CHAPTER III

GENERAL SOLUTIONS OF $f(x + y) = g(x)f(y) + g(y)f(x)$ ON CYCLIC MONOID

<u>DEFINITION 3.1</u> Let S be any semigroup, F be any field. By a <u>solution</u> of the functional equation

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into F, we mean an ordered pair (f,g) where f,g are functions from S into F such that $(*)$ holds for all x,y in S.

It is clear that any ordered pair (f,g) where f is identically zero, and g is an arbitrary function, is a solution of $(*)$. Such a solution will be called a <u>trivial solution</u>, any other solution will be called a <u>non-trivial solution</u>. A solution (f,g) on monoid for which $f(0) = 0$ will be called a <u>zero-type solution</u>, any other solution will be called a <u>non-zero-type solution</u>.

<u>THEOREM 3.2</u> Let S be any monoid, F be a field of characteristic different from 2. Let f,g be functions on S into F. Then (f,g) is a non-zero-type solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S if and only if there exists β in F^* and a homomorphism $\tilde{g}$ on S into F^* such that

$$(3.2.1)\quad \begin{cases} f(x) = \beta\tilde{g}(x), \\ g(x) = \frac{1}{2}\tilde{g}(x) \end{cases}$$

for all x in S.

PROOF Let (f,g) be a non-zero-type solution of (*) on S into F. Observe that

$$\begin{aligned} f(0) &= f(0+0), \\ &= g(0)f(0) + g(0)f(0), \\ &= 2g(0)f(0). \end{aligned}$$

It follows that $[2g(0) - 1]f(0) = 0$. Since $f(0) \neq 0$, hence

$$(3.2.2)\quad g(0) = \frac{1}{2}.$$

Hence for any x in S

$$\begin{aligned} f(x) &= f(x+0), \\ &= g(0)f(x) + g(x)f(0), \\ &= \tfrac{1}{2}f(x) + g(x)f(0), \end{aligned}$$

so that

$$f(x) = 2f(0)g(x).$$

Let $\beta = f(0)$ and $\tilde{g}(x) = 2g(x)$. Hence we have

$$f(x) = \beta\,\tilde{g}(x),$$

$$g(x) = \frac{1}{2}\tilde{g}(x)$$

for all x in S. Hence, for arbitrary x,y in S we have

$$f(x+y) = \beta\,\tilde{g}(x+y),$$

Hence

$$\beta\tilde{g}(x + y) = f(x + y),$$

$$= g(x)f(y) + g(y)f(x),$$

$$= \frac{\beta}{2}\tilde{g}(x)\tilde{g}(y) + \frac{\beta}{2}\tilde{g}(x)\tilde{g}(y),$$

$$= \beta\,\tilde{g}(x)\tilde{g}(y).$$

Therefore, we have

$$\tilde{g}(x + y) = \tilde{g}(x)\tilde{g}(y),$$

which shows that $\tilde{g}$ is a homomorphism.

To show the converse, let $\beta \in F^*$ and $\tilde{g}$ be a homomorphism on S into F^*. Then

(3.2.3) $\tilde{g}(0) = 1$

Define functions f,g on S into F as (3.2.1). Then for any x,y in S we have

$$g(x)f(y) + g(y)f(x) = \frac{1}{2}\tilde{g}(x).\beta\tilde{g}(y) + \frac{1}{2}\tilde{g}(y).\beta\tilde{g}(x),$$

$$= \beta\tilde{g}(x)\tilde{g}(y),$$

$$= \beta\tilde{g}(x + y),$$

$$= f(x + y).$$

Furthermore, we see that $f(0) = \beta\tilde{g}(0) = \beta \neq 0$. Therefore, (f,g) is a non-zero-type solution of (*) on S.

COROLLARY 3.3 Let $(S,+)$ be a cyclic monoid with generator a, F be a field of characteristic different from 2. Let f,g be functions on S into F. Then (f,g) is a non-zero-type solution of

$$(*) \quad f(x+y) = g(x)f(y) + g(y)f(x)$$

on S if and only if f, g are of the form

$$(3.3.1) \quad \begin{cases} f(na) = \beta q^n, \\ g(na) = \frac{1}{2} q^n \end{cases}$$

for all n in $\mathbb{N}$ for some β, q in F^*.

PROOF From theorem 3.2, it follows that (f,g) is a non-zero-type solution of $(*)$ on S if and only if there exists β in F^* and a homomorphism $\tilde{g}$ on S into F^* such that

$$(3.3.2) \quad \begin{cases} f(na) = \beta\, \tilde{g}(na), \\ g(na) = \frac{1}{2}\, \tilde{g}(na) \end{cases}$$

for all n in $\mathbb{N}$. By remark 2.9, we have

$$(3.3.3) \quad \tilde{g}(na) = \tilde{g}(a)^n$$

for all n in $\mathbb{N}$. From (3.3.2), (3.3.3), and let $q = \tilde{g}(a)$, we have f,g will be of the form (3.3.1).

