CHAPTER IV

VILLE THEOREM ON DISCRETE GAME

Among the several procedures which lead to the existence of the value of a discrete two-person zero sum game - player I is based on a theorem of Ville which deals with matrices a_{ik}, $i = 1,2,\ldots m$, $k = 1,2,\ldots,n$ and it states that if to each $q_k \geqslant 0$, $\sum_{k=1}^{n} q_k = 1$ there exists corresponding $p_i \geqslant 0$, $\sum_{i=1}^{n} p_i = 1$ such that the bilinear form $\sum_{k=1}^{n} \sum_{i=1}^{m} a_{ik} p_i q_k$ is non-negative, then there exists fixed $p_i \geqslant 0$, $\sum_{i=1}^{m} p_i = 1$ such that for all $q_k \geqslant 0$, $\sum_{k=1}^{n} q_k = 1$ the bilinear form is non-negative.

In this chapter we shall prove this theorem by using application of the Hahn Banach theorem suggested in chapter III, and then extend this theorem to a discrete, more than two-person, zero sum game.

4.1 Two person zero-sum game

4.1.1 <u>Theorem</u> : Given matrices a_{ik}, $i = 1,2,\ldots, m$, $k = 1,2,\ldots n$.

If to each $q_k \geqslant 0$, $\sum_{k=1}^{n} q_k = 1$ there exists corresponding $p_i \geqslant 0$, $\sum_{i=1}^{m} p_i = 1$ such that the bilinear form $\sum_k \sum_i a_{ik} p_i q_k$ is non-negative, then there exists fixed $p_i \geqslant 0$, $\sum_{i=1}^{m} p_i = 1$ such that for all $q_k \geqslant 0$, $\sum_{k=1}^{n} q_k = 1$, the bilinear form is non-negative.

Proof : Let $P = \left\{ (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m : p_i \geqslant 0 \text{ for } i = 1,2,\ldots,m \text{ and } \sum_{i=1}^{m} p_i = 1 \right\}$

and $Q = \left\{ (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n : q_k \geqslant 0 \text{ for } k = 1,2,\ldots,n \text{ and } \sum_{k=1}^{n} q_k = 1 \right\}$.

Let A be a mapping from P into $\mathbb{R}^{n*}$ (see note) defined by

$$A(p_1, p_2, \ldots, p_m) = \left(\sum_{i=1}^{m} a_{i1} p_i, \sum_{i=1}^{m} a_{i2} p_i, \ldots, \sum_{i=1}^{m} a_{in} p_i \right).$$

We can see that A is a linear mapping on P which has finite dimension. So A is continuous on P.

Let A(P) be the image of P under transformation A. Since P is closed and bounded in $\mathbb{R}^m$, P is compact. By continuity of A, we have A(P) is compact in $\mathbb{R}^{n*}$. Hence A(P) is weak*-compact in $\mathbb{R}^{n*}$. Since $A(P)$ is convex, by 3.3.2, $A(P)$ is regularly convex.

Let $X = \{(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n : x_i \geqslant 0 \text{ for } i = 1,2,\ldots,n\}$.

Then X is closed convex cone. From the hypothesis we have : for each $q \in Q$ there exists $p \in P$ such that the bilinear form

$$(1) \qquad (Ap,q) = \sum_{k=1}^{n}\sum_{i=1}^{m} a_{ik}p_iq_k \geqslant 0.$$

For each $x \in X$ and $x \neq 0$, ie. $x = (x_1,x_2,\ldots,x_n)$, $x_i \neq 0$ for some $i = 1,2,\ldots,n$, let $a = x_1+x_2+\ldots+x_n$, $q_1 = \frac{x_1}{a}$, $q_2 = \frac{x_2}{a}$, ..., $q_n = \frac{x_n}{a}$. Then $(q_1,q_2,\ldots,q_n) \in Q$ and $x = (aq_1,aq_2,\ldots aq_n)$

$$= aq$$

From (1) we have

$$(Ap,aq) \geqslant 0 \text{ , ie. } (Ap,x) \geqslant 0. \text{ For } x = 0, (Ap,x) = 0.$$

By 3.3.7, there exists $p_0 \in P$ such that $(Ap_0,x) \geqslant 0$ for all $x \in X$. That is, there exists $p_0 \in P$ such that $(Ap_0,aq) \geqslant 0$ for all $q \in Q$.

