CHAPTER II

PRELIMINARIES

In this chapter, we will recall some definitions and theorems from topology, real and functional analysis.

The materials of this chapter are drawn from references [1], [2], [4], [5], [7], [8], [10], [12].

2.1 Topological Vector Spaces

2.1.1 Definition. Let K be a scalar field with elements α, β, ..., with zero element 0 and identity element 1.

A vector space over K (or linear space over K) is a set X with element x, y, z, ..., which has the following properties :

(i) For every two elements $x, y \in X$ a sum $x+y$ is defined in X ; under this addition, X is an abelian group, i.e. for all $x, y, z \in X$ we have

(a) $x+y = y+x$

(b) $x+(y+z) = (x+y)+z$

(c) there exists $0 \in X$ with $x+0 = x$ for all $x \in X$

(d) there exists for each $x \in X$ an $x' \in X$ with $x+x' = 0$

(ii) For every $\alpha \in K$ and every $x \in X$ the product $\alpha x = x\alpha$ of α with x is defined as an element of X, and all $x, y \in X$, α, $\beta \in K$ we have

(e) $x(\alpha + \beta) = x\alpha + x\beta$

(f) $(x+y)\alpha = x\alpha + x\beta$

(g) $x(\alpha\beta) = (x\alpha)\beta$

(h) $x.1 = x$.

If K is the field $\mathbb{R}$ of real numbers, then X is called a <u>real vector space</u>.

A subset F of elements of a vector space X is a vector space provided that whenever it contains x and y it also contains $\alpha x + \beta y$ for arbitrary α, β in K. F is called a <u>linear subspace</u> (or simply <u>subspace</u>) of E.

2.1.2 <u>Definition</u>. Let X be a nonempty set and let $\mathcal{T}$ be a family of subsets of X having the properties :

(i) The empty set $\emptyset$ and X itself belong to $\mathcal{T}$

(ii) The intersection of any finite collection of sets in $\mathcal{T}$ is a set in $\mathcal{T}$

(iii) The union of any collection of sets in $\mathcal{T}$ is in $\mathcal{T}$.

The collection $\mathcal{T}$ is said to be a <u>topology</u> for X. Also, X together with $\mathcal{T}$ is said to be a <u>topological space</u>, which we denote by $(X, \mathcal{T})$. The members of $\mathcal{T}$ are called the <u>open sets</u> for this topology.

A set $S \subset E$ is <u>closed</u> if its complement is open.

A <u>neighborhood</u> of a point $p \in X$ is any open set that contains p.

A collection $\mathcal{T}' \subset \mathcal{T}$ is a <u>base</u> for $\mathcal{T}$ if every member of $\mathcal{T}$ (that is, every open set) is a union of members of $\mathcal{T}'$.

A collection γ of neighborhoods of a point $p \in X$ is a <u>local base at p</u> if every neighborhood of p contains a member of γ.

2.1.3 <u>Notation</u>. Let X be a vector space over K, $A \subset X$, $B \subset X$, $x \in X$ and $\lambda \in K$.

$$x + A = \{x + a : a \in A\}$$

$$x - A = \{x - a : a \in A\}$$

$$A + B = \{a + b : a \in A,\ b \in B\} = \bigcup_{a \in A} (a+B) = \bigcup_{b \in B} (A+b)$$

$$\lambda A = \{\lambda a : a \in A\}.$$

In particular (taking $\lambda = -1$) $-A$ denotes the set of all additive inverse of members of A.

2.1.4 <u>Definition</u>. Suppose τ is a topology on a vector space X such that

(i) every point of X is a closed set

(ii) the vector space operation are continuous with respect to τ, i.e., the mapping $(x,y) \longmapsto x + y$ of cartesian product $X \times X$ into X and $(\alpha, x) \longmapsto \alpha x$ of $K \times X$ into X are continuous.

Under these conditions, τ is said to be a <u>vector topology</u> on X and X is a <u>topological vector space</u>.

2.1.5 <u>Remark</u>.

(i) Every topological vector space is a Hausdorff space.

(ii) Every Banach space is a topological vector space.

2.1.6 Lemma. If W is a neighborhood of O in X, then there is a neighborhood U of O which is symmetric (in the sense that $U = -U$) and which satisfies $U + U \subset W$.

Proof : Note that the mapping $(x,y) \longmapsto x+y$ is continuous at $(0,0)$. Since W is a neighborhood of $O = O + O$, we can find neighborhood V_1, V_2 of O such that $V_1 + V_2 \subset W$

$$\text{Take} \quad U = V_1 \cap V_2 \cap (-V_1) \cap (-V_2)$$

Since $O \in V_1$, $O \in (-V_1)$ and since $O \in V_2$, $O \in (-V_2)$. So U is a neighborhood of O and $U \subset V_1$, $U \subset V_2$. Hence $U + U \subset V_1 + V_2 \subset W$.

Note that if we apply this lemma twice we can find a symmetric neighborhood U of O such that $U+U+U+U \subset W$.

2.1.7 Theorem. Suppose K and C are subsets of a topological vector space X, K is compact, C is closed, and $K \cap C = \emptyset$. Then O has a neighborhood V such that $(K+V) \cap (C+V) = \emptyset$.

