N}$ $(z_n^{(\alpha)})_{n\in\mathbb{N}}$ is a cauchy sequence in $\mathbb{H}$.

Since $\mathbb{H}$ is complete, there exists $z_0^{(\alpha)} \in \mathbb{H}$ such that $\lim_{n\to\infty} z_n^{(\alpha)} = z_0^{(\alpha)}$.

Let $z_0 = (z_0^{(\alpha)})_{\alpha \in \mathbb{N}}$. Claim that $\lim_{n\to\infty} z_n = z_0 \in \ell_{\mathbb{H}}^{\infty}$. To prove this, note that $\left|z_m^{(\alpha)} - z_n^{(\alpha)}\right| < \varepsilon/2$ for all m, n $> N_\varepsilon$ and for all $\alpha \in \mathbb{N}$.

Now fix $\alpha \in \mathbb{N}$ and $m > N_\varepsilon$ then let $n \to \infty$ in $|z_m^{(\alpha)} - z_n^{(\alpha)}| < \varepsilon/2$ we get that $\left|z_m^{(\alpha)} - z_0^{(\alpha)}\right| \leqslant \varepsilon/2$. Since α is arbitrary, $\left|z_m^{(\alpha)} - z_0^{(\alpha)}\right| \leqslant \varepsilon/2$ for all $\alpha \in \mathbb{N}$ and for all $m > N_\varepsilon$, hence $\lim_{m\to\infty} z_m = z_0$. Since $\|z_m - z_0\| < \varepsilon$ for all $m > N_\varepsilon$, for each $m > N_\varepsilon$ $z_m - z_0 \in \ell_{\mathbb{H}}^{\infty}$ which is a vector space so $z_0 \in \ell_{\mathbb{H}}^{\infty}$. ⋇

iv) In $C_{\mathbb{H}}[a,b] = \left\{f: [a,b] \longrightarrow \mathbb{H} \;/\; f \text{ is continuous}\right\}$ and $\|f\| = \sup_{x\in[a,b]} \left\{|f(x)|\right\}$. The proof that $C_{\mathbb{H}}[a,b]$ is complete is simi similar to that $\ell_{\mathbb{H}}^{\infty}$ is complete.

<u>Example of LNLS(RNLS) which is not complete</u>

In $(\ell_{\mathbb{H}}^{1}, \| \; \|_\infty)$ where $\|x\| = \sup_{n\in\mathbb{N}} \left\{|x_n|\right\}$

<u>Proof</u>: Note that $\ell_{\mathbb{H}}^{1}$ is a linear subspace of $\ell_{\mathbb{H}}^{\infty}$. Let $\ell_1 = (1,0,0,\dots)$, $\ell_2 = (1,1/_2,0,0,\dots)$, $\ell_3 = (1,1/_2,1/_3,0,0,\dots)$, $\ell_n = (1,1/_2,1/_3,\dots,1/_n,0,0,\dots) \in \ell_{\mathbb{H}}^{1}$ $\forall n \in \mathbb{N}$. Clearly $(\ell_n)_{n\in\mathbb{N}}$ is a cauchy sequence with respect to $\| \; \|_\infty$ in $\ell_{\mathbb{H}}^{1}$ and $\lim_{n\to\infty} \ell_n = (1,1/_2,1/_3,\dots) \notin \ell_{\mathbb{H}}^{1}$. ⋇

Definition 2.6 Let V be a left (right) vector space over $\mathbb{H}$ and $\| \|$, $\| \|'$ are left (right) norms on V. Then these left (right) norm are said to be equivalent if and only if there exist m_1, $m_2 > 0$ such that $m_1 \|x\| \leqslant \|x\|' \leqslant m_2 \|x\|$ for all $x \in V$. Clearly equeivalent norms gives the same topology on V.

Remark: Since $\mathbb{H}^n \cong \mathbb{R}^{4n}$, all left (right) norms on a finite dimensional left (right) vector space over $\mathbb{H}$ are equivalent see [5].

Remark: All subsequent the theorem for LNLS's true for RNLS's and the proof is same. So we shall only prove theorems for LNLS case.

Let V, W be LNLS's (RNLS's). Then the left (right) vector space $V \times W$ over $\mathbb{H}$ is a LNLS(RNLS) by defining

$\|(v,w)\| = (\|v\|^p + \|w\|^p)^{1/p}$ $1 \leqslant p < \infty$. In fact $\|(v,w)\| = \max\{\|v\|, \|w\|\}$.

Since $V \times W$ is a LNLS(RNLS), $V \times W$ is a topological space. Also as a topological space, $V \times W$ has the product topology.

Proposition 2.7 These two topologies are equivalent.

Proof: Standard. ⋇

Remark: If V, W are NLS's then the vector space $V \times W$ over $\mathbb{H}$ is a NLS defined as above.

<u>Proposition</u> 2.8 Let V be LNLS (RNLS). Then the map $(x,y) \mapsto x+y$ and $(\alpha,x) \mapsto \alpha x (x\alpha)$ are continuous with respect to the product topology. Also, the map $x \longmapsto \|x\|$ is continuous. In fact the map $(x,y) \longmapsto x+y$ and $x \longmapsto \|x\|$ are uniformly continuous with respect to the norm topology.

<u>Proof</u>: Standard. ⋇

<u>Remark</u>: Since the map $(x,y) \longmapsto x+y$ and $(\alpha,x) \longmapsto \alpha x$ $(x\alpha)$ are continuous in both variable they are continuous in each variable separately. Therefore if $x_0 \in V$ and $\alpha \in \mathbb{H}$ are fixed then the map $(x,x_0) \longmapsto x+x_0$, $(\alpha_0,x) \longmapsto \alpha_0 x (x\alpha_0)$ and $(\alpha,x_0) \longmapsto \alpha x_0 (x_0\alpha)$ are all continuous.

<u>Corollary</u> 2.9 Let V be a LNLS(RNLS) and $W \subseteq V$ a left (right) linear subspace. Then $\overline{W}$ is a left (right) linear subspace.

<u>Proof</u>: Let x, $y \in \overline{W}$ and $\alpha, \beta \in \mathbb{H}$. Want to show that $\alpha x + \beta y \in \overline{W}$. Then $\exists$ sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in W such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$. Since W is a left linear subspace, $\alpha x_n + \beta y_n \in W$ $\forall n \in \mathbb{N}$. By the above proposition, $\lim_{n \to \infty} (\alpha x_n + \beta y_n) = \alpha x + \beta y$. Hence $\alpha x + \beta y \in \overline{W}$. ⋇

<u>Theorem</u> 2.10 Let V be a LNLS(RNLS) and let W be a closed subspace of V. Then a left (right) linear subspace generated by W and a finite number of elements for V is closed in V.

Proof: Use induction on the number of generators of W. If the number of generator is 0 then the theorem is true. By induction, suppose the theorem is true for n-1 i.e. every left linear subspace of V generated by W and n-1 vector is closed. We must prove the theorem for n. Let U be left linear subspace generated by W and $w_1, w_2, \ldots, w_n$. Must show that U is closed. Let U' be the left linear subspace generated by W and $w_1, w_2, \ldots, w_{n-1}$. Then U' is closed by induction. If $w_n \in U'$, then $U = U'$ therefore U is closed so done. Hence we may assume that $w_n \notin U'$. Therefore U is the left linear subspace by w_n and U'. So every vector in W can be written uniquely in the form $\lambda w_n + w'$ where $w' \in U'$ and $\lambda \in \mathbb{H}$. Let $z \in \bar{U}$. Must show that $z \in U$. Since $z \in \bar{U}$, $z = \lim_{\alpha \to \infty} x_\alpha$ for some sequence $(x_\alpha)_{\alpha \in \mathbb{N}}$ in U. Since $x_\alpha \in U$ for all $\alpha \in \mathbb{N}$, $x_\alpha = \lambda_\alpha w_n + w'_\alpha$ for some $\lambda_\alpha \in \mathbb{H}$ and $w'_\alpha \in U'$. Since $(x_\alpha)_{\alpha \in \mathbb{N}}$ is convergent, $(x_\alpha)_{\alpha \in \mathbb{N}}$ is a bounded sequence i.e. there exists a $\eta > 0$ such that $\|x_\alpha\| < \eta$ for all $\alpha \in \mathbb{N}$. Therefore $\|\lambda_\alpha w_n + w'_\alpha\| < \eta$ for all $\alpha \in \mathbb{N}$. Claim that $(\lambda_\alpha)_{\alpha \in \mathbb{N}}$ is bounded in $\mathbb{H}$. To prove this, suppose not therefore there exists a subsequence $(\lambda_{\alpha_\beta})_{\beta \in \mathbb{N}}$ of $(\lambda_\alpha)_{\alpha \in \mathbb{N}}$ such that $\lambda_{\alpha_\beta} \neq 0$ for all $\beta \in \mathbb{N}$ and $|\lambda_{\alpha_\beta}| \longrightarrow \infty$ as $\beta \longrightarrow \infty$. Now $\left\| 1/_{\lambda_{\alpha_\beta}} \cdot x_{\alpha_\beta} \right\| < \eta/_{|\lambda_{\alpha_\beta}|}$ for all $\beta \in \mathbb{N}$ therefore $\left\| w_n + 1/_{\lambda_{\alpha_\beta}} \cdot w'_{\alpha_\beta} \right\| < \eta/_{|\lambda_{\alpha_\beta}|} \longrightarrow 0$. So

$\lim_{\beta \to \infty} 1/_{\lambda_{\alpha_\beta}} \cdot w'_{\alpha_\beta} = -w_n$. Since $1/_{\lambda_{\alpha_\beta}} \cdot w'_{\alpha_\beta} \in U'$ which is closed and

$\lim_{\beta \to \infty} 1/_{\lambda_{\alpha_\beta}} \cdot w'_{\alpha_\beta} = -w_n$, $-w_n \in \bar{U}' = U'$, hence $w_n \in U'$, a contradiction.

