CHAPTER I

INTRODUCTION

The Algebra of Quaternions

Let 1, i, j and k denote the elements of the standard basis for $\mathbb{R}^4$ The quaternion product on $\mathbb{R}^4$ is then the $\mathbb{R}$-bilinear product with 1 as its multiplicative identity by the formulae $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$ and $ki = j = -ik$. In this thesis we shall denote the $\mathbb{R}$-algebra of all quaternions by " $\mathbb{H}$ ". See [3].

Each quaternion $q = a_0 \cdot 1 + a_1 \cdot i + a_2 \cdot j + a_3 \cdot k$ ($a_n \in \mathbb{R}$ for all n) is uniquely expressible in the form Req + Puq, where Req = $a_0 \cdot 1 \in \mathbb{R}$ and Puq = $a_1 \cdot i + a_2 \cdot j + a_3 \cdot k \in \mathbb{R}^3$, Req being call the real quaternion part of q and Puq the pure quaternion of q.

The conjugate $\bar{q}$ of a quaternion q is defined to be the quaternion Req - Puq. Hence $\overline{a+b} = \bar{a}+\bar{b}$, $\overline{\lambda a} = \lambda \bar{a}$, $\bar{\bar{a}} = a$ and $\overline{ab} = \bar{b}\bar{a}$ for all a, b $\in \mathbb{H}$ and $\lambda \in \mathbb{R}$. Moreover, $a \in \mathbb{R}$ if and only if $\bar{a} = a$, while Rea = $\frac{1}{2}(a+\bar{a})$ and Pua = $\frac{1}{2}(a-\bar{a})$. See[3].

<u>Definition</u> 1.1 Let $C(\mathbb{H}) = \{a \in \mathbb{H} \,/\, ax = xa \quad \forall x \in \mathbb{H}\}$. Then $C(\mathbb{H})$ is said to be the center of $\mathbb{H}$. The center of $\mathbb{H}$ is $\mathbb{R}$.

<u>Proposition</u> 1.2 Let $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$. Suppose that $ix - \bar{x}i = jx - \bar{x}j = kx - \bar{x}k = 0$. Then $x_1 = x_2 = x_3 = 0$.

<u>Proof</u>: Obvious. #

Proposition 1.3 Let $x = x_0 + x_1 i + x_2 j + x_3 k$, $y = y_0 + y_1 i + y_2 j + y_3 k \in \mathbb{H}$. Suppose $x + \bar{x} - (y + \bar{y}) = 0$. Then $x_0 = y_0$

Proof: Obvious. ⋇

Let $a = a_0 + a_1 i + a_2 j + a_3 k$ the non-negative number $|a| = (a\bar{a})^{1/2} = (\sum_{n=0}^{3} a_n^2)^{1/2}$ is call the absolute value of the quaternion a. If $a \neq 0$, then $|a| \neq 0$ and $\frac{a\bar{a}}{|a|^2} = \frac{\bar{a}a}{|a|^2} = 1$. So we have:

Proposition 1.4 $\mathbb{H}$ is a division ring

Proof. See[3].

Proposition 1.5 $|a.b| = |a||b|$ for all a, b $\in \mathbb{H}$.

Proof. See[3].

Proposition 1.6 $\mathbb{H}$ is complete with respect this absolute value.

Proof. Standard. ⋇

Proposition 1.7 Let $\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n)/x_\alpha \in \mathbb{H} \; \forall \alpha \leqslant n\}$.

Let $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{H}^n$. Then

$$\left(\sum_{\beta=1}^{n} |x_\beta + y_\beta|^p\right)^{1/p} \leqslant \left(\sum_{\beta=1}^{n} |x_\beta|^p\right)^{1/p} + \left(\sum_{\beta=1}^{n} |y_\beta|^p\right)^{1/p}$$ where

$1 \leqslant p < \infty$. (Minkowski's Inequality) and

$$\sum_{\beta=1}^{n} |x_\beta| \, |y_\beta| \leqslant \left(\sum_{\beta=1}^{n} |x_\beta|^p\right)^{1/p} \left(\sum_{\beta=1}^{n} |y_\beta|^q\right)^{1/q}$$ where

$1 < p < \infty$. and $\frac{1}{p} + \frac{1}{q} = 1$ (Holder's Inequalily).