Thus $(Ap_0,q) \geqslant 0$ for all $q \in Q$. The proof is complete.

<u>Note</u> : The dual of $\mathbb{R}^n$ ($\mathbb{R}^{n*}$).

$\mathbb{R}^n$ is a vector space which has n dimension. Let $B = \{u_1,u_2,\ldots, u_n\}$ be a basis for $\mathbb{R}^n$. Let $f_1,f_2,\ldots,f_n$

be real-valued functions such that for each $i = 1, 2, \ldots, n$

$u = c_1u_1 + c_2u_2 + \ldots + c_uu_n$

$$f_i(u) = c_i .$$

These functions f_i are linear and therefore belong to $\mathbb{R}^{n*}$.

They are specified by their values at the basis elements :

$$f_i(u_j) = \delta_{ij} \qquad i,j = 1,2,\ldots,n \qquad (*)$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Let us show that $B^* = \{f_1, f_2, \ldots, f_n\}$ is a basis for $\mathbb{R}^{n*}$.

Suppose $b_1f_1 + b_2f_2 + \ldots + b_nf_n = 0$ (zero function). Then, for every $u \in \mathbb{R}^n$

$$b_1f_1(u) + b_2f_2(u) + \ldots + b_nf_n(u) = 0 .$$

Taking $u = u_i$ and applying formula (*), $b_i = 0$ for each $i = 1,2,\ldots,n$. Thus B^* is linearly independent set. To show that B^* spans $\mathbb{R}^{n*}$, given $f \in \mathbb{R}^{n*}$, let $f(u_i) = a_i$. If $u = c_1u_1 + c_2u_2 + \ldots + c_nu_n$, then since f is linear

$$f(u) = f(\sum_{i=1}^{n} c_iu_i) = \sum_{i=1}^{n} c_if(u_i) = \sum_{i=1}^{n} a_ic_i = \sum_{i=1}^{n} c_if_i(u).$$

Since this is true for every $u \in \mathbb{R}^n$,

$$f = a_1f_1 + a_2f_2 + \ldots + a_nf_n ,$$

which show that B^* spans $\mathbb{R}^{n*}$. Thus $\mathbb{R}^{n*}$ has n dimension.

If $\{u_1, u_2, \ldots, u_n\}$ is a basis for $\mathbb{R}^n$ and $\{f_1, f_2, \ldots, f_n\}$ is a basis for $\mathbb{R}^{n*}$, then the linear function $\emptyset$ such that $\emptyset(u_i) = f_i$ for $i = 1,2,\ldots,n$ is an isomorphism for $\mathbb{R}^n$ onto $\mathbb{R}^{n*}$.

4.2 Three-person zero sum game

In this section, discrete 3-person zero-sum game is a game that amounts to a choice, by player I, player II and player III respectively, of elements i,j and k from preassigned sets

$$I = \{1,2,\ldots, m\}, \quad J = \{1,2,\ldots, n\} \quad \text{and}$$

$K = \{1,2,\ldots,l\}$. At the end of a game resulting in the choice (i,j,k), player I gains an amount F(i,j,k), F being a real-valued function on $I \times J \times K$. That is, F is a payoff function for player I. At each stage, the three players are playing independently and the sum of each player's gain or loss is zero.

Next we shall state and prove Ville Theorem for this game.