Hence $(\lambda_\alpha)_{\alpha\in\mathbb{N}}$ is bounded in $\mathbb{H}$. So we have the claim. By the Bolzano-weiertrass theorem, there exists a convergent subsequence $(\lambda_{\alpha_\beta})_{\beta\in\mathbb{N}}$ of $(\lambda_\alpha)_{\alpha\in\mathbb{N}}$. Let $\lambda = \lim_{\beta\to\infty} \lambda_{\alpha_\beta}$. Hence $\lambda w_n = (\lim_{\beta\to\infty} \lambda_{\alpha_\beta})w_n$. Also, $\lambda_{\alpha_\beta} w_n + w'_{\alpha_\beta}$ converges as $\beta\to\infty$ [Since it is a subsequence of the convergent sequence $(\lambda_\alpha w_n + w'_\alpha)_{\alpha\in\mathbb{N}}$]. Since $w'_{\alpha_\beta} = (\lambda_{\alpha_\beta} w_n + w'_{\alpha_\beta}) - (\lambda_{\alpha_\beta} w_n)$ for all $\beta\in\mathbb{N}$, $(w'_{\alpha_\beta})_{\beta\in\mathbb{N}}$ is convergent. Let $w' = \lim_{\beta\to\infty} w'_{\alpha_\beta}$. Then $w' \in \bar{U}' = U'$. Since $z = \lim_{\alpha\to\infty} x_\alpha = \lim_{\alpha\to\infty} (\lambda_\alpha w_n + w'_\alpha) = \lim_{\beta\to\infty} (\lambda_{\alpha_\beta} w_n + w'_{\alpha_\beta}) = \lambda w_n + w' \in$ the left linear subspace generated by w_n and U' which is U, $z \in U$. ⋇

<u>Theorem</u> 2.11 Let V, W be LNLS's (RNLS's) and F is a left (right) linear map. If F is continuous at one point, then F is continuous everywhere.

<u>Proof</u>: Let $x_0 \in V$ be the point where F is continuous. Let $x \in V$ and let $\lim_{n\to\infty} x_n = x$. To show that $\lim_{n\to\infty} F(x_n) = F(x)$. Since $\lim_{n\to\infty} (x_n - x + x_0) = x_0$, $\lim_{n\to\infty} F(x_n - x + x_0) = F(x_0)$. Therefore $\lim_{n\to\infty} F(x_n)$ $= \lim_{n\to\infty} F(x + x_n - x + x_0 - x_0) = \lim_{n\to\infty} F(x_n - x + x_0) + F(x - x_0) = F(x_0) + F(x) - F(x_0) =$ $F(x)$. ⋇

<u>Corollary</u> 2.12 Let V be a LNLS(RNLS). If F is a left (right) conjugate map and F is continuous at one point, then F is continuous everywhere.

<u>Definition</u> .13 Let V, W be LNLS's (RNLS's) and F: $V \longrightarrow W$ a left (right) linear map. Then F is said to be bounded if and only if

$$\sup_{x \neq 0} \left\{ \frac{\|F(x)\|}{\|x\|} \right\} < \infty .$$

<u>Definition</u> .14 Let V be LNLS (RNLS) and F: $V \longrightarrow \mathbb{H}$ a left (right) conjugate map. Then F is said to be bounded if and only if

$$\sup_{x \neq 0} \left\{ \frac{|F(x)|}{\|x\|} \right\} < \infty .$$

<u>Theorem</u> 2.15 Let V, W be LNLS's (RNLS's) and F: $V \longrightarrow W$ a left (right) linear map. Then F is continuous if and only if F is bounded.

<u>Proof</u>: Let $M = \sup_{x \neq 0} \left\{ \frac{\|F(x)\|}{\|x\|} \right\} < \infty$. Hence $\|F(x)\| \leqslant M\|x\|$ for all $x \in V \setminus \{0\}$. To prove F is continuous. We shall show that F is continuous at 0. Let $\lim_{n \to \infty} x_n = 0$ must show that $\lim_{n \to \infty} F(x_n) = F(0) = 0$. Since $0 \leqslant \|F(x_n)\| \leqslant M \|x_n\|$ and $\lim_{n \to \infty} x_n = 0$, $\lim_{n \to \infty} \|F(x_n)\| = 0$, hence $\lim_{n \to \infty} F(x_n) = 0$. So done.

Conversely, suppose not, so for each $m \in \mathbb{N}$ there exists an $x_m \in V \setminus \{0\}$ such that $\frac{\|F(x_m)\|}{\|x_m\|} > m$. Let $y_m = \frac{x_m}{m\|x_m\|}$ therefore $\|y_m\| = 1/_m$ for all $m \in \mathbb{N}$ and $\lim_{m \to \infty} \|y_m\| = 0$. Hence $\lim_{m \to \infty} y_m = 0$. Since F is continuous, $\lim_{m \to \infty} F(y_m) = F(0) = 0$. Hence $\|F(y_m)\| = \left\| \frac{F(x_m)}{m\, x_m} \right\| = \frac{\|F(x_m)\|}{m\,\|x_m\|} > 1$ for all $m \in \mathbb{N}$ and $\lim_{m \to \infty} \|F(y_m)\| = 0$, a contradiction. #

Corollary 2.16 Let V be LNLS(RNLS) and $F: V \longrightarrow \mathbb{H}$ a left (right) conjugate map. Then F is continuous if and only if F is bounded.

Let V be a LNLS(RNLS) and W a NLS. The set of all continuous left (right) linear operator which map V into W, is denoted by C(V,W). We also denote $V^* = C(V,\mathbb{H})$. The set of all continuous left (right) conjugate operators which map V in to $\mathbb{H}$, is denoted by $\bar{V}$. Then C(V,W) and $\bar{V}$ can be made into a RNLS(LNLS) and LNLS(RNLS) respectively as follows: Given $F, G \in C(V,W) \cup \bar{V}$ and $\alpha \in \mathbb{H}$ define $(F\alpha)(x) = (F(x))\alpha$, $(\alpha F)(x) = \alpha(F(x))$, $(F+G)(x) = F(x)+G(x)$ and $\| F \| = \sup_{x \neq 0} \left\{ \frac{\|F(x)\|}{\|x\|} \right\}$.

Remarks: (i) $\|F(x)\| \leqslant \|F\| \|x\|$ for all $x \in V$.

(ii) If V, W are LNLS's (RNLS's) then C(V,W) is left (right) vector space over $\mathbb{R} = \text{Cent}(\mathbb{H})$ and are also left (right) norm linear space over R using the above norm i.e. $\| F \| = \sup_{x \neq 0} \left\{ \frac{\|F(x)\|}{\|x\|} \right\}$.

Theorem 2.17 Let V be a LNLS and W a NLS which is also a Banach space. Then C(V,W) is complete.

Proof: Let $(F_n)_{n \in \mathbb{N}}$ be a cauchy sequence in C(V,W). Hence given $\varepsilon > 0$ there exists a N_ε such that $\| F_m - F_n \| < \varepsilon/2$ whenever m, $n > N_\varepsilon$. Hence for each $x \in V$.

$0 \leqslant \|F_m(x) - F_n(x)\| \leqslant \|F_m - F_n\| \|x\|$. Since $(F_n)_{n \in \mathbb{N}}$ is cauchy in C(V,W), $(F_n(x))_{n \in \mathbb{N}}$ is cauchy in W. Since W is complete, there exists a $v_x \in W$ such that $\lim_{n \to \infty} F_n(x) = v_x$. Define $F: V \longrightarrow W$ by $F(x) = v_x$. Claim that $\lim_{n \to \infty} F_n = F \in C(V,W)$. Clearly F is left linear.

To show that F is continuous. Choose $N \in \mathbb{N}$ such that $\|F_N - F_{N+p}\| < 1$ for all $p \in \mathbb{N}$. Hence $\|F_{N+p}\| < \|F_N\| + 1$ for all $p \in \mathbb{N}$. Thus $\|F_{N+p}(x)\| \leqslant \|F_{N+p}\| \, \|x\| \leqslant (\|F_N\| + 1)\|x\|$ for all $x \in V$ and for all $p \in \mathbb{N}$. Now, taking the limit as $p \to \infty$ we get that $\lim_{p\to\infty} \|F_{N+p}(x)\| = \|F(x)\|$. Therefore $\|F(x)\| < (\|F_N\|+1)\|x\|$ for all $x \in V$. Hence $\|F\| \leqslant \|F_N\|+1 < \infty$. Therefore F is bounded, so F is continuous.

To show that $\lim_{n\to\infty} F_n = F$, note that $F_n - F \in C(V,W)$. Fix $n \in \mathbb{N}$. Hence given $\varepsilon > 0$ there exists an $x_\varepsilon \in V \setminus \{0\}$ such that

$$\|F_n - F\| \leqslant \frac{\|(F_n - F)(x_\varepsilon)\|}{\|x_\varepsilon\|} + \varepsilon/2 = \frac{\|F_n(x_\varepsilon) - F(x_\varepsilon)\|}{\|x_\varepsilon\|} + \varepsilon/2$$

$$= \left\|F_n\left(\frac{1}{\|x_\varepsilon\|}\cdot x_\varepsilon\right) - F\left(\frac{1}{\|x_\varepsilon\|}\cdot x_\varepsilon\right)\right\| + \varepsilon/2 .$$

Since $F\left(\frac{1}{\|x_\varepsilon\|}\cdot x_\varepsilon\right) = \lim_{m\to\infty} F_m\left(\frac{1}{\|x_\varepsilon\|}\cdot x_\varepsilon\right)$, there exists a N_ε such that

$$\left\|F_n\left(\frac{1}{\|x_\varepsilon\|}\cdot x_\varepsilon\right) - F\left(\frac{1}{\|x_\varepsilon\|}\cdot x_\varepsilon\right)\right\| < \varepsilon/2 \quad \text{for all } n > N_\varepsilon .$$

Hence $\lim_{n\to\infty} F_n = F$. Therefore $C(V,W)$ is complete. ⋇

<u>Remark</u>: The same proof that if V,W are LNLS and W is complete, then then C(V,W) is complete.