Proof: If p = 1 then Minkowski's Inequalily is trivial.

Suppose $1 < p < \infty$. Then

$$\sum_{\beta=1}^{n} (|x_\beta|+|y_\beta|)^p = \sum_{\beta=1}^{n} (|x_\beta|+|y_\beta|)^{p-1}|x_\beta| + \sum_{\beta=1}^{n} (|x_\beta|+|y_\beta|)^{p-1}|y_\beta|.$$

Since

$$\sum_{\beta=1}^{n} |a_\beta b_\beta| \leqslant (\sum_{\beta=1}^{n} |a_\beta|^p)^{1/p} (\sum_{\beta=1}^{n} |y_\beta|^q)^{1/q} \quad \text{for all}$$

$(a_1,a_2,\ldots,a_n), (b_1,b_2,\ldots b_n) \in \mathbb{R}^n$ [4],

$$\sum_{\beta=1}^{n} (|x_\beta|+|y_\beta|)^p \leqslant (\sum_{\beta=1}^{n} |x_\beta|^p)^{1/p} (\sum_{\beta=1}^{n} |x_\beta|+|y_\beta|)^{(p-1)q})^{1/q} +$$

$$(\sum_{\beta=1}^{n} |y_\beta|^p)^{1/p} (\sum_{\beta=1}^{n} |x_\beta|+|y_\beta|)^{q(p-1)})^{1/q}.$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, $pq - q = p$: we get that

$$\sum_{\beta=1}^{n} (|x_\beta|+|y_\beta|)^p \leqslant (\sum_{\beta=1}^{n} (|x_\beta|+|y_\beta|)^p)^{1/q} (\sum_{\beta=1}^{n} |x_\beta|^p)^{1/p} + (\sum_{\beta=1}^{n} |y_\beta|^p)^{1/p}.$$

Divide both sides by $(\sum_{\beta=1}^{n} (|x_\beta|+|y_\beta|)^p)^{1/q}$ we get that

$$(\sum_{\beta=1}^{n} |x_\beta + y_\beta|^p)^{1/p} \leqslant (\sum_{\beta=1}^{n} |x_\beta|^p)^{1/p} + (\sum_{\beta=1}^{n} |y_\beta|^p)^{1/p}.$$ Clearly

$$\sum_{\beta=1}^{n} |x_\beta|\,|y_\beta| \leqslant (\sum_{\beta=1}^{n} |x_\beta|^p)^{1/p} (\sum_{\beta=1}^{n} |y_\beta|^q)^{1/q}.$$ ⋇

Proposition 1.8. $\mathbb{Q}^4 = \{(x_1,x_2,x_3,x_4)/x_\alpha \in \mathbb{Q} \ \alpha = 1,2,3,4\}$ is dense in $\mathbb{H}$.

<u>Proof:</u> Standard.

⋇

<u>Linear Algebra over $\mathbb{H}$</u>

<u>Definition</u> 1.9 A left vector space V over $\mathbb{H}$ is a set of elements which the operation of addition and scalar multiplication on the left defined. If x and y are in V and α, $\beta \in \mathbb{H}$. Then

1. V is an abelian group under addition
2. $\alpha(x+y) = \alpha x+\alpha y$
3. $\alpha(\beta x) = (\alpha\beta)x$
4. $1.x = x$
5. $(\alpha+\beta)x = \alpha x+\beta x$.