<u>THEOREM 3.4</u> Let S be any monoid, F be a field. If (f,g) is a non-trivial zero-type solution of

$$(*) \quad f(x+y) = g(x)f(y) + g(y)f(x)$$

on S into F, then

$$g(0) = 1$$

<u>PROOF</u> Let (f,g) be a non-trivial zero-type solution of (*) on S into F. For any x in S,

$$\begin{aligned} f(x) &= f(x+0), \\ &= g(x)f(0) + g(0)f(x), \\ &= g(0)f(x), \end{aligned}$$

$$[1 - g(0)]\, f(x) = 0.$$

Since $f \neq 0$, hence $g(0) = 1$.

<u>LEMMA 3.5</u> Let S be cyclic monoid with generator a, F be a field. If (f,g) is a non-trivial zero-type solution of

$$(*) \quad f(x+y) = g(x)f(y) + g(y)f(x)$$

on S into F, then

$$(3.5.1) \quad f(na) = \left\{ g((n-1)a) + \sum_{i=1}^{n-1} g(a)^i g((n-1-i)a) \right\} f(a)^{\bigstar}$$

for all n in $\mathbb{P}$.

($\bigstar$) Here and in the sequel, $\sum_{i=1}^{0} g(a)^i g(-ia) = 0$.

<u>PROOF</u> Let (f,g) be a non-trivial zero-type solution of (*) on S into F. By theorem 3.4, we can verify that (3.5.1) holds for the case $n = 1,2$.

Assume that (3.5.1) holds for the case $n = k$, i.e

$$(3.5.2)\quad f(ka) = \left[g((k-1)a) + \sum_{i=1}^{k-1} g(a)^i g((k-1-i)a) \right] f(a).$$

From (*), we have

$$f((k+1)a) = g(ka)f(a) + g(a)f(ka).$$

From (3.5.2), it follows that

$$f((k+1)a) = g(ka)f(a) + g(a)\left\{ g((k-1)a) + \sum_{i=1}^{k-1} g(a)^i g((k-1-i)a) \right\} f(a),$$

$$= \left\{ g(ka) + \sum_{i=1}^{k} g(a)^i g((k-i)a) \right\} f(a).$$

Therefore, the lemma is proved.

<u>REMARK 3.6</u> From theorem 3.4 and lemma 3.5, we can see that if S is a cyclic monoid with generator a, F is a field, (f,g) is a non-trivial zero-type solution of

$$(*)\quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into F, then $g(0) = 1$ and $f(a) \neq 0$.

<u>LEMMA 3.7</u> Let S be a cyclic monoid with generator a, F be a field of characteristic different from 2. Let (f,g) be a non-trivial zero-type solution of

$$(*) \qquad f(x+y) = g(x)f(y) + g(y)f(x)$$

on S into F, then

$$(3.7.1) \quad g(na) = g(a)g((n-1)a) + \left[g(2a)-g(a)^2\right]\left[g((n-2)a) + \sum_{i=1}^{n-2} g(a)^i g((n-2-i)a)\right]$$

for all n in $\mathbb{P}$ such that $n \geq 2$.

<u>PROOF</u> Let (f,g) be a non-trivial zero-type solution of (*) on S into F. For any n in $\mathbb{P}$ such that $n \geq 2$ we have

$$\begin{aligned}(3.7.2) \quad f((n+1)a) &= f((n-1)a + 2a), \\ &= g((n-1)a)f(2a) + g(2a)f((n-1)a), \\ &= g((n-1)a)2g(a)f(a) + g(2a)\left[g((n-2)a) + \sum_{i=1}^{n-2} g(a)^i g((n-2-i)a)\right] f(a), \\ &= \left[2g(a)g((n-1)a) + g(2a)\left\{g((n-2)a) + \sum_{i=1}^{n-2} g(a)^i g((n-2-i)a)\right\}\right]f(a).\end{aligned}$$

Here the second and third equalities follows from (*) and lemma 3.5 respectively. On the other hand, by lemma 3.5, we have

$$(3.7.3)\qquad f((n+1)a) = \left[g(na) + \sum_{i=1}^{n} g(a)^i g((n-i)a)\right] f(a)$$

for all n in $\mathbb{P}$. Hence by equating (3.7.2), (3.7.3), and cancelling f(a) on both sides of the equation, we have

$$g(na) + \sum_{i=1}^{n} g(a)^i g((n-i)a) = 2g(a)g((n-1)a) + g(2a)\left[g((n-2)a) + \sum_{i=1}^{n-2} g(a)^i g((n-2-i)a)\right]$$

for all n in $\mathbb{P}$ such that $n \geq 2$. From this, it follows that

$$g(na) = g(a)g((n-1)a) + \left[g(2a)-g(a)^2\right]\left[g((n-2)a) + \sum_{i=1}^{n-2} g(a)^i g((n-2-i)a)\right]$$

for all n in $\mathbb{P}$ such that $n \geq 2$.