<u>Corollary</u> 2.18 Let V be a LNLS(RNLS). Then $\bar{V}$ is complete.

<u>Proposition</u> 2.19 Let $F \in C(V,W)$. Then $\|F\| = \sup_{x\neq 0} \left\{ \frac{\|F(x)\|}{\|x\|} \right\} = \sup_{\|x\|=1} \{\|F(x)\|\} = \sup_{\|x\|\leqslant 1} \{\|F(x)\|\} = \frac{1}{r} \sup_{\|x\|=r} \{\|F(x)\|\} = \frac{1}{r} \sup_{\|x\|\leqslant r} \{\|F(x)\|\}$.

Proof: Let $x \in V \setminus \{0\}$ then $x = \|x\|(\frac{1}{\|x\|}\circ x)$. Let $U = \frac{1}{\|x\|}\circ x$. Then $\|U\| = 1$ therefore $\frac{\|F(x)\|}{\|x\|} = \frac{F(\|x\|U)}{\|x\|} = \frac{\|x\|\|F(U)\|}{\|x\|} = \|F(U)\|$. Hence $\|F\| = \sup_{\|x\|=1}\{\|F(x)\|\}$ so obvious $\|F\| = \sup_{\|x\|\leqslant 1}\{\|F(x)\|\}$. Also suppose that $\|x\| = r > 0$ therefore $\|\frac{1}{r}\circ x\| = 1$. In addition, if $\|y\| = 1$, then $y = \frac{1}{r}\circ(ry)$ and $\|ry\| = r\|y\| = r$. Hence we see that $\|F\| = \sup_{\|x\|=r}\{\|F(\frac{1}{r}x)\|\} = \frac{1}{r}\sup_{\|x\|=r}\{\|F(x)\|\}$. Also, it is clear that $\|F\| = \frac{1}{r}\sup_{\|x\|\leqslant r}\{\|F(x)\|\}$.

Theorem 2.20 (Hahn-Banach) Let V be a LNLS(RNLS) and W is a left (right) linear subspace of V. Let $f: W \longrightarrow \mathbb{H}$ be a continuous left (right) linear function. Then there exists a continuous left (right) linear function $F: V \longrightarrow \mathbb{H}$ such that $F/_W = f$ and $\|f\| = \|F\|$.

Proof: Since V is a LNLS over $\mathbb{H}$, V is a left norm linear space over $\mathbb{C}$ where $\mathbb{C} \cong \{a+bi/a,b \in \mathbb{R}\}$, hence W is a left norm linear space over $\mathbb{C}$. For each $x \in W$ let $f(x) = a_0 + a_1 i + a_2 j + a_3 k$ for some $a_\alpha \in \mathbb{R}$ where $\alpha = 0,1,2,3$. Then define $U(x) = a_0 + a_1 i$ therefore $U: W \longrightarrow \mathbb{C}$. Claim that U is a $\mathbb{C}$-linear map. Let x,y W and $\alpha \in \mathbb{C}$ therefore $f(\alpha x) = \alpha f(x)$. Let $f(x) = a_0 + a_1 i + a_2 j + a_3 k$ and $f(y) = b_0 + b_1 i + b_2 j + b_3 k$. Then $U(x+y) = a_0 + b_0 + (a_1+b_1)i = a_0 + a_1 i + b_0 + b_1 i = U(x) + U(y)$ and $U(\alpha x) = \alpha a_0 + \alpha a_1 i = \alpha(a_0 + a_1 i) = \alpha U(x)$. Hence we have the claim. Claim that U is continuous. Let

$x \in W \setminus \{0\}$ therefore $f(x) = a_0 + a_1 i + a_2 j + a_3 k$ for some $a_\alpha \in \mathbb{R}$ where $\alpha \in \{0,1,2,3\}$, so $\frac{|U(x)|}{\|x\|} = \frac{|a_0 + a_1 i|}{\|x\|} < \frac{|f(x)|}{\|x\|} \leqslant \|f\| < \infty$.

Hence U is continuous and in fact, $\|U\| \leqslant \|f\|$. Claim that $f(x) = U(x) - kU(kx)$ for all $x \in W \setminus \{0\}$. Let $x \in W \setminus \{0\}$ therefore $f(x) = a_0 + a_1 i + a_2 j + a_3 k$ for some $a_\alpha \in \mathbb{R}$ where $\alpha \in \{0,1,2,3\}$. i.e. $f(x) = a_0 + a_1 i + k(a_3 + a_2 i)$ and $f(kx) = kf(x) = a_0 k + a_1 j - a_2 i - a_3$, so $U(kx) = -a_3 - a_2 i$. i.e. $-U(kx) = a_3 + a_2 i$. Hence $f(x) = U(x) - kU(kx)$. Thus we have the claim. Since U is continuous $\mathbb{C}$-linear map, By the Hahn-Banach theorem for complex norm linear space, there exists a continuous $\mathbb{C}$-linear map $U' : V \longrightarrow \mathbb{C}$ such that $U'/_W = U$ and $\|U'\| = \|U\|$. Define $F: V \longrightarrow \mathbb{H}$ by $F(x) = U'(x) - kU'(kx)$. Claim that F is left linear. Let $x, y \in V$ and $\alpha \in \mathbb{C}$ therefore $\alpha = a+bi$ for some $a, b \in \mathbb{R}$. Then

$$
\begin{aligned}
F(x+y) &= U'(x+y)-kU'(k(x+y)) = U'(x)+U'(y)-kU'(kx)-kU'(ky) \\
&= U'(x)-kU'(kx)+U'(y)-kU'(ky) = F(x)+F(y), \\
F(\alpha x) &= U'(\alpha x)-kU'(k(\alpha x)) = \alpha U'(x)-kU'(k(a+bi)x) \\
&= \alpha U'(x)-kU'(akx)-kU'(bkix) = \alpha U'(x)-akU'(kx)-bkU'(-ikx) \\
&= \alpha U'(x)-akU'(kx)+b(ki)U'(kx) = \alpha U'(x)-akU'(kx)-bikU'(kx) \\
&= \alpha U'(x)-(a+bi)kU'(kx) = \alpha U'(x)-\alpha kU'(kx) \\
&= \alpha(U'(x)-kU'(kx)) = \alpha F(x), \\
F(kx) &= U'(kx)-kU'(k(kx)) = U'(kx)-kU'(-x) \\
&= U'(kx)+kU'(x) = kU'(x)+U'(kx) = k(U'(x)-kU'(kx)) = kF(x)
\end{aligned}
$$

and

$$F(jx) = U'(jx)-kU'(k(jx)) = U'(jx)+kU'(ix) = U'(jx)+kiU'(x)$$
$$= U'(jx)+jU'(x) = jU'(x)+U'(jx) = jU(x)-iU'(ijx)$$
$$= jU'(x)-jkU'(kx) = j(U'(x)-kU'(kx)) = jF(x).$$ Hence F is left linear over $\mathbb{H}$. Claim that F is continuous and extends f. Since U' is continuous and the map $x \mapsto kx$ is continuous, F is con continuous, F is continuous. Let $x \in W$. Then $F(x) = U'(x)-kU'(kx) = U(x)-kU(kx) = f(x)$. Must show that $\|F\| = \|f\|$. First claim that $\|f\| = \|U\|$. We have that $\|U\| \leqslant \|f\|$. Let $x \in W$ if $f(x) \neq 0$ let $\alpha_x = \frac{|f(x)|}{f(x)}$ therefore $|\alpha_x| = 1$ and $\alpha_x f(x) = |f(x)|$ therefore if $x \neq 0$ and $f(x) \neq 0$, then $\frac{|f(x)|}{\|x\|} = \frac{\alpha_x f(x)}{\|x\|} = \frac{f(\alpha_x x)}{\|\alpha_x x\|} = \frac{U(\alpha_x x)}{\|\alpha_x x\|} \leqslant \|U\|$. If $x \neq 0$ and $f(x) = 0$, then $\frac{|f(x)|}{\|x\|} = 0 \leqslant \|U\|$. Hence $\|f\| \leqslant \|U\|$. Thus $\|f\| = \|U\|$. Similarly $\|U'\| = \|F\|$. Hence $\|f\| = \|U\| = \|U'\| = \|F\|$. Hence we have the theorem. ※

<u>Corollary</u> 2.21 If $f: W \longrightarrow \mathbb{H}$ is a left((right) conjugate function then there exists a continuous left (right) conjugate function $F: V \longrightarrow \mathbb{H}$ such that $F/_W = f$ and $\|F\| = \|f\|$

<u>Theorem</u> .22 Let V be a LNLS (RNLS) which is also a Banach space and $A \in C(V,V)$ such that $\|A\| < 1$. Then $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$.

Proof: First we must show that $(\sum_{k=0}^{n} A^k)$ converges. Note that $\sum_{k=0}^{n} \|A^k\| \leqslant \sum_{k=0}^{\infty} \|A\|^k$ for all $n \in \mathbb{N}$ therefore $\sum_{k=0}^{\infty} \|A^k\| \leqslant \sum_{k=0}^{\infty} \|A\|^k < \infty$.