A right vector space V over $\mathbb{H}$ is defined dually. A vector space V over $\mathbb{H}$ is a left vector and right vector space over H such that $\alpha(x\beta) = (\alpha x)\beta$ for all $x \in V$ and for all α, $\beta \in \mathbb{H}$.

<u>Proposition</u> 1.10 Let V be a vector space over $\mathbb{H}$. Then $\alpha x = x\alpha$ for all $x \in V$ and for all $\alpha \in \mathbb{Q}$. Furthermore if V has a topology such that both scalar multiples are continuous then $\alpha x = x\alpha$ for all $\alpha \in \mathbb{R}$.

<u>Proof:</u> Standard.

⋇

<u>Example of vector space over $\mathbb{H}$</u>

(i) $\mathbb{H}^n$. (ii) $S = \{(x_n)_{n\in\mathbb{N}} \text{ in } \mathbb{H}\}$.

(iii) $C = \{(x_n)_{n\in\mathbb{N}} \,/\, x_n \in \mathbb{H} \;\forall n \in \mathbb{N} \text{ and } (x_n)_{n\in\mathbb{N}} \text{ converges}\}$

<u>Definition</u> 1.11 Let V be a left vector space over $\mathbb{H}$ and $A \subseteq V$. Then A is said to be left linear subspace of V if and only if $\alpha x+\beta y \in A$ for all $x, y \in V$ and for all $\alpha, \beta \in \mathbb{H}$.

Right linear subspace are defined dually. If V is vector space over $\mathbb{H}$. Then A is said to be linear subspace of V if and only if A is a left and right linear subspace.

<u>Definition</u> 1.12 Let V be a left vector space over $\mathbb{H}$ and $A \subseteq V$. Then A is said to be left convex if and only if $\alpha x+\beta y \in A$ for all $x, y \in V$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha, \beta \geqslant 0$ and $\alpha+\beta = 1$.

Right convex is defined dually. Hence every left (right) linear subspace is left (right) convex. If V is a vector space over $\mathbb{H}$. Then a subspace V is convex if and only if V is both left and right convex.

<u>Definition</u> 1.13 A subset $(v_\alpha)_{\alpha \in I}$ of a left vector space V is said to be left linearly independent if and only if for any finite

$v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_n}$ $\sum_{m=1}^{n} \beta_m v_{\alpha_m} = 0$ implies $\beta_m = 0$ for all m.

$(v_\alpha)_{\alpha \in I}$ is left linearly dependent if and only if it is not linear independent.

Right linearly independent and right linearly dependent are defined dually .

<u>Definition</u> 1.14 A left (right) linear independent set spanning a left (right) vector space V is called a left (right) basis or

left (right) base of V.

Then every left (right) nonzero vector space has a basis

Definition 1.15. Let V, W be left vector space over $\mathbb{H}$ and $f: V \longrightarrow W$ a map. Then f is said to be a left linear map if and only if $f(\alpha x+\beta y) = \alpha f(x)+\beta f(y)$ for all x, $y \in V$ and for all α, $\beta \in \mathbb{H}$.

If V, W are right vector space over $\mathbb{H}$ then right linear is defined dually.

If V, W are vector space over $\mathbb{H}$ $f: V \longrightarrow W$ is a map then f is said to be a linear map if and only if f is both left and right linear.

Definition 1.16 Let V be a left vector space over $\mathbb{H}$ and $f: V \longrightarrow \mathbb{H}$ a map. Then f is said to be left-conjugate if and only if $f(\alpha x+\beta y) = f(x)\bar{\alpha} + f(y)\bar{\beta}$ for all x, $y \in V$ and for all α, $\beta \in \mathbb{H}$.

If V is a right vector space over $\mathbb{H}$. Then f is said to be right-conjugate if and only if $f(x\alpha+y\beta) = \bar{\alpha}f(x)+\bar{\beta}f(y)$ for all x, $y \in V$ and for all α, $\beta \in \mathbb{H}$.