4.3.1 <u>Theorem</u> : Given $I = \{1,2,\ldots,m\}$, $J = \{1,2,\ldots,n\}$, $K = \{1,2,\ldots,l\}$,

$$F : I \times J \times K \longrightarrow \mathbb{R} \quad \text{define by}$$

$$F(i,j,k) = a_{ijk} \quad \text{where } i \in I,\ j \in J,\ k \in K.$$

If to each $q_j \geqslant 0$, $r_k \geqslant 0$, $\sum_{j=1}^{n} q_j = 1$, $\sum_{k=1}^{l} r_k = 1$, there exists corresponding $p_i \geqslant 0$, $\sum_{i=1}^{m} p_i = 1$ such that the trilinear form $\sum_k \sum_j \sum_i a_{ijk} p_i q_j r_k \geqslant 0$, then there exists fixed $p_i \geqslant 0$, $\sum_{i=1}^{m} p_i = 1$ such that for all $q_j \geqslant 0$, $\sum_{j=1}^{n} q_j = 1$, $r_k \geqslant 0$, $\sum_{k=1}^{l} r_k = 1$, the trilinear form is non-negative.

<u>Proof</u> : Let

$$P = \left\{ (p_1, p_2, \ldots, p_m) \in \mathbb{R}^m : p_i \geqslant 0 \text{ for } i = 1,2,\ldots m, \text{ and } \sum_{i=1}^{m} p_i = 1 \right\},$$

$$Q = \left\{ (q_1, p_2, \ldots, q_n) \in \mathbb{R}^n : q_j \geqslant 0 \text{ for } j = 1,2,\ldots,n \text{ and } \sum_{j=1}^{n} q_j = 1 \right\},$$

$$R = \left\{ (r_1, r_2, \ldots, r_l) \in \mathbb{R}^l : r_k \geqslant 0 \text{ for } k = 1,2,\ldots,l \text{ and } \sum_{k=1}^{l} r_k = 1 \right\}.$$

Let A be a mapping from P into $(\mathbb{R}^n \times \mathbb{R}^l)^*$ defined by

$$A(p_1, p_2, \ldots, p_m) = \Big(\sum_{i=1}^{m} \big(\sum_{k=1}^{l} a_{i1k} r_k\big) p_i, \sum_{i=1}^{m} \big(\sum_{k=1}^{l} a_{i2k} r_k\big) p_i, \ldots \sum_{i=1}^{m} \big(\sum_{k=1}^{l} a_{ink} r_k\big) p_i, \sum_{i=1}^{m} \big(\sum_{j=1}^{n} a_{ij1} q_j\big) p_i, \ldots, \sum_{i=1}^{m} \big(\sum_{j=1}^{n} a_{ijl} q_j\big) p_i \Big),$$

We can see that A is a linear mapping on P which has finite dimension. So A is continuous on P.

Let A(P) be the image of P under transformation A. Since P is closed and bounded in $\mathbb{R}^m$, P is compact in $\mathbb{R}^m$. By continuity of A, A(P) is compact in $(\mathbb{R}^n \times \mathbb{R}^l)^*$. Hence A(P) is weak*-compact in $(\mathbb{R}^n \times \mathbb{R}^l)^*$. Since A(P) is convex, by 3.3.2, A(P) is regularly convex.

Let $X = \Big\{(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+l}) : x_i \geqslant 0$

$\quad$ for $i = 1,2,\ldots,n,n+1,\ldots n+l$, and $\displaystyle\sum_{i=1}^{n} x_i = \sum_{i=n+1}^{n+l} x_i \Big\}$.

Then X is closed convex cone. From the hypothesis we have : for each $s = (q,r)$ where $q \in Q$, $r \in R$, there exists $p \in P$ such that the bilinear form

$$(2) \qquad (Ap,s) = \sum_k \sum_j \sum_i a_{ijk} p_i q_j r_k \geqslant 0.$$

For each $x \in X$ and $x \neq 0$, ie. $x = (x_1, x_2, \ldots, x_{n+1}, \ldots x_{n+l})$ and $x_i \neq 0$ for some $i = 1,2,\ldots,n,n+1,\ldots,n+l$, let

$a = x_1 + x_2 + \ldots + x_n = x_{n+1} + x_{n+2} + \ldots + x_{n+l}$, and

$$q_1 = \frac{x_1}{a}, \; q_2 = \frac{x_2}{a}, \ldots, \; q_n = \frac{x_n}{a},$$

$$r_1 = \frac{x_{n+1}}{a}, \; r_2 = \frac{x_{n+2}}{a}, \ldots, \; r_l = \frac{x_{n+l}}{a}.$$