Let m, n then $\left\| \sum_{k=0}^{n} A^k - \sum_{k=0}^{m} A^k \right\| = \left\| \sum_{k=m+1}^{n} A^k \right\| \leqslant \sum_{k=m+1}^{n} \|A^k\| < \sum_{k=m+1}^{n} \|A\|^k \longrightarrow 0$ as $m, n \longrightarrow \infty$ therefore $(\sum_{k=0}^{n} A^k)_{n \in \mathbb{N}}$ is cauchy sequence in C(V,V), hence $(\sum_{k=0}^{n} A^k)_{n \in \mathbb{N}}$ converges. $(I-A)(\sum_{k=0}^{n} A^k) = I - A^{n+1}$ for all $n \in N$. Therefore $\lim_{n\to\infty} (I-A)(\sum_{k=0}^{n} A^k) = \lim_{n\to\infty} I - A^{n+1} = I$. Hence $(I-A)^{-1} = \sum_{n=0}^{\infty} A^n$. ⁂

Definition 2.23 Let $(V, \| \;\|)$, $(V', \| \;\|')$ be LNLS's (RNLS's) then we say that $(V, \| \;\|)$ is left (right) isomorphic to $(V', \| \;\|')$ if and only if there exists a 1-1 onto left linear map F such that $\|F(v)\|' = \|v\|$ for all $v \in V$.

Example 2.24 (i) $(\ell^p_{\mathbb{H}}, \| \;\|)^*$ is right (left) isomorphic to $(\ell^q_{\mathbb{H}}, \| \;\|_q)$ where $1/_p + 1/_q = 1$ and $1 < p < \infty$.

Proof: Let $y = (\beta_n)_{n \in \mathbb{N}} \in \ell^q_{\mathbb{H}}$. Define $F_y : \ell^p_{\mathbb{H}} \longrightarrow \mathbb{H}$ by $F_y(x) = \sum_{n=1}^{\infty} \alpha_n \beta_n$ where $x = (\alpha_n)_{n \in \mathbb{N}} \in \ell^p_{\mathbb{H}}$. Since $(\sum_{n=1}^{k} \alpha_n \beta_n)_{k \in \mathbb{N}}$ is is cauchy sequence in $\mathbb{H}$, $(\sum_{n=1}^{k} \alpha_n \beta_n)_{k \in \mathbb{N}}$ converges, so F_y is well-defined. Clearly F_y is left linear. Claim that F_y is continuous. Let $x = (\alpha_n)_{n \in \mathbb{N}} \in \ell^p_{\mathbb{H}}$. Then

$$|F_y(x)| = \left|\sum_{n=1}^{\infty}\alpha_n\beta_n\right| \leqslant \sum_{n=1}^{\infty}|\alpha_n\beta_n| \leqslant \left(\sum_{n=1}^{\infty}|\alpha_n|^p\right)^{1/p}\left(\sum_{n=1}^{\infty}|\beta_n|^q\right)^{1/q}$$

$= \|x\|_p \|y\|_q$. Therefore $\|F_y\| < \|y\|_q$. Hence F_y is continuous. In fact $\|F_y\| \leqslant \|y\|_q$. Thus $F_y \in \ell_{\mathbb{H}}^{p^*}$. Define a map $F: \ell_{\mathbb{H}}^{q} \longrightarrow \ell_{\mathbb{H}}^{p^*}$ by $F(y) = F_y$ for all $y \in \ell_{\mathbb{H}}^{q}$. Claim that F is right linear. Let α, $\beta \in \mathbb{H}$ and $y_1 = (\alpha_n)_{n\in\mathbb{N}}$, $y_2 = (\beta_n)_{n\in\mathbb{N}}$. Then

$$F_{y_1\alpha+y_2\beta}(x) = \sum_{n=1}^{\infty} x_n(\alpha_n\alpha+\beta_n\beta) = \left(\sum_{n=1}^{\infty} x_n\alpha_n\right)\alpha+\left(\sum_{n=1}^{\infty} x_n\beta_n\right)\beta$$

$$= F_{y_1}(x)\alpha+F_{y_2}(x)\beta = F_{y_1}\alpha(x)+F_{y_2}\beta(x) \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in \ell_{\mathbb{H}}^{p}.$$

Hence $F(y_1\alpha+y_2\beta) = F_{y_1\alpha+y_2\beta} = F_{y_1}\alpha + F_{y_2}\beta = F(y_1)\alpha+F(y_2)\beta$.

Therefore F is right linear. Claim that $\|y\|_q = \|F(y)\| = \|F_y\|$ for all $y \in \ell_{\mathbb{H}}^{q}$. If $y = 0$, then $\|y\|_q = \|0\|_q = \|F(0)\| = \|F_y\| = \|F_0\|$ so done. Assume that $0 \neq y = (\beta_n)_{n\in\mathbb{N}} \in \ell_{\mathbb{H}}^{q}$. Define $x_N \in \ell_{\mathbb{H}}^{p}$ by

$x_N = \sum_{n=1}^{N} \alpha_n e_n$ where $e_n = (0,0,\dots,\underset{n^{th}\text{ place}}{1},0,0,\dots)$ and for $n = 1,2,\dots,N$

$$\alpha_n = \begin{cases} 0 & \text{if } \beta_n = 0 \\ |\beta_n|^{q-1}\cdot\dfrac{\bar{\beta}_n}{|\beta_n|} & \text{if } \beta_n \neq 0. \end{cases}$$

Then $\|x_N\|_p = \left(\sum_{n=1}^{N}|\alpha_n|^p\right)^{1/p} = \left(\sum_{n=1}^{N}|\beta_n|^q\right)^{1/q}$. Therefore

$$\|F_y(x_N)\| = \left|\sum_{n=1}^{N} {}_n\beta_n\right| = \sum_{n=1}^{N}|\beta_n|^q = \left(\sum_{n=1}^{N}|\beta_n|^q\right)^{1/q}\left(\sum_{n=1}^{N}|\beta_n|^q\right)^{1/p}$$

$$= \left(\sum_{n=1}^{N}|\beta_n|^q\right)^{1/q}\|x_N\|_p .$$

Thus for N sufficiently large to ensure that $\| x_N \|_p \neq 0$ we have that $\left| F_y\left(\frac{x_N}{\|x_N\|_p}\right)\right| = \left(\sum_{n=1}^{N} |\beta_n|^q\right)^{1/q}$. Hence $\| F_y \| \geqslant \left(\sum_{n=1}^{N} |\beta_n|^q\right)^{1/q}$.

Therefore $\| F_y \| \geqslant \left(\sum_{n=1}^{\infty} |\beta_n|^q\right)^{1/q} = \| y \|_q$. Claim that F is onto. Let $e_n = (0,0,\ldots,\overset{n^{th}\text{place}}{1},0,0,\ldots)$ for all $n \in \mathbb{N}$. Let $F' \in \ell_{\mathbb{H}}^{p*}$ we denote $F'(e_n) = \beta_n$ for all $n \in \mathbb{N}$. We must show that $(\beta_n)_{n\in\mathbb{N}} \in \ell_{\mathbb{H}}^q$. If $\beta_n = 0$ for all $n \in \mathbb{N}$, then $(\beta_n)_{n\in\mathbb{N}} \in \ell_{\mathbb{H}}^q$. So assume $(\beta_n)_{n\in\mathbb{N}} \neq 0$.

Define $x_N \in \ell_{\mathbb{H}}^p$ as follows: $x_N = \sum_{n=1}^{N} \alpha_n e_n$ for $n = 1,2,\ldots,N$

$$\alpha_n = \begin{cases} 0 & \text{if if } \beta_n = 0 \\ |\beta_n|^{q-1} \cdot \dfrac{\bar{\beta}_n}{|\beta_n|} & \text{if } \beta_n \neq 0. \end{cases}$$

Therefore $|F'(x_N)| = \left|F'\left(\sum_{n=1}^{N} {}_n e_n\right)\right| = \sum_{n=1}^{N} |\beta_n|^q$ and $\| x_N \|_p = \left(\sum_{n=1}^{N} |\beta_n|^q\right)^{1/p}$

for N sufficiently large we have that

$$\frac{|F'(x_N)|}{\|x_N\|_p} = \frac{\sum_{n=1}^{N} |\beta_n|^q}{\|x_N\|_p} = \left(\sum_{n=1}^{N} |\beta_n|^q\right)^{1-1/p}, \text{ so } \| F' \| \geqslant \left(\sum_{n=1}^{N} |\beta_n|^q\right)^{1/q}.$$

Since N is arbitrary we get that $\left(\sum_{n=1}^{\infty} |\beta_n|^q\right)^{1/q} \leqslant \| F' \|$. Therefore $y = (\beta_n)_{n\in\mathbb{N}} \in \ell_{\mathbb{H}}^q$. Claim $F' = F_y$. Let $(x_n)_{n\in\mathbb{N}} = x \in \ell_{\mathbb{H}}^q$. Then

$$F'(x) = F'\left(\lim_{k\to\infty} \sum_{n=1}^{k} x_n e_n\right) = \lim_{k\to\infty} \sum_{n=1}^{k} x_n F'(e_n) = \lim_{k\to\infty} \sum_{n=1}^{k} x_n \beta_n$$

$= \sum_{n=1}^{\infty} x_n\beta_n = F_y(x)$. Hence $F' = F_y$. Therefore F is onto. Claim F is 1-1. Let $y = (\alpha_n)_{n\in\mathbb{N}} \in \ell^q_{\mathbb{H}}$ such that $F(y) = 0 = F_y$ therefore $F_y(x) = 0$ for all $x \in \ell^p_{\mathbb{H}}$, hence $F_y(e_n) = \alpha_n = 0$ for all $n \in \mathbb{N}$, so $y = \sum_{n=1}^{\infty} \alpha_n e_n = 0$. Hence F is 1-1. Thus $(\ell^p_{\mathbb{H}}, \| \ \|)^*$ is right isomorphic to $(\ell^q_{\mathbb{H}}, \| \ \|_q)$.