Topological Prerequisites

Theorem 1.17 (Baire's theorem) Let X be a complete metric space, and assume that X is the union of closed subset of S_n $n \in \mathbb{N}$. Then some S_n contain a non-empty open ball

Proof: See [4].

Corollary 1.18 Let X be a complete metric space and $(M_n)_{n \in \mathbb{N}}$ a sequence of open dense sets. Then the intersection $\bigcap_{n \in \mathbb{N}} M_n$ is not empty.

Proof: See [4].

Corollary 1.19 Let X be a metric space and $(M_n)_{n \in \mathbb{N}}$ a sequence of subset of V such that $V = \bigcup_{n \in \mathbb{N}} M_n$. Then there exists a closed ball B in V and there exists an $n_0 \in \mathbb{N}$ such that $B \cap M_{n_0}$ is dense in B.

Proof: See [4].

Definition 1.20 Let (X,d) be a metric space. Let $A = \{ f: X \longrightarrow \mathbb{H} \}$ is said to be equicontinuous if and only if given $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x, $y \in X$ $d(x,y) < \delta$ implies $|f(x)-f(y)| < \varepsilon$ for all $f \in A$.

Definition 1.21 A metric space X is said to be totally bounded if for any $\varepsilon > 0$, there is a finite covering of X by sets of dimemeter less than ε.

Definition 1.22 Let $(X,\mathcal{T})$ is said to have the BW property if and only if every sequence in X has a convergent subsequence which converges to a point of X.

Theorem 1.23 Let (X,d) be a metric space. Then the following statements are equivalent:

(i) (X,d) has the BW property

(ii) (X,d) is complete and totally bounded

(iii) (X,d) is compact

Proof: See [4].

Theorem 1.24 Let (X,d) be a compact metric space. Let $C(X) = \{f: X \longrightarrow \mathbb{H} \ / \ f \text{ is continuous}\}$. The function d^* defined on $C(X) \times C(X)$ by

$$d^*(f,g) = \sup \ \{|f(x)-g(x)| \ / \ x \in X\}$$ is a metric on $C(X)$

and $(C(X),d^*)$ is complete.

Proof: Let $(f_n)_{n \in \mathbb{N}}$ be a cauchy sequence in $C(X)$ i.e. for each $\varepsilon > 0$ there exists a N_ε such that $d^*(f_n,f_m) < \varepsilon$ for all $m, n > N_\varepsilon$. Hence for each $x \in X$, $(f_n(x))$ is a cauchy sequence of $\mathbb{H}$. Since $\mathbb{H}$ is complete, there exists an $y_x \in \mathbb{H}$ such that

$\lim\limits_{n \to \infty} f_n(x) = y_x$. Define $f: X \longrightarrow \mathbb{H}$ by $f(x) = y_x$ for all $x \in X$.

To show 1) $\lim\limits_{n \to \infty} f_n = f$ with respect to d^*.

2) $f \in C(X)$.

Let $\varepsilon > 0$ there exists a N_0 such that $|f_n(x)-f_m(x)| < \varepsilon/2$ for all $m, n > N_0$ and for all $x \in X$. Fix $n > N_0$ and let $n \to \infty$ in $|f_n(x)-f_m(x)|$ we get that $|f_n(x)-f(x)| \leqslant \varepsilon/2$ for all $n > N_0$ and for all $x \in X$. Hence $\sup \{|f_n(x)-f(x)| \ / x \in X\} < \varepsilon$ for all $n > N_0$. This prove 1). To prove 2) Let $x \in X$ and let $\varepsilon > 0$ be given. By 1) there exists an $n_1 \in \mathbb{N}$ such that $d^*(f_n - f) < \varepsilon/3$ for all $n \geqslant n_1$ therefore $|f_{n_1}(t)-f(t)| < \varepsilon/3$ for all $t \in X$. But $f_{n_1} \in C(X)$, hence it is continuous at x therefore there exists a $\delta > 0$ such that for all $y \in X$ $d(x,y) < \delta$ implies $|f_{n_1}(x)-f_{n_1}(y)| < \varepsilon/3$. If $d(x,y)<\delta$ then

$$|f(x)-f(y)| \leq |f(x)-f_{n_1}(x)| + |f_{n_1}(x)-f_{n_1}(y)| + |f_{n_1}(y)-f(y)|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .$$

\#\#

<u>Definition</u> 1.25 A subset A of a metric space is said to be bounded if diameter A is finite.