Proof : If $K = \emptyset$, then $K+V = \emptyset$ and the conclusion of the theorem is obvious. We therefore assume that $K \neq \emptyset$ and consider a point $x \in K$. Since $K \cap C = \emptyset$, $x \notin C$, i.e. $x \in C^c$ (complement of C). Since C is closed, C^c is open. Therefore, there exists a neighborhood W_x of O such that $x + W_x \subset C^c$. By 2.1.6, O has a neighborhood V_x such that

$$x + V_x + V_x + V_x + V_x \subset x + W_x \subset C^c ,$$

$$x + V_x + V_x + V_x \subset x + W_x \subset C^c ,$$

i.e. $\quad x + V_x + V_x + V_x \cap C = \emptyset .$

The symmetric property of V_x shows that

(1) $$(x + V_x + V_x) \cap (C + V_x) = \emptyset .$$

Since K is compact, there are finitely many points $x_1, x_2, \ldots, x_n$ in K such that

$$K \subset \bigcup_{i=1}^{n} (x_i + V_{x_i}) .$$

Put $V = \bigcap_{i=1}^{n} V_{x_i}$. Then

$$K + V \subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + V)$$

$$\subset \bigcup_{i=1}^{n} (x_i + V_{x_i} + V_{x_i}) ,$$

and no term in this last union intersects C+V, by (1). This completes the proof.

2.1.8 **Definition.** A set $C \subset X$ is said to be <u>convex</u> if

$$tC + (1-t)C \subset C \qquad \text{for} \quad 0 \leq t \leq 1.$$

2.1.9 **Remark.** Every convex set in $\mathbb{R}$ is an interval.

2.1.10 **Definition.** A set $B \subset X$ is said to be <u>balanced</u> if $\alpha B \subset B$ for all $\alpha \in$ scalar field K with $|\alpha| \leq 1$

2.1.11 **Remark.** If E is balanced, then $sE \subset tE$ for $s < t, s > 0, t > 0$

2.1.12 **Theorem.** If B is a balanced subset of a topological vector space X and $0 \in B^0$ (interior of B) then B^0 is balanced.

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Proof : If $0 < |\alpha| \leq 1$, then $\alpha B^{o} = (\alpha B)^{o}$, since $x \longmapsto \alpha x$ is a homeormorphism. Hence $\alpha B^{o} \subseteq \alpha B \subseteq B$, since B is balanced. But αB^{o} is open. So $\alpha B^{o} \subset B^{o}$. If B^{o} contains the origin, then $\alpha B^{o} \subset B^{o}$ even $\alpha = 0$. Thus B^{o} is balanced.

2.1.13 Theorem. In a topological vector space X,

(i) every neighborhood of 0 contains a balanced neighborhood of 0 and

(ii) every convex neighborhood of 0 contains a balanced convex neighborhood of 0.

Proof : (i) Let U be a neighborhood of 0 in X. Since the mapping $(\alpha, x) \longmapsto \alpha x$ is continuous, there is a $\delta > 0$ and there is a neighborhood V of 0 in X such that $\alpha V \subset U$ whenever $|\alpha| < \delta$

$$\text{Take} \quad W = \bigcup_{|\alpha|<\delta} \alpha V.$$

Since for any β with $|\beta| \leq 1$ and for any $w \in W$, $\beta w = \beta\alpha v$ for some α, v such that $|\alpha| < \delta$ and $v \in V$. Since $|\beta\alpha| < \delta$, $\beta w \in W$. So we have $\beta W \subset W$. Thus W is balanced and $W \subset U$.

(ii) Let U be a convex neighborhood of 0 in X. Take $A = \bigcap_{|\alpha|=1} \alpha U$. Choose W as in part (i). Since W is balanced, $\alpha^{-1} W = W$ when $|\alpha| = 1$; hence $W \subset \alpha U$. Thus $W \subset A$ which implies that the interior A^{o} of A is a neighborhood of 0. Clearly $A^{o} \subseteq U$. Being an intersection of convex set A is convex; hence so is A^{o}. To prove that A^{o} is a neighborhood with desired properties, we have to show that A^{o} is balanced; for this its suffices to prove that A is balance. Choose γ and β so that $0 \leq \gamma \leq 1$, $|\beta| = 1$. Then

$$\gamma\beta A = \bigcap_{|\alpha|=1} \gamma\beta\alpha U \subset \bigcap_{|\beta\alpha|=1} \beta\alpha U = A$$

This completesthe proof.

2.1.14 <u>Theorem.</u> Suppose V is a neighborhood of 0 in a topological vector space X. If $0 < r_1 < r_2 < \cdots$ and $r_n \longrightarrow \infty$ as $n \longrightarrow \infty$, then $X = \bigcup_{n=1}^{\infty} r_n V$

<u>Proof</u> : Fix $x \in X$. By continuity of $\alpha \longmapsto \alpha x$ at 0, we can find $\delta > 0$ such that $\alpha x \in V$ whenever $|\alpha| < \delta$. Since $\frac{1}{r_n} \longrightarrow 0$ as $n \longrightarrow \infty$, there exists N such that $\frac{1}{r_N} < \delta$. Hence $\frac{1}{r_N} x \in V$, i.e., $x \in r_n V$. Thus $X = \bigcup_{n=1}^{\infty} r_n V$.