(ii) $(\ell^1_{\mathbb{H}}, \| \ \|)^*$ is a right (left) isomorphic to $(\ell^\infty_{\mathbb{H}}, \| \ \|)$

<u>Proof</u>: Let $y = (\beta_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{H}}$. Define $F_y : \ell^1_{\mathbb{H}} \to \mathbb{H}$ by

$F_y(x) = \sum_{n=1}^{\infty} \alpha_n\beta_n$ where $x = (\alpha_n)_{n\in\mathbb{N}} \in \ell^1_{\mathbb{H}}$. Claim that F_y is well defined i.e. we must to show $\left(\sum_{n=1}^{k} \alpha_n\beta_n\right)_{k\in\mathbb{N}}$ converges. Fix $N \in \mathbb{N}$ therefore

$$\sum_{n=1}^{N} |\alpha_n\beta_n| = \sum_{n=1}^{N} |\alpha_n|\,|\beta_n| \leqslant \left(\sum_{n=1}^{N} |\alpha_n| \left(\max_{1\leqslant n\leqslant N} \{|\beta_n|\}\right)\right) \leqslant \|x\|\|y\|_\infty .$$ Since N is arbitrary, $\sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty$, so $\left(\sum_{n=1}^{k} \alpha_n\beta_n\right)_k$ converges. Hence F_y is well-define. Clearly F_y is a left linear map. Claim that F_y is continuous. Let $x = (\alpha_n)_{n\in\mathbb{N}} \in \ell^1_{\mathbb{H}}$. Then

$$|F_y(x)| = \left|\sum_{n=1}^{\infty} \alpha_n\beta_n\right| \leqslant \sum_{n=1}^{\infty} |\alpha_n\beta_n| = \sum_{n=1}^{\infty} |\alpha_n|\,|\beta_n| \leqslant \|x\|\|y\|_\infty < \infty.$$ Hence F_y is continuous. In fact $\|F_y\| \leqslant \|y\|_\infty$. Hence $F_y \in \ell^{1*}_{\mathbb{H}}$. Define a map $F : \ell^\infty_{\mathbb{H}} \to \ell^{1*}_{\mathbb{H}}$ by $F(y) = F_y$ for all $y \in \ell^\infty_{\mathbb{H}}$. The same proof as in

(i) show that F is right linear. Claim that $\|y\|_\infty = \|F(y)\| = \|F_y\|$ for all $y \in \ell^\infty_{\mathbb{H}}$. If $y = 0$, then $\|y\|_\infty = \|0\|_\infty = \|F(0)\| = \|F_0\|$. So done. Assume that $0 \neq y = (\beta_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{H}}$. Given $\varepsilon > 0$ there exists an $n_\varepsilon \in N$ such that $\beta_{n_\varepsilon} > \|y\|_\infty - \varepsilon$. Now let

$$x_m = \begin{cases} 0 & \text{if } m \neq n_\varepsilon \\ \dfrac{\overline{\beta}_{n_\varepsilon}}{|\beta_{n_\varepsilon}|} & \text{if } m = n_\varepsilon . \end{cases}$$

Therefore $\dfrac{|F_y(x)|}{\|x\|} = |F_y(x)| = \left|\sum_{m=1}^{\infty} x_m\beta_m\right| = |\beta_{n_\varepsilon}| > \|y\|_\infty - \varepsilon$.

Hence given $\varepsilon > 0$ there exists an $x_0 \in \ell^1_{\mathbb{H}} \setminus \{0\}$ such that $\dfrac{|F_y(x_0)|}{\|x_0\|} > \|y\|_\infty - \varepsilon$. Hence $\|F_y\| \geq \|y\|_\infty$. Thus $\|F_y\| = \|y\|$. Claim that F is onto. Let $e_n = (0,0,\ldots,\underset{n^{th}\text{ place}}{1},0,0,\ldots)$ for all $n \in \mathbb{N}$. Let $F' \in \ell^1_{\mathbb{H}}$ we denote $F'(e_n) = \beta_n$ for all $n \in \mathbb{N}$. If $\beta_n = 0$ for all $n \in \mathbb{N}$ then $(\beta_n)_{n\in\mathbb{N}} \in \ell^\infty_{\mathbb{H}}$. So assume that $y = (\beta_n)_{n\in\mathbb{N}} \neq 0$. Suppose that $\sup_{n\in\mathbb{N}}\{|\beta_n|\} = \infty$. Given $M > 0$ there exists an $N_M \in \mathbb{N}$ such that $|\beta_{N_M}| > M$. For each $n \in \mathbb{N}$ let

$$d_n = \begin{cases} 0 & \text{if } n \neq N_M \\ \dfrac{\overline{\beta}_{N_M}}{|\beta_{N_M}|} & \text{if } n = N_M . \end{cases}$$

Then $d = (d_n)_{n \in \mathbb{N}} \in \ell^1_{\mathbb{H}}$. Also, $\| d \| = 1$. Since

$$|F'(d)| = \left|F'\left(\sum_{n=1}^{\infty} d_n e_n\right)\right| = \left|\sum_{n=1}^{N_M} d_n F'(e_n)\right| = |\beta_{N_M}| > M, \quad \frac{|F(d)|}{\| d \|} > M$$

a contradiction since $F' \in \ell^{1*}_{\mathbb{H}}$. Hence $\sup_{n \in \mathbb{N}} \{|\beta_n|\} < \infty$ i.e.

$y = (\beta_n)_{n \in \mathbb{N}} \in \ell^{\infty}_{\mathbb{H}}$. The same proof (i) show that $F' = F_y$ and F is 1-1 i.e. F is onto and 1-1. Hence $(\ell^1_{\mathbb{H}}, \| \ \|)^*$ is right isomorphic to $(\ell^{\infty}_{\mathbb{H}}, \| \ \|_{\infty})$. ⋇

(iii) Let $C_0 = \left\{(x_n)_{n \in \mathbb{N}} / x_n \in \mathbb{H} \ \forall n \in \mathbb{N} \text{ and } \lim_{n \to \infty} x_n = 0\right\}$ and $\| x \| = \sup_{n \in \mathbb{N}} \{|x_n|\}$ where $x = (x_n)_{n \in \mathbb{N}} \in C_0$. Then $(C_0, \| \ \|)^*$ is right (left) isomorphic to $\ell^1_{\mathbb{H}}$.

<u>Proof</u>: Claim that if $\varphi \in C_0^*$ then there exists an $a = (a_n)_{n \in \mathbb{N}} \in \ell^1_{\mathbb{H}}$ such that $\varphi(x) = \sum_{n=1}^{\infty} x_n a_n$ for all $x = (x_n)_{n \in \mathbb{N}} \in C_0$.

Let $e_n = (0, 0, \ldots, \overset{n^{th} \text{ place}}{1}, 0, 0, \ldots)$ and let $\varphi \in C_0^*$. Suppose $\varphi(e_n) = a_n$ for all $n \in \mathbb{N}$. Must show that $a = (a_n)_{n \in \mathbb{N}} \in \ell^1_{\mathbb{H}}$ i.e. $\sum_{n=1}^{\infty} |a_n| < \infty$.

Suppose not then $\sum_{n=1}^{\infty} |a_n| = \infty$. Therefore given $M > 0$ there exists an $N_M \in \mathbb{N}$ such that $\sum_{n=1}^{N_M} |a_n| > M$. Given $n \in \mathbb{N}$ let

$$d_n = \begin{cases} 0 & \text{if } n > N_M \\ 0 & \text{if } n \leqslant N_M \text{ and } a_n = 0. \\ \dfrac{\bar{a}_n}{|a_n|} & \text{if } n \leqslant N_M \text{ and } a_n \neq 0. \end{cases}$$

Let $d = (d_n)_{n\in\mathbb{N}}$ then $d \in C_0$ also, $\|d\| = 1$. Then

$|\varphi(d)| = |\varphi(\sum_{n=1}^{N_M} d_n e_n)| = \sum_{n=1}^{N_M} |a_n| > M.\|d\|$. Therefore $\frac{|\varphi(d)|}{\|d\|} > M$.

Hence we see that give $M > 0$ there exists an $d \in C_0$ such that

$\frac{|\varphi(d)|}{\|d\|} > M$, a contradiction since $\varphi \in C_0^*$. So $\sum_{n=1}^{\infty} |a_n| < \infty$ therefore $a \in \ell^1_{\mathbb{H}}$. Next we must show that $\varphi(x) = \sum_{n=1}^{\infty} x_n a_n$. Let $x = (x_n)_{n\in\mathbb{N}} \in C_0$.