<u>Theorem</u> 1.26 (Alzela'-Asoli) A closed subspace of C(X) is compact if and only if it is bounded and equicontinuous.

<u>Proof:</u> Let $F \subseteq C(X)$ be compact therefore F is totally bounded, so F is bounded. Let $\varepsilon > 0$ therefore there exist $F_1, F_2, \ldots, F_n$ such that $\bigcup_{m=1}^{n} F_m = F$ and $\text{diam}(F_m) < \varepsilon/3$ for all $m = 1,2,\ldots,n$. Assume $F_m \neq 0$ for all $m = 1,2,\ldots,n$. For each $m \leqslant n$ choose an element $f_m \in F_m$. Therefore for any $f \in F$, $f \in F_m$ for some m, hence $d^*(f,f_m) < \varepsilon/3$. Since $f_1, f_2, \ldots, f_n$ belong to $F \subseteq C(X)$ they are uniformly continuous on X. For $m = 1,2,\ldots,n$ there exists a $\delta_m > 0$ such that for each $x, y \in X$ $d(x,y) < \delta_m$ implies $|f_m(x)-f_m(y)| < \varepsilon/3$. Choose $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_m\}$ therefore $\delta > 0$. Let $f \in F$ and let $x, y \in X$ such that $d(x,y) < \delta$. Since $f \in F$, there exists an $k \in \{1,2,\ldots,n\}$ such that $d^*(f,f_k) < \varepsilon/3$.

Hence

$$|f(x)-f(y)| \leqslant |f(x)-f_k(x)|+|f_k(x)-f_k(y)|+|f_k(y)-f(y)|$$

$$< \varepsilon/3+ \varepsilon/3+ \varepsilon/3 = \varepsilon \quad \text{so F is equicontinuous.}$$