Then $q = (q_1, q_2, \ldots q_n) \in Q$, $r = (r_1, r_2, \ldots, r_l) \in R$,

and $x = (aq_1, aq_2, \ldots, aq_n, ar_1, ar_2, \ldots, ar_{n+l})$

$\quad = as \quad$ where $s = (q,r)$

From (2) we have

$$(Ap, as) \geqslant 0 , \quad \text{i.e}, (Ap, x) \geqslant 0.$$

For $x = 0$, we have $(Ap, x) = 0$.

By 3.3.7, there exists $p_0 \geqslant P$ such that $(Ap_0, x) \geqslant 0$ for all $x \in X$. That is, there exists $p_0 \in P$ such that $(Ap_0, as) \geqslant 0$ for all $s \in Q \times R$. Hence $(Ap_0, s) \geqslant 0$ for all $s \in Q \times R$, i.e,

$\sum_k \sum_j \sum_i a_{ijk} p_i q_j r_k \geqslant 0$ for all $q \in Q$, $r \in R$. The proof is complete.

4.3 n-person zero-sum game

The discrete, n-person zero-sum game, in this section, is similar to discrete, 3-person zero-sum game in section 4.2. That is, it amounts to a choice by player 1, player 2,....... player n respectively, of elements $i_1, i_2, \ldots, i_n$ from preassigned sets

$$I_1 = \{1, 2, \ldots, m_1\} ,$$

$$I_2 = \{1, 2, \ldots, m_2\} ,$$

$$\vdots$$

$$I_n = \{1, 2, \ldots, m_n\} .$$

At the end of a game resulting in the choice $(i_1, i_2, \ldots, i_n)$, player 1 gains an amount $F(i_1, i_2, \ldots, i_n)$, F being a real-valued function on $I_1 \times I_2 \times \ldots \times I_n$. That is, F is a pay off function for player 1. At each stage, all players are playing independently and the sum of each player's gain or loss is zero. The Ville theorem for this game is as follow.

4.3.1 Theorem :

Given $I_1 = \{1,2,\ldots,m_1\}$, $I_2 = \{1,2,\ldots,m_2\}, \ldots\ldots$

$I_n = \{1,2,\ldots, m_n\}$.

Let $F : I_1 \times I_2 \times \ldots, \times I_n \longrightarrow \mathbb{R}$ define by

$F(i_1,i_2,\ldots,i_n) = a_{i_1 i_2,\ldots,i_n}$ where $i_1 \in I_1, i_2 \in I_2, \ldots, i_n \in I_n$.

Let $P^{(1)} = \{(p_1^{(1)}, p_2^{(1)}, \ldots, p_{m_1}^{(1)}) \in \mathbb{R}^{m_1} : p_{i_1}^{(1)} \geqslant 0$ for $i_1 = 1,2,\ldots,m_1$

and $\sum_{i_1=1}^{m_1} p_{i_1}^{(1)} = 1\}$

$P^{(2)} = \{(p_1^{(2)}, p_2^{(2)}, \ldots, p_{m_2}^{(2)}) \in \mathbb{R}^{m_2} : p_{i_2}^{(2)} \geqslant 0$ for $i_2 = 1, 2, \ldots m_2$

and $\sum_{i_2=1}^{m_2} p_{i_2}^{(2)} = 1\}$

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$P^{(n)} = \{(p_1^{(n)}, p_2^{(n)}, \ldots, p_{m_n}^{(n)}) \in \mathbb{R}^{m_n} : p_{i_n}^{(n)} \geqslant 0$ for $i_n = 1,2,\ldots,m_n$

and $\sum_{i_n=1}^{m_n} p_{i_n}^{(n)} = 1\}$.