2.1.15 <u>Definition.</u> A subset A of a topological vector space X is said to be <u>absorbing</u> if for any $x \in X$, then exists a positive number $t = t(x)$ such that $x \in tA$.

2.1.16 <u>Remark.</u>

(i) Every neighborhood of 0 in a topological vector space is absorbing

(ii) Every absorbing set contains 0

<u>Proof</u> : (i) Let V be a neighborhood of 0. By 2.1.4, $X = \bigcup_{n=1}^{\infty} r_n V$ where $0 < r_1 < r_2 \ldots$, and $r_n \longrightarrow \infty$ as $n \longrightarrow \infty$. For each $x \in X$, there exists r_n such that $x \in r_n V$. Thus V is absorbing.

(ii) It is obvious.

2.2 Linear Mapping

2.2.1 Definition. Let X and Y be a vector spaces over $\mathbb{R}$.

A linear map of X into Y is simply a function f of X into Y such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{R}$. In the special case in which $Y = \mathbb{R}$, we speak of a linear functional on X.

2.2.2 Definition. Let X be a topological space. The space X^*, called the dual space of X, is the set of all continuous linear functionals of X.

2.2.3 Remark. Let Λ be a linear functional on a topological vector space X.

(i) $\Lambda 0 = 0$

(ii) if A is convex set, $\Lambda(A)$ is convex

(iii) if A is subspace of X, $\Lambda(A)$ is subspace of $\mathbb{R}$.

2.2.4 Theorem. Let X and Y be a topological vector spaces. If $\Lambda : X \longrightarrow Y$ is a linear and continuous at 0, then Λ is continuous. In fact, Λ is uniformly continuous in the following sense : To each neighborhood W of 0 in Y corresponds a neighborhood V of 0 in X such that $y - x \in V$ implies $\Lambda y - \Lambda x \in W$.

Proof : Given a neighborhood W of 0 in Y, by continuity of Λ at 0, we can find a neighborhood V of 0 in X such that $\Lambda x \in W$ whenever $x \in V$. If now $y - x \in V$, the linearity of Λ shows that

$\Lambda y - \Lambda x = \Lambda(y-x) \in W$. Thus Λ maps the neighborhood $x + V$ of x into the preassigned neighborhood $\Lambda x + W$ of Λx, which says that Λ is continuous at x.

2.2.5 Theorem. Let Λ be a linear functional on a topological vector space X. Assume $\Lambda x \neq 0$ for some $x \in X$.

(i) Λ is a continuous implies the null space $\mathcal{N}(\Lambda)$ is closed.

(ii) Λ is bounded in some neighborhood V of 0 implies Λ is continuous.

Proof : (i) Since $\mathcal{N}(\Lambda) = \Lambda^{-1}(\{0\})$ and $\{0\}$ is closed subset of $\mathbb{R}$. By continuity of Λ, we have $\Lambda^{-1}(\{0\})$ is closed. Thus $\mathcal{N}(\Lambda)$ is closed set.

(ii) There exists a real number $M > 0$ and a neighborhood V of 0 such that $|\Lambda x| \leq M$ for x in V. If $\varepsilon > 0$ and if $W = \frac{\varepsilon}{M} V$, then $|\Lambda x| < \varepsilon$ for every x in W. Hence Λ is continuous at 0. By 2.2.4, the proof is complete.

2.3 Seminorms and Local Convexity

2.3.1 <u>Definition.</u> A <u>seminorm</u> on a vector space X is a real valued function p on X such that

(i) $p(x+y) \leq p(x)+p(y)$

(ii) $p(\alpha x) = |\alpha| \, p(x)$

for all $x, y \in X$ and all scalars α .

Property (i) is called <u>subadditivity.</u> A seminorm is a norm if it is satisfies

(iii) $p(x) \neq 0$ if $x \neq 0$.

A family $\mathcal{P}$ of seminorms on X is said to be <u>separating</u> if to each $x \neq 0$ corresponds at least one $p \in \mathcal{P}$ with $p(x) \neq 0$.

Next, consider a set $A \subset X$ which is absorbing. The Minkowski functional μ_A of A is defined by

$$\mu_A(x) = \inf\{ t > 0 : x \in t A \} \quad \text{for all } x \text{ in } X.$$

Note that $\mu_A(x) < \infty$ for all $x \in X$, since A is absorbing. The seminorms of X will turn out to be precisely the Minkowski functionals of balanced convex absorbing sets.

2.3.2 <u>Theorem.</u> Suppose p is a seminorm on a vector space X. Then

(i) $p(0) = 0$

(ii) $p(x-y) = p(y-x)$

(iii) $|p(x) - p(y)| \leq p(x-y)$

Proof : (i) By definition of a seminorm, we have $p(\alpha x) = |\alpha|\, p(x)$. Let $\alpha = 0$. Thus $p(0) = 0$.

(ii) $p(x-y) = p(-1(y-x)) = |-1|\, p(y-x) = p(y-x)$

(iii) Since $p(x) = p(x-y+y)$ and $p(y) = p(y-x+x)$. By 2.3.1 we have

$$p(x) \leq p(x-y) + p(y) \quad \text{and}$$
$$p(y) \leq p(y-x) + p(x) .$$

So

$$p(x) - p(y) \leq p(x-y)$$
$$p(y) - p(x) \leq p(y-x) = p(x-y) \quad \text{by (ii).}$$

Thus $|p(x) - p(y)| \leq p(x-y)$.