Give $m \in \mathbb{N}$ let $x_m = \sum_{n=1}^{m} x_n e_n$. Then $\varphi(x_m) = \sum_{n=1}^{m} x_n a_n$. Since $\lim_m x_m = x$ and $\varphi \in C_0^*$, $\lim_{m\to\infty} \varphi(x_m) = \varphi(x)$. Thus $\varphi(x) = \sum_{n=1}^{\infty} x_n a_n$. Define $F: C_0^* \to \ell^1_{\mathbb{H}}$ by $F(\varphi) = (a_n)_{n\in\mathbb{N}}$ where $a_n = \varphi(e_n)$ for all $n \in \mathbb{N}$. Clearly F is right linear. To show that F is 1-1. Let $\varphi \in C_0^*$ be such that $F(\varphi)=0$ therefore $\varphi(e_n) = 0$ for all $n \in \mathbb{N}$. Hence $\varphi(x) = \varphi(\sum_{n=1}^{\infty} x_n e_n) =$

$\sum_{n=1}^{\infty} x_n \varphi(e_n) = 0$ for all $x = (x_n)_{n\in\mathbb{N}} \in C_0$ i.e. $\varphi \equiv 0$. Hence F is 1-1. To show F is onto. Given $(a_n)_{n\in\mathbb{N}} \in \ell^1_{\mathbb{H}}$ define $\varphi(x) = \sum_{n=1}^{\infty} x_n a_n$ for all $x = (x_n)_{n\in\mathbb{N}} \in C_0$. Then φ is well-defined, left linear and continuous since

$$|\varphi(x)| = |\sum_{n=1}^{\infty} x_n a_n| \leqslant \sum_{n=1}^{\infty} |a_n||x_n| \leqslant \|a\|_1 \|x\|.$$ In fact, $\|\varphi\| \leqslant \|a\|_1$.

Since $\varphi(e_n) = a_n$ for all $n \in \mathbb{N}$, $F(\varphi) = (a_n)_{n\in\mathbb{N}}$. Lastly we must show that F preserves norm i.e. $\|\varphi\| = \|F(\varphi)\|$ for all $\varphi \in C_0^*$. If

$\varphi \equiv 0$ then $F(\varphi) = 0$ therefore $\|\varphi\| = \|F(\varphi)\|$ so done. Hence assume that $\varphi \not\equiv 0$. We have already show that $\|\varphi\| \leqslant \|a\|_1$. Give $(a_n)_{n\in\mathbb{N}} \in \ell^1_{\mathbb{H}}$ and $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $\sum_{n=1}^{N_\varepsilon} |a_n| > \|a\|_1 - \varepsilon$.

Now let

$$x_n = \begin{cases} 0 & \text{if } n > N_\varepsilon . \\ 0 & \text{if } n \leqslant N_\varepsilon \text{ and } a_n = 0. \\ \dfrac{\bar{a}_n}{|a_n|} & \text{if } n \leqslant N_\varepsilon \text{ and } a_n \neq 0 \end{cases}$$

Let $x = (x_n)_{n\in\mathbb{N}}$. Then $x \in C_0$ also, $\|x\| = 1$. Now,

$$\frac{|\varphi(x)|}{\|x\|} = |\varphi(x)| = \left|\sum_{n=1}^{\infty} x_n a_n\right| = \left|\sum_{n=1}^{N_\varepsilon} |a_n|\right| = \sum_{n=1}^{N_\varepsilon} |a_n| > \|a\|_1 - \varepsilon .$$

Hence given $\varepsilon > 0$ there exists an $x \in C_0 \setminus \{0\}$ such that $\dfrac{|\varphi(x)|}{\|x\|} > \|a\|_1 - \varepsilon$ therefore $\|\varphi\| \geqslant \|a\|_1$. Hence $\|\varphi\| = \|a\|_1 = \|F(\varphi)\|$. Thus $(C_0, \|\ \|^*)$ is right isomorphic to $\ell^1_{\mathbb{H}}$. ⋇

Let V be a LNLS(RNLS) and $W \subsetneq V$ a closed left(right) linear subspace. Then $V/_W$ is a left (right) vector space over $\mathbb{H}$. $V/_W$ has a left (right) norm define as follows: let $\alpha \in V/_W$ define $\|\alpha\| = \inf_{x\in\alpha} \|x\|$. Let $P: V \longrightarrow V/_W$ be the natural projection i.e. if $x \in V$ define $P(x) = x+W = [x]$. Then P is left (right) linear.

Let $x \in V \setminus \{0\}$. Then $\frac{\|P(x)\|}{\|x\|} = \frac{\|[x]\|}{\|x\|} \leqslant \frac{\|x\|}{\|x\|} = 1$ therefore P is continuous.

<u>Remark</u>: If V is a NLS and $W \subseteq V$ a closed linear subspace. Then $V/_W$ is a vector space over $\mathbb{H}$. $V/_W$ has a norm define as above.

<u>Theorem</u> 2.25 Let V be a LNLS(RNLS) which is also a Banach space. Then $V/_W$ is a Banach space where W is closed left (right) linear subspace of V.

<u>Proof</u>: Let $(y_n)_{n \in \mathbb{N}}$ be a cauchy sequence in $V/_W$ there exists a $n_1 \in \mathbb{N}$ such that $\|y_{n_1} - y_n\| < 1/_2$ for all $n \geqslant n_1$. Choose $n_2 > n_1$ such that $\|y_{n_2} - y_n\| < 1/_{2^2}$ for all $n \geqslant n_2$. By induction we can find $n_1 < n_2 < \dots$ such that $\|y_{n_k} - y_n\| < 1/_{2^k}$ for all $n \geqslant n_k$ In particular, $\|y_{n_k} - y_{n_{k+1}}\| < 1/_{2^k}$. Let x_1 be an element in V such that $P(x_1) = y_{n_1}$. Since P is onto, there exists a $x \in V$ such that $P(x) = y_{n_2}$. Since $\|y_{n_1} - y_{n_2}\| = \|P(x_1) - P(x)\| = \|P(x_1 - x)\| = \inf\left\{\|x_1 - x + z\| \,/\, z \in W\right\} < 1/_2$, there exists a $z_0 \in W$ such that $\|x_1 - x + z_0\| < 1/_2$. Let $x_2 = x - z_0$ then $P(x_2) = P(x - z_0) = P(x) - P(z_0) = P(x) = y_{n_2}$ and $\|x_1 - x_2\| < 1/_2$. Suppose there exists a $x_k \in V$ such that $P(x_k) = y_{n_k}$ then there exists an $x_{k+1} \in V$ such that $\|x_k - x_{k+1}\| \leqslant 1/_{2^k}$ by the same proof as above. For any positive integer l we get that

$$\|x_k - x_{k+l}\| \leq \|x_k - x_{k+1}\| + \|x_{k+1} - x_{k+2}\| + \dots + \|x_{k+l-1} - x_{k+l}\|$$

$< 1/_2{}^k + 1/_2{}^{k+1} + \dots + 1/_2{}^{k+l-1} < 1/_2{}^k\,(1+1/_2+\dots) = 1/_2{}^{k-1}$ therefore

$\lim\limits_{k\to\infty} \|x_k - x_{k+l}\| = 0$, so $(x_k)_{k\in\mathbb{N}}$ is a cauchy sequence in V.

Since V is complete, there exists an $x \in V$ such that $\lim\limits_{k\to\infty} x_k = x$.

By the continuity of P we have that $\lim\limits_{k\to\infty} P(x_k) = P(x)$ i.e.

$\lim\limits_{k\to\infty} y_{n_k} = P(x)$ therefore $(y_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(y_n)_{n\in\mathbb{N}}$ hence $(y_n)_{n\in\mathbb{N}}$ itself must be convergent. ⋕

<u>Theorem</u> 2.26 Let V,W be a LNLS(RNLS) which are also Banach space. Then $V \times W$ is a Banach space

<u>Proof</u>: Same proof as in $(\mathbb{H}^n, \|\ \|_2)$.

<u>Theorem</u> 2.27 Let V be a LNLS(RNLS). Then V is left (right) isomorphic to a subspace of V^{**}

<u>Proof</u>: For each $x \in V$ define $\psi_x : V^* \to \mathbb{H}$ by $\psi_x(\varphi) = \varphi(x)$ for all $\varphi \in V^*$. Then ψ_x is right linear. Claim that ψ_x is continuous. To prove this let $\varphi \in V^* \setminus \{0\}$ therefore

$$\frac{|\psi_x(\varphi)|}{\|\varphi\|} = \frac{|\varphi(x)|}{\|\varphi\|} \leq \frac{|\varphi(x)|}{\frac{|\varphi(x)|}{\|x\|}} = \|x\|, \text{ so } \|\psi_x\| \leq \|x\| < \infty.$$

If $x = 0$, then $\psi_0 = 0$ which is continuous. In fact $\|\psi_0\| = \|0\| = 0$ therefore $\|\psi_x\| \leq \|x\|$ for all $x \in V$. Define a map $F : V \to V^{**}$ by

$F(x) = \psi_x$. Then F is left linear. Claim that F is 1-1 and preserves norms. To show that F is 1-1, suppose that F is not 1-1. There exists an $x \in V \setminus \{0\}$ such that $F(x) = 0$, so $\psi_x = 0$ i.e. $\psi_x(\varphi) = 0$ for all $\varphi \in V^*$, hence $\varphi(x) = 0$ for all $\varphi \in V^*$. Let W be the left linear subspace of V generated by x i.e. $W = \{\alpha x / \alpha \in \mathbb{H}\}$. Define $\eta : W \longrightarrow \mathbb{H}$ by $\eta(\alpha x) = \alpha$ for all $\alpha \in \mathbb{H}$. Then η is left linear. η is continuous since dim W is 1. By the Hahn-Banach Theorem we can extend η to a continuous left linear map $\emptyset : V \longrightarrow \mathbb{H}$ such that $\|\emptyset\| = \|\eta\|$ therefore $\emptyset \in V^*$ and $\psi_x(\emptyset) = 0 = \emptyset(x) = \eta(x) = 1$,a contradiction. Hence F is 1-1. We must show that $\|\psi_x\| \geqslant \|x\|$ for all $x \in V$. To prove this, we shall first show that given $x \in V \setminus \{0\}$ and $M > 0$ then there exists an $\varphi \in V^*$ such that $\|\varphi\| = M$ and $\varphi(x) = \|\varphi\| \|x\|$. To prove this, let U be the left linear subspace of V generated by x. Define $W : U \longrightarrow \mathbb{H}$ by $W(\ x) = \alpha M\|x\|$. Then W is left linear and continuous. By the Hahn Banach Theorem, there exists a continuous left linear map $\Gamma : V \longrightarrow \mathbb{H}$ such that $\|\Gamma\| = \|W\| = M$ and $\Gamma(x) = W(x) = M\|x\| = \|\Gamma\| \|x\|$. Hence for each $x \in V \setminus \{0\}$ and $M > 0$ there exists a $\varphi \in V^*$ such that $\|\varphi\| = M$ and $\varphi(x) = \|\varphi\| \|x\|$. Given $x \in V \setminus \{0\}$ there exists an $\varphi \in V^*$ such that $\|\varphi\| = 1$ and $\varphi(x) = \|x\|$. Hence $\dfrac{|\psi_x(\varphi)|}{\|\varphi\|} = |\psi_x(\varphi)| = |\varphi(x)| = \|x\|$, so $\|\psi_x\| \geqslant \|x\|$. Hence $\|\psi_x\| = \|x\|$.