If to each $p^{(2)} \in P^{(2)}$, $p^{(3)} \in P^{(3)}, \ldots, p^{(n)} \in P^{(n)}$ there exists corresponding $p^{(1)} \in P^{(1)}$ such that the multilinear form

$\sum_{i_1,\ldots,i_n} a_{i_1 i_2 \ldots i_n} p^{(1)}_{i_1} p^{(2)}_{i_2} \ldots p^{(n)}_{i_n} \geqslant 0$, then there exists fixed $p^{(1)}_0 \in P^{(1)}$ such that for all $p^{(2)} \in P^{(2)}, p^{(3)} \in P^{(3)}, \ldots, p^{(n)} \in P^{(n)}$,

$$\sum_{i_1,\ldots,i_n} a_{i_1 i_2, \ldots i_n} p^{(1)}_{i_1} p^{(2)}_{i_2} \ldots p^{(n)}_{i_n} \geqslant 0 .$$

<u>Proof</u> : Let A be a mapping from $P^{(1)}$ into $(\mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \times \ldots \times \mathbb{R}^{m_n})^*$ defined by

$$Ap^{(1)} = y \quad \text{where } p^{(1)} \in P^{(1)},$$

$$y = (y^{(2)}_1, y^{(2)}_2, \ldots y^{(2)}_{m_2}, y^{(3)}_1, y^{(3)}_2, \ldots, y^{(3)}_{m_3}, \ldots, y^{(n)}_1, y^{(n)}_2 \ldots, y^{(n)}_{m_n})$$

and $$y^{(2)}_{i_2} = \sum_{i_1=1}^{m_1} \Big(\sum_{i_3,\ldots,i_n} a_{i_1 i_2, \ldots i_n} p^{(3)}_{i_3} p^{(4)}_{i_4} \ldots p^{(n)}_{i_n} \Big) p^{(1)}_{i_1}, i_2 = 1,2,\ldots m_2$$

$$y^{(3)}_{i_3} = \sum_{i_1=1}^{m_1} \Big(\sum_{i_2, i_4 \ldots, i_n} a_{i_1 i_2, \ldots i_n} p^{(2)}_{i_2} p^{(4)}_{i_4} \ldots p^{(n)}_{i_n} \Big) p^{(1)}_{i_1}, i_3 = 1,2,\ldots m_3$$

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$$y^{(n)}_{i_n} = \sum_{i_1=1}^{m_1} \Big(\sum_{i_2 i_3 \ldots i_{n-1}} a_{i_1 i_2 i_3 \ldots i_n} p^{(2)}_{i_2} p^{(3)}_{i_3} \ldots p^{(n-1)}_{i_{n-1}} \Big) p^{(1)}_{i_1},$$

$$i_n = 1,2,\ldots m_n$$

Then A is a linear mapping on $P^{(1)}$ which has finite dimension. Thus A is continuous on $P^{(1)}$. Let A(P) be the image of P under transformation A. Since P is closed and bounded in $\mathbb{R}^{m_1}$, P is compact in $\mathbb{R}^{m_1}$. By continuity of A, A(P) is compact in

$(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \cdots \times \mathbb{R}^{m_n})^*$. Hence A(P) is weak*-compact in $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \cdots \times \mathbb{R}^{m_n})^*$.

Since A(P) is convex, by 3.3.2, A(P) is regularly convex.

Let $X = \Big\{(x_1^{(2)}, x_2^{(2)}, \ldots x_{m_2}^{(2)}, x_1^{(3)}, x_2^{(3)} \ldots, x_{m_3}^{(3)}, \ldots\ldots\ldots x_1^{(n)}, x_2^{(n)}, \ldots x_{m_n}^{(n)}) : x_{i_2}^{(2)} \geqslant 0$ for $i_2 = 1, 2, \ldots, m_2$, $x_{i_3}^{(3)} \geqslant 0$ for $i_3 = 1, 2, \ldots, m_3, \ldots, x_{i_n}^{(n)} \geqslant 0$, $i_n = 1, 2, \ldots, m_n$, and $\sum_{i_1=1}^{m_2} x_{i_2} = \sum_{i_2=1}^{m_2} x_{i_3} = \ldots = \sum_{i_n=1}^{m_n} x_{i_n}\Big\}$. Then X is closed convex cone. From the hypothesis we have : for each $s = (p^{(2)}, p^{(3)}, \ldots, p^{(n)})$ where $p^{(2)} \in P^{(2)}$, $p^{(3)} \in P^{(3)}, \ldots, p^{(n)} \in P^{(n)}$, there exists $p^{(1)} \in P^{(1)}$ such that the bilinear form