2.3.3 Theorem. Let p be a seminorm on a vector space X such that p is continuous at 0. Then p is continuous on X.

2.3.4 Remark. If $\mu_A(x) < \alpha$ then $x \in \alpha A$.

2.3.5 Definition. A set E in a topological vector space X is said to be bounded if for any neighborhood V of 0 there exists scalar α such that $E \subset \alpha V$.

2.3.6 Definition. A topological vector space X is said to be locally convex if there is a local base $\mathcal{B}$ whose members are convex.

2.3.7 Theorem. Suppose $\mathcal{P}$ is a separating family of seminorm on a vector space X. Associate to each $p \in \mathcal{P}$ and to each positive integer n the set

$$V(p,n) = \left\{ x : p(x) < \frac{1}{n} \right\}.$$

Let $\mathcal{B}$ be the collection of all finite intersections of the sets $V(p,n)$. Then $\mathcal{B}$ is a convex balanced local base for a topology on X which turns X into a locally convex space such that

(i) every $p \in \mathcal{P}$ is continuous, and

(ii) a set $E \subset X$ is bounded if and only if every $p \in \mathcal{P}$ is bounded on E.

<u>Proof</u> : Let τ be the collection of all subset of X such that each of which is a union of translation of member of $\mathcal{B}$. Then τ is a topology on X ; each member of $\mathcal{B}$ is convex and balanced, and $\mathcal{B}$ is a local base for τ.

Suppose $x \in X$, $x \neq 0$. Then, by separating property of $\mathcal{P}$, $p(x) > 0$ for some $p \in \mathcal{P}$; hence there exists an n such that $p(x) > \frac{1}{n}$. That is $x \notin V(p,n)$. Since $p(x) = p(-x)$, $V(p,n) = -V(p,n)$. So we have $-x \notin V(p,n)$ and $0 \notin V(p,n) + x$, which is a neighborhood of x. Thus $V(p,n) + x \subset \{0\}^c$ (complement of $\{0\}$). This say that $\{0\}^c$ is open, i.e. $\{0\}$ is closed set, and since τ is translation invarient topology, every point of X is closed set.

Next we show that addition and scalar multiplication are continuous. Let U be a neighborhood of 0 in X. Then, there exists $p_1, p_2, \ldots, p_m$ of $\mathcal{P}$ and positive integer $n_1, n_2, \ldots, n_m$ such that

(1) $V(p_1,n_1) \cap V(p_2,n_2) \cap \ldots \cap V(p_m,n_m) \subset U$. Put

(2) $V = V(p_1,2n_1) \cap \ldots\ldots\ldots \cap V(p_m,2n_m)$. Since every $p \in \mathcal{P}$ is subadditive, $V + V \subset U$. This shows that the mapping $(x,y) \longmapsto x + y$ is continuous at $(0,0)$ and hence, it is continuous at every point of X.

Suppose now that $x \in X$, α is a scalar, and U and V are as above. Let $p_i(x) = \alpha_i$ where $\alpha_i \geqslant 0$ for $1 \leq i \leq m$. Then $p_i(\frac{x}{\alpha_i}) = 1$, $p_i(\frac{x}{3n_i\alpha_i}) < \frac{1}{2n_i}$. Hence $x \in 3n_i\alpha_i\, V(p_i,\, 2n_i)$.

Take $t = \max\left\{ 3n_1\alpha_1, 3n_2\alpha_2, \ldots, 3n_m\alpha_m \right\}$.

Then $t > 0$. Since $V(p_i, 2n_i)$ is balanced, $x \in t\,V(p_i, 2n_i)$ for all $1 \leq i \leq m$. Thus $x \in tV$, i.e. there exists $t > 0$ such that $x \in tV$. Put $s = t/(1 + |\alpha|t)$. If $y \in x + sV$ and $|\beta - \alpha| < \frac{1}{t}$, then

$$\beta y - \alpha x = \beta(y - x) + (\beta - \alpha)x$$

which lies in

$$|\beta|sV + |\beta - \alpha|\,tV \subset V + V \subset U$$

since $|\beta|s \leq 1$ and V is balanced. This proves that scalar multiplication is continuous.

Thus X is a locally convex space. The definition of $V(p,n)$ shows that every $p \in \mathcal{P}$ is continuous at 0. Hence p is continuous on X, by 2.3.3 .

Finally, suppose $E \subset X$ is bounded : Fix $p \in \mathcal{P}$. Since $V(p,1)$ is a neighborhood of 0, $E \subset kV(p,1)$ for some $k < \infty$. Hence $p(x) < k$ for every $x \in E$. It follows that $p \in \mathcal{P}$ is bounded on E.

Conversely, suppose U is a neighborhood of 0, and (1) holds. Since each $p \in \mathcal{P}$ is bounded on E, there are number $M_i < \infty$ such that $p_i < M_i$ on E for $1 \leq i \leq m$. If $n > M_i n_i$ for $1 \leq i \leq m$, it follows that $E \subset n\,U$, so that E is bounded.