For convenience, we shall classify non-trivial zero-type solutions of

$$(*)\qquad f(x + y) = g(x)f(y) + g(y)f(x)$$

on a cyclic monoid according to the following definitions.

<u>DEFINITION 3.8</u> Let S be a cyclic monoid with generator a, F be a field. A non-trivial zero-type solution (f,g) of (*) is said to be a <u>type I solution</u> of (*) if and only if

$$2g(a)^2 - g(2a) \neq 0 \quad \text{and} \quad g(a)^2 \neq g(2a).$$

It is said to be a <u>type II solution</u> of (*) if and only if

$$2g(a)^2 - g(2a) \neq 0 \quad \text{and} \quad g(a)^2 = g(2a).$$

It is said to be a <u>type III solution</u> of (*) if and only if

$$2g(a)^2 - g(2a) = 0 \quad \text{and} \quad g(a) \neq 0.$$

It is said to be a <u>type IV solution</u> of (*) if and only if

$$2g(a)^2 - g(2a) = 0 \quad \text{and} \quad g(a) = 0.$$

Observe that any non-trivial zero-type solution of (*) must fall in one of the above four types.

<u>LEMMA 3.9</u> Let S be a cyclic monoid with generator a, F be an algebraically closed field of characteristic different from 2. Then (f,g) is a type I solution of

$$(*) \qquad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into F if and only if f, g are functions on S into F of the form

$$(3.9.1) \quad \begin{cases} f(na) = \beta\,(q_1^n - q_2^n)\,, \\ g(na) = \frac{1}{2}\,(q_1^n - q_2^n) \end{cases}$$

for all n in $\mathbb{N}$, where $\beta, q_1, q_2 \in F^*$ such that $q_1 \neq q_2$.

<u>PROOF</u> Let (f,g) be a type I solution of (*) on S into F, hence we have

$$(3.9.2) \qquad 2g(a)^2 - g(2a) \neq 0$$

and

(3.9.3) $\quad g(a)^2 \neq g(2a).$

Let

$$(3.9.4) \qquad \beta = \frac{f(a)}{2\sqrt{g(2a)-g(a)^2}} \quad , \quad q_1 = g(a) + \sqrt{g(2a) - g(a)^2} \ ,$$

$$q_2 = g(a) - \sqrt{g(2a) - g(a)^2} \ .$$

Since F is algebraically closed and characteristic different from 2, hence

(3.9.5) $\quad \beta \in F^*$

and

(2.9.6) $\quad q_1 q_2 = 2g(a)^2 - g(2a)$.

From (3.9.2) and (3.9.6), we can see that

(3.9.7) $\quad q_1, q_2 \in F^*$

From (3.9.4), it can be verified directly that (3.9.1) holds in the case $n = 0,1,2$.

Let $n \geq 2$. Assume that

$$(3.9.8) \qquad \begin{cases} f(ka) = \beta\,(q_1^k - q_2^k) \ , \\ g(ka) = \frac{1}{2}(q_1^k + q_2^k) \end{cases}$$

for all $k < n$. From (✱) and the assumption (3.9.8), we have

$$f(na) = g((n-1)a)f(a) + g(a)f((n-1)a),$$

$$= \frac{1}{2}(q_1^{n-1} + q_2^{n-1})\beta(q_1 - q_2) + \frac{1}{2}(q_1 + q_2)\beta(q_1^{n-1} - q_2^{n-1}),$$

$$= \frac{\beta}{2}\Big[q_1^n - q_1^{n-1}q_2 + q_1q_2^{n-1} - q_2^n + q_1^n - q_1q_2^{n-1}$$

$$+ q_1^{n-1}q_2 - q_2^n\Big],$$

$$= \beta\,(q_1^n - q_2^n).$$

From lemma 3.7, we have

$$g(na) = g(a)g((n-1)a) + \Big[g(2a) - g(a)^2\Big]\Big[g((n-2)a)$$

$$+ \sum_{i=1}^{n-2} g(a)^i g((n-2-i)a)\Big],$$

By assumption (3.9.8), it follows that

$$g(na) = \frac{1}{2}(q_1 + q_2)\,\frac{1}{2}(q_1^{n-1} + q_2^{n-1}) + \Big[\frac{1}{2}(q_1^2 + q_2^2) - \frac{1}{4}(q_1 + q_2)^2\Big]\Big[\frac{1}{2}(q_1^{n-2} + q_2^{n-2})$$