$$(3) \qquad (Ap^{(1)}, s) = \sum_{i_1, i_2 \ldots i_n} a_{i_1 i_2, \ldots, i_n} p_{i_1}^{(1)} p_{i_2}^{(2)} \ldots, p_{i_n}^{(n)} \geqslant 0.$$

For each $x \in X$ and $x \neq 0$, let

$$a = \sum_{i_2=1}^{m_2} x_{i_2} = \sum_{i_3=1}^{m_3} x_{i_3} = \ldots = \sum_{i_n=1}^{m_n} x_{i_n},$$

$$p_1^{(2)} = \frac{x_1^{(2)}}{a}, \; p_2^{(2)} = \frac{x_2^{(2)}}{a}, \ldots, p_{m_2}^{(2)} = \frac{x_{m_2}^{(2)}}{a},$$

$$\vdots$$

$$p_1^{(n)} = \frac{x_1^{(n)}}{a}, \; p_2^{(n)} = \frac{x_2^{(n)}}{a}, \ldots, p_{m_n} = \frac{x_{m_n}^{(n)}}{a}.$$

Then $p^{(2)} = (p_1^{(2)}, p_2^{(2)}, \ldots, p_{m_2}^{(2)}) \in \mathbb{R}^{m_2}$,

$$p^{(3)} = (p_1^{(3)}, p_2^{(3)}, \ldots, p_{m_3}^{(3)}) \in \mathbb{R}^{m_3},$$

$$\vdots$$

$$p^{(n)} = (p_1^{(n)}, p_2^{(n)}, \ldots, p_{m_n}^{(n)}) \in \mathbb{R}^{m_n}, \text{ and}$$

$$x = (ap_1^{(2)}, ap_2^{(2)}, \ldots, ap_{m_2}^{(2)}, ap_1^{(3)} \ldots, ap_{m_3}^{(3)}, \ldots, ap_1^{(n)}, \ldots, ap_{m_n}^{(n)})$$

$= as$ where $s = (p^{(2)}, p^{(3)}, \ldots, p^{(n)})$.

From (3) we have

$$(Ap^{(1)}, as) \geqslant 0 \text{ , i.e, } (Ap^{(1)}, x) \geq 0 \text{ .}$$

For $x = 0$, we have $(Ap^{(1)}, x) = 0$.

By 3.3.7, there exists $p_0^{(1)} \in P^{(1)}$ such that $(Ap_0^{(1)}, x) \geqslant 0$ for all $x \in X$. That is, there exists $p_0^{(1)} \in P^{(1)}$ such that $(Ap_0^{(1)}, as) \geqslant 0$ for all $s \in \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \times \ldots \times \mathbb{R}^{m_n}$. Hence $(Ap_0^{(1)}, s) \geqslant 0$ for all $s \in \mathbb{R}^{m_2} \times \mathbb{R}^{m_3} \times \ldots \times \mathbb{R}^{m_n}$,

i.e., $\sum_{i_1, i_2, \ldots i_n} a_{i_1, i_2, \ldots, i_n} p_{i_1}^{(1)} p_{i_2}^{(2)} \ldots p_{i_n}^{(n)} \geqslant 0$ for all

$p_{i_2}^{(2)} \geqslant 0$ for $i_1 = 1, 2, \ldots m_2$ and $\sum_{i_2=1}^{m_2} p_{i_2} = 1, \ldots, p_{i_n}^{(n)} \geqslant 0, \ldots,$

for $i_n = 1, 2, \ldots, m_n$ and $\sum_{i_n=1}^{m_n} p_{i_n} = 1$. The proof is complete.