2.4 Function of bounded variation

2.4.1 Definition. Let f be a real-valued function on $[a, b]$. Partition $[a,b]$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$ and form the sum

$V = \sum_{k=0}^{n-1} \left| f(x_{k+1}) - f(x_k) \right|$. The total variation of f on $[a,b]$ is defined by

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$$\overset{b}{\underset{a}{V}}(f) = \sup\{V\}$$

where the supremum is taken over all partition of $[a,b]$. If $\overset{b}{\underset{a}{V}}(f) < \infty$, f is said to be of bounded variation on $[a,b]$. Denote the space of bounded variation functionson $[a,b]$ by $BV[a,b]$.

If addition and scalar multiplication are defined by

$$(f + g)(t) = f(t) + g(t),$$

$$\alpha f(t) = f(\alpha t)$$

where $f, g \in BV[a,b]$, α is scalar and $t \in [a,b]$. Then $BV[a,b]$ is vector space over scalar field $\mathbb{K}$.

If we define the norm of $f \in BV[a,b]$ by

$$\|f\| = |f(a)| + \overset{b}{\underset{a}{V}}(f).$$

Then $BV[a,b]$ is a Banach space.

2.4.2 Theorem. If $f : [a,b] \longrightarrow \mathbb{R}$ is a non-decreasing function. Then f is of bounded variation on $[a,b]$.

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2.4.3 Theorem. If f is a continuous function on the interval $[a,b]$ and g is bounded variation function on $[a,b]$, then $\int_a^b f(x)\,dg(x)$ exists and $\left|\int_a^b f(x)\,dg(x)\right| \leq M \overset{b}{\underset{a}{V}}(g)$ a where $M = \sup_{x\in[a,b]} \{|f(x)|\}$

2.4.4 Remark. If f, f_1, f_2 are continuous functions on $[a,b]$ g, g_1, g_2 are bounded variation functions on $[a,b]$ and k, l are real numbers. Then

$$(1) \quad \int_a^b (f_1+f_2)(x)dg(x) = \int_a^b f_1(x)dg(x) + \int_a^b f_2(x)dg(x),$$

$$(2) \quad \int_a^b f(x)d(g_1+g_2)(x) = \int_a^b f(x)dg_1(x) + \int_a^b f(x)dg_2(x),$$

$$(3) \quad \int_a^b kf(x)d(lg)(x) = kl \int_a^b f(x)dg(x) ,$$

$$(4) \quad \int_a^b g(x)df(x) \text{ exists} \Longrightarrow \int_a^b f(x)dg(x) \text{ exists.}$$

2.4.5 Remark. Let f be a continuous real-valued function of x and y in $a \leq x \leq b$, $a \leq y \leq b$. We can define a linear bounded mapping F on BV $[a,b]$ by the equation

$$Fg = \varphi \quad \text{where}$$

$$\varphi(y) = \int_a^b f(x,y)dg(x) \quad (x \in [a,b],\ g \in BV\,[a,b]).$$

By 2.4.3 and 2.4.4, we can see that for each g, $Fg \in C\,[a,b]$.

2.5 Measure Theory

2.5.1 Definition. A collection $\mathcal{M}$ of subsets of a set X is said to be a σ-algebra in X if $\mathcal{M}$ has the following three properties :

(1) $X \in \mathcal{M}$

(2) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$ where A^c is the complement of A relative to X.

(3) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathcal{M}$ for $n = 1,2,\ldots,$ then $A \in \mathcal{M}$.

If $\mathcal{M}$ is a σ-algebra in X, then X is called a measurable space, and the members of $\mathcal{M}$ are called the measurable sets in X.

2.5.2 Definition. If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

2.5.3 Theorem. If $\mathcal{F}$ is any collection of subsets of X. There exists a smallest σ-algebra $\mathcal{M}^*$ in X such that $\mathcal{F} \subset \mathcal{M}^*$. This $\mathcal{M}^*$ is sometimes called the σ-algebra generated by $\mathcal{F}$.

Proof : Let Ω be the family of all σ-algebras $\mathcal{M}$ in X which contain $\mathcal{F}$. Since the collection of all subsets of X is such a σ-algebra, Ω is not empty. Let $\mathcal{M}^*$ be the intersection of all $\mathcal{M} \in \Omega$. It is clear that $\mathcal{F} \subset \mathcal{M}^*$ and that $\mathcal{M}^*$ lies in every σ-algebra in X which contains $\mathcal{F}$. To complete the proof, we have to show that $\mathcal{M}^*$ is itself a σ-algebra.

If $A_n \in \mathcal{M}^*$ for $n = 1,2,\ldots$, and if $\mathcal{M} \in \Omega$, then $A_n \in \mathcal{M}$, so $\bigcup_n A_n \in \mathcal{M}$, since $\mathcal{M}$ is σ-algebra. Since $\bigcup_n A_n \in \mathcal{M}$ for every $\mathcal{M} \in \Omega$ we conclude that $\bigcup_n A_n \in \mathcal{M}^*$. The other two properties of a σ-algebra are verified in the same manner.

2.5.4 <u>Definition.</u> Let X be a topological space. By 2.5.3, there exists a smallest σ-algebra $\mathcal{B}$ in X such that every open set in X belongs to $\mathcal{B}$. The members of $\mathcal{B}$ are called the <u>Borel set</u>.

2.5.5 <u>Remark.</u> If $f : X \longrightarrow Y$ is a continuous function of X, where Y is any topological space, then it is evident from the definition that $f^{-1}(V) \in \mathcal{B}$ for every open set V in Y. In other words, every continuous mapping of X is Borel measurable.