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<u>Remarks</u>: (i) Since F is norm preserving F is an isometry, hence F is a homeomorphic onto its image

(ii) In the finite dimensional case V is left (right) isomorphic to V^{**}

(iii) In the ∞-dimensional case V may not to left (right) isomorphic to V^{**}.

<u>Example</u> 2.28 Let $C_0 = \{(x_n)_{n\in\mathbb{N}} / x_n \in \mathbb{H}\ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 0\}$ and $\|x\| = \sup_{n\in\mathbb{N}} \{|x_n|\}$ where $x = (x_n)_{n\in\mathbb{N}} \in C_0$. Claim that C_0 is separable. To prove this, let

$$D = \{(x_n)_{n\in\mathbb{N}} \in C_0 / x_n \in \mathbb{Q}^4\ \forall n\in\mathbb{N} \text{ and } \exists N \in \mathbb{N} \ni x_n = 0\ \forall n > N\}$$

therefore D is countable. To show $\bar{D} = C_0$. Let $z = (z_n)_{n\in\mathbb{N}} \in C_0$. Let $\varepsilon > 0$ be given there exists a $N_\varepsilon > 0$ such that $|z_n| < \varepsilon/2$ for all $n > N_\varepsilon$. For each $n = 1,2,\ldots,N_\varepsilon$ there exists an $q_n \in \mathbb{Q}^4$ such that $|q_n - z_n| < \varepsilon/2$ therefore $\sup_{1\leq n\leq N_\varepsilon} \{|q_n - z_n|\} \leq \varepsilon/2$. Take $x_\varepsilon = (q_1, q_2, \ldots, q_{N_\varepsilon}, 0, 0, \ldots)$ therefore $x_\varepsilon \in D$ and $\|x_\varepsilon - z\| \leq \varepsilon/2 < \varepsilon$. Hence $\bar{D} = C_0$ i.e. C_0 is separable. Next, claim that $\ell^\infty_{\mathbb{H}}$ is non separable. To prove this, note that the set of sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_n = 0$ or 1 is uncountable and belong to $\ell^\infty_{\mathbb{H}}$. Let $z = (z_n)_{n\in\mathbb{N}}$, $w = (w_n)_{n\in\mathbb{N}}$ are distinct sequences such that $w_n = 0$ or 1 and $z_n = 0$ or 1 for all $n \in \mathbb{N}$. Then

$$d(z,w) = \|z-w\| = \sup_{n\in\mathbb{N}} \{|z_n - w_n|\} = 1.$$

Claim that $B(z,1/4) \cap B(w,1/4) = \emptyset$. Suppose not therefore there exists an $x \in B(z,1/4) \cap B(w,1/4)$. So $\|x-z\| < 1/4$ and $\|x-w\| < 1/4$.

Therefore

$$1 = \|z-w\| \leqslant \|z-x\| + \|x-w\| < 1/_2 \text{ , a contradiction.}$$

Hence if $z = (z_n)_{n\in\mathbb{N}}$, $w = (w_n)_{n\in\mathbb{N}}$ are distinct sequence sucht that $w_n = 0$ or 1 and $z_n = 0$ or 1 then $B(z,1/_4) \cap B(w,1/_4) = \emptyset$. If $\ell^{\infty}_{\mathbb{H}}$ has a countable dense subset D then $B(z,1/_4) \cap D \neq \emptyset$ for all $z = (z_n)_{n\in\mathbb{N}} \in \ell^{\infty}_{\mathbb{H}}$ where $z_n = 0$ or 1. Hence D is uncountable, a contradiction. Hence we have the claim. Thus C_0 is not left isomorphic to $\ell^{\infty}_{\mathbb{H}}$. Hence C_0 is not left isomorphic to C_0^{**}.

<u>Theorem</u> 2.29 (Open Mapping Theorem) Let V, W be LNLS's (RNLS's) which are also Banach space and $F: V \longrightarrow W$ a continuous left (right) linear map which is onto. Then F is open.

<u>Proof</u>: Let $U \subseteq V$ be a nonempty open set. Must show that F(U) is open in W. Let $y \in F(U)$ therefore there exists an $x \in U$ such that $y = F(x)$. Since U is open in V, there exists an open ball $B(x;\delta) \subseteq U$. Hence $F(B(x;\delta)) \subseteq F(U)$ and $y \in F(B(x;\delta))$. If we can show that there exists an open ball $B(y;\varepsilon)$ in W such that $B(y;\varepsilon) \subseteq F(B(x;\delta))$, then we are done. Claim that if we can show that there exists an open ball $B(0;r) \subseteq F(B(0;1))$, then we get that there exists an open ball $B(y;\varepsilon) \subseteq F(B(x;\delta))$. To prove this claim, Note that if there exists an open ball $B(0;r) \subseteq F(B(0;1))$ then given $r' > 0$ therefore $F(B(0;r')) = F(r'(B(0;1)) = r'F(B(0;1))$ $\supseteq r'B(0;r) = B(0;r'r)$ therefore $B(0;r'r) \subseteq F(B(0;r'))$ for all $r' > 0$.

Now, $B(x;\delta)-x = B(0;\delta)$ therefore $F(B(x;\delta)-x) = F(B(x;\delta))-F(x)$. But $F(B(x;\delta)-x) = F(B(0;\delta)) \supseteq B(0;\delta r)$. Hence $F(B(x;\delta)) \supseteq B(0;r\delta) + F(x) = B(F(x);r\delta) = B(y;r\delta)$. Hence to finish the theorem we only to show that there exists an open ball $B(0;r) \subseteq F(B(0;1))$. Since $V = \bigcup_{n\in\mathbb{N}} B(0;n/_2) = \bigcup_{n\in\mathbb{N}} nB(0;1/_2)$ and F is onto, $W = F(V) = F(\bigcup_{n\in\mathbb{N}} nB(0;1/_2)) = \bigcup_{n\in\mathbb{N}} nF(B(0;1/_2))$. By corollary 0.19., there exists a closed ball B in W and there exists an $n_0 \in \mathbb{N}$ such that $B \cap n_0 F(B(0;1/_2))$ is dense in B. Hence $n_0(F(B(0;1/_2)))$ contains an open ball in B. Since $n_0 \neq 0$, $\overline{F(B(0;1/_2))}$ is homeomorphic $\overline{n_0 F(B(0;1/_2)}$ therefore $\overline{F(B(0;1/_2))}$. contains an open ball. Let $B(a;r)$ be the open ball contained in $\overline{F(B(0;1/_2))}$ i.e. $B(a;r) \subseteq \overline{F(B(0;1/_2))}$. Claim that $B(0;r) \subseteq \overline{F(B(0;1))}$. To prove this claim, we'll first show that

$$\overline{F(B(0;1/_2)} \ominus \overline{F(B(0;1/_2))} \subseteq \overline{2F(B(0;1/_2))} \quad \text{where}$$

$A \ominus B = \{a-b \,/ a\in A,\ b\in B\}$. Let $x \in \overline{F(B(0;1/_2))} \ominus \overline{F(B(0;1/_2))}$.

Then $x = y-z$ where $y, z \in \overline{F(B(0;1/_2)}$. Hence there exist sequences $(y_n)_{n\in\mathbb{N}}$ in $F(B(0;1/_2))$ and $(z_n)_{n\in\mathbb{N}}$ in $F(B(0;1/_2))$ such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} z_n = z$. Hence for each $n \in \mathbb{N}$ there exist u_n, $v_n \in B(0;1/_4)$ such that $\frac{y_n}{2} = F(u_n)$ and $\frac{z_n}{2} = F(v_n)$. Since v_n, $u_n \in B(0;1/_4)$ for all $n \in \mathbb{N}$, $-\frac{y_n}{2}, -\frac{z_n}{2} \in F(B(0;1/_4))$ for all $n \in \mathbb{N}$.

Let $u = \frac{y}{2}$ and $v = \frac{z}{2}$ then $u = \lim_{n\to\infty} \frac{y_n}{2} = \lim_{n\to\infty} F(u_n) \in \overline{F(B(0;1/_4))}$.