Conversely, suppose F is bounded and equicontinuous. Must show that F is compact. Since F is closed in a complete metric space, hence F is complete. To show F is totally bounded. F is totally bounded if and only if every sequence in F contains a cauchy sequence[4]. Let $(f_n)_{n\in \mathbb{N}}$ be a sequence in F. Claim that there exists a $K > 0$ such that $|f(x)| \leqslant K$ for all $x \in X$ and for all $f \in F$. Since F is bounded there exists a $K_1 > 0$ such that $d^*(f,g) \leqslant K_1$ for all f, g $\in$ F therefore $|f(x)-g(x)| \leqslant K_1$ for all $x \in X$ and for all f, g $\in$ F. Let $g_0 \in F$ there exists a $K_2 > 0$ such that $|g_0(x)| \leqslant K_2$ for all $x \in X$. Choose $K = K_1+ K_2$. Let $f \in F$, so $|f(x)-g_0(x)| \leqslant K_1$ for all $x \in X$ therefore $|f(x)| \leqslant |f(x)-g_0(x)| + |g_0(x)| \leqslant K_1+K_2 = K$. So we have the claim. Since X is compact, X is totally bounded, hence X separable[4], so there exists a $D = \{x_1,x_2,\dots\}$ such that $\bar{D} = X$. Since $(f_n(x_1))_{n\in \mathbb{N}}$ is a bounded sequence of $\mathbb{H}$ which has the BW property it contains a convergent subsequence, call it $(f_{1n}(x_1))_{n \in \mathbb{N}}$ therefore $(f_{1n})_{n\in \mathbb{N}}$ is a subsequence of $(f_n)_{n\in \mathbb{N}}$. Since $(f_{1n}(x_2))$ is a bounded sequence of $\mathbb{H}$, it contains a convergent subsequence, call it $(f_{2n}(x_2))_{n\in \mathbb{N}}$ therefore $(f_{2n})_{n\in \mathbb{N}}$ is a subsequence of $(f_{1n})_{n\in \mathbb{N}}$. By induction, for each $k \in \mathbb{N}$ we get a sequence $(f_{kn})_{n\in\mathbb{N}}$ which is a subsequence of $(f_{k-1\,n})_{n\in\mathbb{N}}$ (and hence is a subsequence of $(f_n)_{n\in\mathbb{N}}$) such that $(f_{kn}(x_k))_{n\in\mathbb{N}}$ converges to some $P_k \in \mathbb{H}$. Consider $(f_{nn})_{n\in\mathbb{N}}$ which is a subsequence of $(f_n)_{n\in\mathbb{N}}$. For each $x_k \in D$, $(f_{nn}(x_k))_{n\in\mathbb{N}\setminus\{1,2,\dots,k-1\}}$ is a subsequence of the sequence $(f_{kn}(x_k))_{n\in\mathbb{N}}$ therefore $(f_{nn}(x_k))_{n\in\mathbb{N}}$ converges. Hence $(f_{nn}(x_k))_{n\in\mathbb{N}}$ is cauchy. Claim that $(f_{nn})_{n\in\mathbb{N}}$ is cauchy. Let $\varepsilon > 0$ there exists a M_k such that $|f_{nn}(x_k)-f_{mm}(x_k)| < \varepsilon/6$ for all m, $n > M_k$. Since F is equicontinuous, there exists a $\delta > 0$ such that for each x, $y \in X$ and for all $f \in F$ $d(x,y) < \delta$ implies $|f(x)-f(y)| < \varepsilon/6$. Claim that $\{B_d(x_m,\delta)/x_m \in D\}$ is an open cover X. Let $x \in X$ therefore $B_d(x;\delta)\cap D \neq \emptyset$, so there exists an $x_m \in D$ such that $x_m \in B_d(x;\delta)$ hence $x \in B_d(x_m;\delta)$, so $x \in \bigcup_{m\in\mathbb{N}} B_d(x_m,\delta)$ therefore $X \subseteq \bigcup_{m\in\mathbb{N}} B_d(x_m;\delta)$ and thus we have the claim. Since X is compact, there exists an $n_0 \in N$ such that that $\{B_d(x_m;\delta)/\ m = 1,2,\dots,n_0\}$ cover X. Choose $N = \max\{M_1,M_2,\dots,M_{n_0}\}$. Let $m,n > N$ and let $x \in X$ then $x \in B_d(x_\alpha,\delta)$ for some $\alpha \in \{1,2,\dots,n_0\}$.

$$|f_{mm}(x)-f_{nn}(x)| \leq |f_{mm}(x)-f_{mm}(x_\alpha)| + |f_{mm}(x_\alpha)-f_{nn}(x_\alpha)| + |f_{nn}(x_\alpha)-f_{nn}(x)|$$

$$< \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2 . \text{ So } d^*(f_{mm},f_{nn}) < \varepsilon .$$

Hence $(f_n)_{n\in\mathbb{N}}$ contains a cauchy sequence $(f_{nn})_{n\in\mathbb{N}}$. Thus F is totally bounded, hence F is compact.

<u>Definition</u> 1.26 A subset M of a topological space X is said to be relatively compact in X if and only if $\bar{M}$ is compact.

<u>Definition</u> 1.27 Let f be a real (or extended-real) function on a topological space. If $\{x/f(x) > \alpha\}$ is open for every real α, f is said to be lower semicontinuous. If $\{x/f(x) < \alpha\}$ is open for every real , f is said to be upper semicontinuous.