2.5.6 <u>Definition.</u> A Borel measure μ is an extended real-valued function on $\mathcal{B}$ such that

(1) $\mu(\emptyset) = 0$

(2) if $A_1, A_2, \ldots \in \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j).$$

A Borel measure is said to be <u>bounded</u> if $\mu(A) < \infty$ for $A \in \mathcal{B}$.

2.5.7 <u>Definition.</u> Let B be collection of bounded Borel measures. If we define addition and scalar multiplication by

$$(\mu + \sigma)(E) = \mu(E) + \sigma(E)$$

$$(c\mu)(E) = c\,\mu(E)$$

where μ, $\nu \in$ B, c is a scalar and $E \in \mathcal{B}$. Then B is a vector space over scalar field $\mathbb{K}$.

2.5.8 <u>Definition.</u> Let μ be a bounded Borel measure. The set function $|\mu|$ on $\mathcal{B}$ defined by

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \qquad (E \in \mathcal{B}),$$

the supremum being taken over all partitions $\{E_i\}$ of E, is called <u>the total variation of μ</u>.

2.5.9 <u>Remark.</u> Let B be a collection of bounded Borel measures in a set X. Define

$$\|\mu\| = |\mu|(X).$$

Then $(B, \| \ \|)$ is a Banach space.

<u>Proof</u> : First, we have to show that B is a normed linear space.

(1) It is clear that $\|\mu\| \geq 0$.

(2) It is obvious that if $\mu = 0$, $\|\mu\| = 0$. Conversely if $\|\mu\| = 0$, $|\mu|(X) = 0$. Since for $E \in \mathcal{B}$

$$|\mu(E)| \leq |\mu|(E) \leq |\mu|(X) = 0.$$

$$|\mu(E)| = 0 \qquad \text{for } E \in \mathcal{B}.$$

Thus $\mu(E) = 0$, that is $\mu = 0$.

(3) For each $\mu \in B$, scalar α

$$\|\alpha\mu\| = |\alpha\mu|(X)$$

$$= \sup\left\{ \sum_{i=1}^{\infty} |(\alpha\mu)(E_i)| : \bigcup_i E_i = X \right\}$$

$$= |\alpha| \sup\left\{ \sum_{i=1}^{\infty} |\mu(E_i)| : \bigcup_i E_i = X \right\}$$

$$= |\alpha|\,|\mu|(X)$$

$$= |\alpha|\,\|\mu\| .$$

(4) For $\mu_1, \mu_2 \in B$,

$$\|\mu_1 + \mu_2\| = |(\mu_1 + \mu_2)|(X)$$

$$= \sup\left\{ \sum_i |(\mu_1+\mu_2)(E_i)| : \bigcup_i E_i = X \right\}$$

$$= \sup\left\{ \sum_i |\mu_1(E_i) + \mu_2(E_i)| : \bigcup_i E_i = X \right\}$$

$$\leq \sup\left\{ \sum_i |\mu_1(E_i)| + |\mu_2(E_i)| : \bigcup_i E_i = X \right\}$$

$$\leq \sup\left\{ \sum_i |\mu_1(E_i)| : \bigcup_i E_i = X \right\} +$$

$$\sup\left\{ \sum_i |\mu_2(E_i)| : \bigcup_i E_i = X \right\}$$

Thus $\|\mu_1 + \mu_2\| \leq |\mu_1|(X) + |\mu_2|(X)$,

i.e, $\|\mu_1 + \mu_2\| \leq \|\mu_1\| + \|\mu_2\|$.

Let $\{\mu_n\}$ be a sequence in B. Given $\varepsilon > 0$. There exists an N such that for all n, m $\geqslant$ N,

$\|\mu_n - \mu_m\| < \varepsilon$, i.e, $|\mu_n - \mu_m|(X) < \varepsilon$.

Since for $E \in \mathcal{B}$

$$|(\mu_n-\mu_m)(E)| \leq |\mu_n-\mu_m|(E) \leq |\mu_n-\mu_m|(X),$$

$$|(\mu_n-\mu_m)(E)| < \varepsilon \qquad \text{for all } m, n \geqslant N,$$

i.e, $$|\mu_n(E)-\mu_m(E)| < \varepsilon \qquad \text{for all } m, n \geqslant N.$$

Thus $\{\mu_n(E)\}$ is a cauchy sequence in $\mathbb{R}$ which is complete. So $\mu_n(E)$ converges uniformly to $\mu(E)$. That is, there axists N_0 such that for all $n \geqslant N_0$ and for all $E \in \beta$

$$|\mu_n(E) - \mu(E)| < \varepsilon .$$

For all $E_i \in \beta$ such that $\bigcup_{i=1}^{\infty} E_i = X$, we have

$$|\mu_n(E_i) - \mu(E_i)| < 2^{-i}\varepsilon ,$$

therefore $$\sum_{i=1}^{\infty} |\mu_n(E_i) - \mu(E_i)| < \sum_{i=1}^{\infty} 2^{-i}\varepsilon = \varepsilon \sum_{i=1}^{\infty} 2^{-i} = \varepsilon .$$

Hence $$\sup\left\{ \sum_i |(\mu_n-\mu)(E_i)| : \bigcup_i E_i = X \right\} < \varepsilon ,$$

i.e, $\|\mu_n - \mu\| < \varepsilon$. This completes the proof.