Similarly, $v \in \overline{F(B(0;1/_4))}$. Now $x = y-z = 2(y/_2 - z/_2) = 2(u-v)$. Must show that $u-v \in \overline{F(B(0;1/_2))}$. Note that $u_n, v_n \in B(0;1/_4)$ for all $n \in \mathbb{N}$ therefore $u_n - v_n \in B(0;1/_2)$. So $F(u_n - v_n) \in F(B(0;1/_2))$. Now, $u = \lim_{n\to\infty} F(u_n)$ and $v = \lim_{n\to\infty} F(v_n)$. Given $\varepsilon > 0$ there exists an $N_\varepsilon \in \mathbb{N}$ such that $\|u-F(u_n)\| < \varepsilon/_2$ for all $n > N_\varepsilon$ and there exists an $N'_\varepsilon \in \mathbb{N}$ such that $\|v-F(v_n)\| < \varepsilon/_2$ for all $n > N'_\varepsilon$. Let $M = \max\{N_\varepsilon, N'_\varepsilon\}$. Then if $n > M$ we get that

$$\|u-v-(F(u_n)-F(v_n))\| \leqslant \|u-F(u_n)\|+\|v-F(v_n)\|$$

$$< \varepsilon/_2 + \varepsilon/_2 = \varepsilon.$$

Therefore $u-v = \lim_{n\to\infty} F(u_n - v_n)$. Hence

$$B(0;r) = B(a;r)-a \subseteq \overline{F(B(0;1/_2))}-a$$

$$\subseteq \overline{F(B(0;1/_2))} \ominus \overline{F(B(0;1/_2))}$$

$$\subseteq 2\overline{F(B(0;1/_2))} = \overline{F(B(0;1))}.$$ Claim that

$B(0;r/_4) \subseteq F(B(0,1))$. Let $y \in B(0;r/_4)$ be arbitrary. Then $y \in \overline{F(B(0;1/_4))}$ therefore there exists an $y_1 \in F(B(0;1/_4))$ such that $\|y-y_1\| < r/_{2^3}$. Hence $y-y_1 \in B(0;r/_8) \subseteq \overline{F(B(0;1/_8))}$ therefore there exists an $y_2 \in F(B(0;1/_8)$ such that $\|y-y_1-y_2\| < r/_{2^4}$ therefore $y-y_1-y_2 \in B(0;r/_{16}) \subseteq \overline{F(B(0;1/_{16}))}$. By induction we get that for each $n \in \mathbb{N}$ there exists an $y_n \in F(B(0;1/_{2^{n+1}}))$ such that

$\| y- \sum_{\alpha=1}^{n} y_\alpha \| < r/2^{n+2}$. Since $y_n \in F(B(0;1/2^{n+1}))$ for all $n \in \mathbb{N}$ we get that for each $n \in N$ there exists an $x_n \in B(0;1/2^{n+1})$ such that $F(x_n) = y_n$. Since $x_n \in B(0;1/2^{n+1})$, $\|x_n\| < 1/2^{n+1}$ for all $n \in \mathbb{N}$. Consider the sequence $(\sum_{\alpha=1}^{n} x_\alpha)_{n\in\mathbb{N}}$. This is a cauchy sequence in V. Since $m < n$ implies.

$$\| \sum_{\alpha=1}^{n} x_\alpha - \sum_{\alpha=1}^{m} x_\alpha \| = \| \sum_{\alpha=m+1}^{n} x_\alpha \| \leqslant \sum_{\alpha=m+1}^{n} \| x_\alpha \| \leqslant \sum_{\alpha=m+1}^{n} 1/2^{\alpha+1} \to 0$$

as $m, n \longrightarrow \infty$. Since V is a Banach space, there exists an $x \in V$ such that $x = \lim_{n\to\infty} \sum_{\alpha=1}^{n} x_\alpha$. Since

$$\| x \| = \| \lim_{n\to\infty} \sum_{\alpha=1}^{n} x_\alpha \| = \lim_{n\to\infty} \| \sum_{\alpha=1}^{n} x_\alpha \| \leqslant \lim_{n\to\infty} \sum_{\alpha=1}^{n} \| x_\alpha \| \leqslant \lim_{n\to\infty} \sum_{\alpha=1}^{n} 1/2^{\alpha+1}$$

$= 1/2 < 1$, $x \in B(0;1)$. Then $F(x) = F(\sum_{\alpha=1}^{\infty} x_\alpha) = F(\lim_{n\to\infty} \sum_{\alpha=1}^{\infty} x_\alpha)$

$= \lim_{n\to\infty} F(\sum_{\alpha=1}^{n} x_\alpha) = \lim_{n\to\infty} \sum_{\alpha=1}^{n} F(x_\alpha) = \sum_{\alpha=1}^{\infty} y_\alpha = y$. Hence $y = F(x) \in F(B(0;1))$. Thus $B(0;r/4) \subseteq F(B(0;1))$ and so we are done.

⋇

<u>Corollary</u> 2.30 Let V, W be LNLS's (RNLS's) which are also Banach space and $T: V \longrightarrow W$ a 1-1, onto, continuous and left linear map. Then T^{-1} is continuous.

<u>Theorem</u> 2.31 (Closed Graph Theorem). Let V,W be LNLS's (RNLS's) which iare also Banach apace and $\varphi : V \longrightarrow W$ a left(right) linear map and assume that $G = \{(x,\varphi(x))/x \in V\}$ is closed in the product topology. Then φ is continuous.

Proof: Give $V \times W$ the $\| \|_2$ left norm. Then this norm gives the product topology. Since G is closed in $V \times W$, G is complete. Let $F: G \longrightarrow V$ be define by $F(v,\varphi(v)) = v$ for all $v \in V$. Then F is 1-1, onto and left linear. Claim that F is continuous. Let $v \in V$. Given $\varepsilon > 0$. Choose $\delta = \varepsilon$ therefore for all $w \in V$

$$\| (v,\varphi(v)-(w,\varphi(w)) \| = \| (v-w, \varphi(v)-\varphi(w)) \|$$

$$= (\|v-w\|^2+\|\varphi(v)-\varphi(w)\|^2)^{1/2} < \delta$$

implies $\|v-w\| = \| F(v,\varphi(v))-F(w,\varphi(w))\| < \delta = \varepsilon$.

Hence F is continuous. Therefore F^{-1} is continuous [Corollary 1.27]. Let $F_1: G \longrightarrow W$ be defined by $F_1(v,\varphi(v)) = \varphi(v)$ for all $v \in V$. By the same proof as above, F_1 is continuous left linear. Hence the map $F_1 \circ F^{-1}$ is continuous. But $F_1 \circ F^{-1} = \varphi$. So φ is continuous. ⋕

Theorem 2.32 (Banach-Sticnhaus) Let V be a LNLS (RNLS) which is also a Banach space, W a LNLS (RNLS) and $(F_\alpha)_{\alpha \in I}$ a family of continuous left (right) linear and $(F_\alpha)_{\alpha \in I}$ a family of continuous left (right) linear map from V to W. Then either there exists a $M > 0$ such that $\| F_\alpha \| \leqslant M$ for all $\alpha \in I$ or $\sup_{\alpha \in I} \{\| F_\alpha(x) \|\} = \infty$ for all $x \in V$ in some dense G_δ set in V (where G_δ is a countable intersection of open set).

Proof: Define $\varphi : V \longrightarrow [0, \infty]$ by $\varphi(x) = \sup_{\alpha \in I} \{\|F(x)\|\}$. Given $n \in \mathbb{N}$, let $V_n = \{ x \in V / \varphi(x) > n \}$. Since the map taking

$x \longmapsto \|F_\alpha(x)\|$ is continuous for all $\alpha \in I$, the map taking $x \longmapsto \|F_\alpha(x)\|$ is lower semicontinuous. Since suppremum of set of lower semicontinuous is lower semicontinuous. By the definition of lower semicontinuous V_n is open for all $n \in \mathbb{N}$.

Case 1. There exists an $n_0 \in \mathbb{N}$ such that V_{n_0} is not dense in V. Hence there exists a nonempty open set U in V such that $U \cap V_{n_0} = \emptyset$. Hence there exists an open ball B in V such that $B \cap V_{n_0} = \emptyset$. Hence there exists a closed ball $\overline{B(x_0;r)}$ such that $\overline{B(x_0;r)} \cap V_{n_0} = \emptyset$. Hence if $\|x\| \leqslant r$, then $x_0 + x \notin V_{n_0}$, so $\varphi(x+x_0) \leqslant n_0$, hence $\|F_\alpha(x+x_0)\| \leqslant n_0$ for all $\alpha \in I$. Since $x = (x_0 + x) - x$ we get that

$$\|F_\alpha(x)\| = \|F_\alpha[(x_0+x)-x_0]\| = \|F_\alpha(x_0+x)-F_\alpha(x_0)\|$$

$$\leqslant \|F_\alpha(x_0+x)\| + \|F_\alpha(x_0)\| \leqslant 2n_0 \text{ for all } \alpha \in I \text{ and for all}$$

$\|x\| \leqslant r$. Let $M = \frac{2n_0}{r}$. Then we get that for all $\alpha \in I$

$$\|F_\alpha\| = \sup\left\{\|F_\alpha(x)\| / \|x\| = 1\right\} = \frac{1}{r} \sup\left\{\|F_\alpha(x)\| / \|x\| = r\right\}$$

$$= \frac{1}{r} \sup\left\{\|F_\alpha(x)\| / \|x\| \leqslant r\right\} \leqslant \frac{2n_0}{r} = M \quad \text{i.e.}$$

for all $\alpha \in I$ $\|F_\alpha\| \leqslant M$. So done.

Case 2. V_n is dense in V for all $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} V_n$ is dense in V by Baire's theorem. Let $x \in \bigcap_{n \in \mathbb{N}} V_n$ then $x \in V_n$ for all $n \in \mathbb{N}$ therefore $\varphi(x) = \infty$ i.e. $\sup_{\alpha \in I}\left\{\|F_\alpha(x)\|\right\} = \infty$. So done. #