2.5.10 <u>Definition.</u> Let f be a measurable function on A. f is said to be <u>simple</u> if f(A) is countable, i.e, $f(A) = \{y_1, y_2, \ldots\}$.

2.5.11 <u>Definition.</u> Let f be a simple function. We define the integral of f over a set A by

$$\int_A f(x)\, d\mu = \sum_{n=1}^{\infty} y_n \mu(A_n)$$

where $A_n = \{x : x \in A, f(x) = y_n\}$. The integral of f exists if $\sum_{n=1}^{\infty} y_n \mu(A_n)$ is absolutely convergent, and we say that f is <u>integrable</u>.

2.5.12 <u>Lemma.</u> Given a simple function f defined on a set A, suppose A is a union

$$A = \bigcup_k B_k$$

of pairwise disjoint set B_k such that f takes only one value c_k on B_k. Then f is integrable on A if and only if the series $\sum_k c_k \mu(B_k)$ is absolutely convergent, in which case

$$\int_A f(x)\, d\mu = \sum_k c_k \mu(B_k).$$

<u>Proof</u> : Each set

$$A_n = \{x : x \in A,\ f(x) = y_n\}$$

is the union of the sets B_k for which $c_k = y_n$. Therefore

$$\sum_n y_n \mu(A_n) = \sum_n y_n \Big(\sum_{c_k = y_n} \mu(B_k)\Big)$$

$$= \sum_k c_k \mu(B_k).$$

Moreover, since μ is non-negative, we have

$$\sum_n |y_n| \mu(A_n) = \sum_n |y_n| \Big(\sum_{c_k = y_n} \mu(B_k)\Big) = \sum_k c_k \mu(B_k),$$

so that if one series is absolutely convergent, so is the other.

2.5.13 <u>Theorem.</u> Let f and g be simple functions integrable on a set A, and let $k \in \mathbb{R}$. Then f+g and kf are integrable over A and

$$\int_A [f(x) + g(x)]\, d\mu = \int_A f(x) d\mu + \int_A g(x) d\mu,$$

$$\int_A kf(x) d\mu = k \int_A f(x) d\mu.$$

Proof : Suppose that

$$F_i = \left\{ x : x \in A \text{ and } f(x) = y_i \right\} \text{ and}$$

$$G_j = \left\{ x : x \in A \text{ and } g(x) = z_j \right\}$$

where i, j = 1, 2,.... Then

$$(*) \qquad \int_A f(x)\, d\mu = \sum_i y_i \mu(F_i)$$

$$(**) \qquad \int_A g(x)\, d\mu = \sum_j z_j \mu(G_j).$$

Clearly, f+g takes the values $c_{ij} = y_i + z_j$ (not necessarily distinct) on the pairwise disjoint sets $B_{ij} = F_i \cap G_j$. It follows from

$$\mu(F_i) = \sum_j \mu(F_i \cap G_j), \ \mu(G_j) = \sum_i (F_i \cap G_j)$$

and the absolute convergence of the series (*) and (**) the series

$$\sum_i \sum_j c_{ij} \mu(B_{ij}) = \sum_i \sum_j (y_i + z_j) \mu(F_i \cap G_j)$$

is absolutely convergent. Hence by lemma 2.5.11 f+g is integrable on A and

$$\int_A \left[f(x) + g(x) \right] d\mu = \sum_i \sum_j (y_j + z_j) \mu(F_i \cap G_j)$$

$$= \sum_i y_i \mu(F_i) + \sum_j z_j \mu(G_j)$$

$$= \int_A f(x) d\mu + \int_A g(x) d\mu .$$

$$\int_A k\, f(x) d\mu = \sum_i k\, y_i \mu(F_i)$$

$$= k \sum_i y_i \mu(F_i)$$

$$= k \int_A f(x) d\mu .$$

2.5.14 Definition. A measurable function f is said to be integrable on a set A if there exists a sequence $\{f_n\}$ of integrable simple functions converging uniformly to f on A. We shall then say that

$$\lim_{n\to\infty} \int_A f_n(x)d\mu = \int_A f(x)d\mu .$$

2.5.15 Theorem. If $f, g : A \longrightarrow \mathbb{R}$ are integrable function on A and $k \in \mathbb{R}$. Then f+g and kf are also integrable and

$$\int_A (f+g)(x)d\mu = \int_A f(x)d\mu + \int_A g(x)d\mu ,$$

$$\int_A kf(x)d\mu = k\int_A f(x)d\mu .$$

Proof : There exists sequences of simple integrable function $\{f_n\}$ and $\{g_n\}$ such that $f_n \longrightarrow f$ and $g_n \longrightarrow g$ uniformly on A and

$$\lim_{n\to\infty} \int_A f_n(x)d\mu = \int_A f(x)d\mu ,$$

$$\lim_{n\to\infty} \int_A g_n(x)d\mu = \int_A g(x)d\mu .$$

Thus $f_n + g_n \longrightarrow f+g$ uniformly on A; hence f+g is integrable and

$$\lim_{n\to\infty} \int_A (f_n + g_n)(x)d\mu = \lim_{n\to\infty} \int_A f_n(x)d\mu + \lim_{n\to\infty} \int_A g_n(x)d\mu .$$

So $$\int_A (f+g)(x)d\mu = \int_A f(x)d\mu + \int_A g(x)d\mu .$$

Also $kf_n \longrightarrow kf$ uniformly on A. Therefore kf is integrable on A and

$$\int_A kf(x)d\mu = \lim_{n\to\infty} \int_A kf_n(x)d\mu = \lim_{n\to\infty} k\int_A f_n(x)d\mu$$

$$= k\lim_{n\to\infty} \int_A f_n(x)d\mu$$

$$= k\int_A f(x)d\mu .$$

2.5.16 Theorem. If φ is non-negative and integrable function on A and if a measurable function f is bounded by φ almost everywhere. Then f is integrable and $\left|\int_A f(x)d\mu\right| \leq \int_A \varphi(x)d\mu$.

Proof : If f and φ are simple functions, then, by substracting a set of measure zero from A, we get a set A' which can be represented as a finite or countable union

$$A' = \bigcup_n A_n$$

of subset $A_n \subset A'$ such that

$$f(x) = a_n \quad , \quad \varphi(x) = b_n$$

for all $x \in A_n$ and

$$|a_n| \leq b_n \qquad (n = 1,2,\ldots).$$

Since φ is integrable on A, we have

$$\sum_n |a_n|\,\mu(A_n) \leq \sum_n b_n\,\mu(A_n) = \int_{A'}\varphi(x)d\mu = \int_A \varphi(x)d\mu .$$

Therefore f is also integrable on A and

$$\left|\int_A f(x)d\mu\right| = \left|\int_{A'} f(x)d\mu\right| = \left|\sum_n a_n\mu(A_n)\right| \leq \sum_n |a_n|\,\mu(A_n)$$

$$\leq \int_A \varphi(x)d\mu .$$

In the case where f and φ are arbitary measurable functions, let $\{f_n\}$ and $\{\varphi_n\}$ be sequences of simple functions converging uniformly to f and φ, respectively. We can choose the sequence so that

$|f_n(x)| \leq \varphi_n(x)$ for all n and for all $x \in A$. Moreover each φ_n is integrable since φ is integrable. So each f_n is integrable and hence f is integrable. Also $\int_A |f_n(x)| \, d\mu \leq \int_A \varphi_n(x) d\mu$, taking the limit as $n \longrightarrow \infty$ we have $\left| \int_A f(x) d\mu \right| \leq \int_A \varphi(x) d\mu$.

2.5.17 <u>Corollary</u>. If f is bounded measurable function on A, then f is integrable.

<u>Proof</u> : Let $\varphi(x) = \sup_{x \in A} \{|f(x)|\} = M < \infty$. Apply 2.5.16, we get the result.

2.5.18 <u>Theorem</u>. Let f be a measurable function such that f is μ-and μ_1- and μ_2- integrable on A and $k \in \mathbb{R}$. Then

$$\int_A f(x) \, d(\mu_1+\mu_2) = \int_A f(x) d\mu_1 + \int_A f(x) d\mu_2 \quad \text{and}$$

$$k \int_A f(x) d\mu = \int_A f(x) \, d(k\mu).$$

<u>Proof</u> : There exists a sequence of integrable simple functions $\{f_n\}$ converging uniformly to f. First, we shall verify that

$$\int_A f_n(x) d\mu_1 + \int_A f_n(x) d\mu_2 = \int_A f_n(x) d(\mu_1+\mu_2).$$

Since $\int_A f_n(x)d\mu_1 = \sum_k y_{k_n}\mu(A_{k_n})$ and

$$\int_A f_n(x)d\mu_2 = \sum_k y_{k_n}\mu(A_{k_n})$$

where $A_{k_n} = \{x : x \in A \text{ and } f_n(x) = y_{k_n}\}$. So

$$\int_A f_n(x)d\mu_1 + \int_A f_n(x)d\mu_2 = \sum_k y_{k_n}\mu_1(A_{k_n}) + \sum_k y_{k_n}\mu_2(A_{k_n})$$

$$= \sum_k y_{k_n}(\mu_1+\mu_2)(A_{k_n})$$

$$= \int_A f_n(x)d(\mu_1+\mu_2) .$$

By taking the limit as $n \longrightarrow \infty$, we get

$$\int_A f(x)d\mu_1 + \int_A f(x)d\mu_2 = \int_A f(x)d(\mu_1+\mu_2).$$

Similarly we have

$$k\int_A f_n(x)d\mu = k\sum_k y_{k_n}\mu(A_{k_n})$$

$$= \sum_k y_{k_n}(k\mu)(A_{k_n})$$

$$= \int_A f_n(x)d(k\mu).$$

Take limit as $n \longrightarrow \infty$. Thus

$$k\int_A f(x)d\mu = \int_A f(x)d(k\mu).$$

2.5.19 Remark. Let f be a continuous real-valued function of x and y in $a \leq x \leq b$, $a \leq y \leq b$. We can define a linear bounded mapping M on B by the equation

$$M\mu = \psi \quad \text{where}$$

$$\psi(y) = \int_a^b f(x,y)\,d\mu(x) \qquad (x \in [a,b],\ \mu \in B).$$

By 2.16 and 2.18 we can see that for each $\mu \in B$, $M\mu \in C[a,b]$.