$$+ \sum_{i=1}^{n-3} \frac{1}{2^i}(q_1 + q_2)^i\,\frac{1}{2}(q_1^{n-2-i} + q_2^{n-2-i}) + \frac{1}{2^{n-2}}(q_1 + q_2)^{n-2}\Big],$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)^2\Big[\frac{1}{2}(q_1^{n-2} + q_2^{n-2}) +$$

$$\frac{q_1^{n-2}}{2}\sum_{i=1}^{n-3}\left(\frac{q_1 + q_2}{2q_1}\right)^i + \frac{q_2^{n-2}}{2}\sum_{i=1}^{n-3}\left(\frac{q_1 + q_2}{2q_2}\right)^i +$$

$$\frac{1}{2^{n-2}}(q_1 + q_2)^{n-2}\Big],$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)^2\Big[\frac{1}{2}(q_1^{n-2} + q_2^{n-2})$$

$$+ \frac{q_1^{n-2}}{2}\;\frac{(\frac{q_1 + q_2}{2q_1})\left[1 - (\frac{q_1 + q_2}{2q_1})^{n-3}\right]}{1 - (\frac{q_1 + q_2}{2q_1})}$$

$$+ \frac{q_2^{n-2}}{2}\;\frac{(\frac{q_1 + q_2}{2q_2})\left[1 - (\frac{q_1 + q_2}{2q_2})^{n-3}\right]}{1 - (\frac{q_1 + q_2}{2q_2})} + \frac{1}{2^{n-2}}(q_1+q_2)^{n-2}\Big],$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)^2\Big[\frac{1}{2}(q_1^{n-2} + q_2^{n-2})$$

$$+ \frac{q_1^{n-2}}{2}\,(\frac{q_1 + q_2}{q_1 - q_2})\left[1 - (\frac{q_1 + q_2}{2q_1})^{n-3}\right]$$

$$- \frac{q_2^{n-2}}{2}\,(\frac{q_1 + q_2}{q_1 - q_2})\left[1 - (\frac{q_1 + q_2}{2q_2})^{n-3}\right] + \frac{1}{2^{n-2}}(q_1 + q_2)^{n-2}\Big],$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)^2\Big\{\frac{1}{2}(q_1^{n-2} + q_2^{n-2})$$

$$+ \frac{1}{2}(\frac{q_1 + q_2}{q_1 - q_2})(q_1^{n-2} - q_2^{n-2}) - \frac{1}{2^{n-2}}(\frac{q_1 + q_2}{q_1 - q_2})(q_1 + q_2)^{n-3}(q_1 - q_2)$$

$$+ \frac{1}{2^{n-2}}(q_1 + q_2)^{n-2}\Big\},$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)^2 \Big[\frac{1}{2}(q_1^{n-2} + q_2^{n-2})$$

$$+ \frac{1}{2}\left(\frac{q_1 + q_2}{q_1 - q_2}\right)(q_1^{n-2} - q_2^{n-2}) - \frac{1}{2^{n-2}}(q_1 + q_2)^{n-2}$$

$$+ \frac{1}{2^{n-2}}(q_1 + q_2)^{n-2} \Big] ,$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)^2 \Big[\frac{1}{2}(q_1^{n-2} + q_2^{n-2})$$

$$+ \frac{1}{2}\left(\frac{q_1 + q_2}{q_1 - q_2}\right)(q_1^{n-2} - q_2^{n-2}) \Big] ,$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)^2 \Big[\frac{1}{2} q_1^{n-2} \left(1 + \left(\frac{q_1 + q_2}{q_1 - q_2}\right)\right)$$

$$+ \frac{1}{2} q_2^{n-2} \left(1 - \left(\frac{q_1 + q_2}{q_1 - q_2}\right)\right) \Big] ,$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)^2 \Big\{ \frac{1}{2} q_1^{n-2} \cdot \frac{2q_1}{q_1 - q_2}$$

$$+ \frac{1}{2} q_2^{n-2} \cdot \frac{(-2q_2)}{q_1 - q_2} \Big\} ,$$

$$= \frac{1}{4}(q_1 + q_2)(q_1^{n-1} + q_2^{n-1}) + \frac{1}{4}(q_1 - q_2)(q_1^{n-1} - q_2^{n-1}),$$

$$= \frac{1}{4}(q_1^n + q_1 q_2^{n-1} + q_1^{n-1} q_2 + q_2^n) + \frac{1}{4}(q_1^n - q_1 q_2^{n-1} - q_1^{n-1} q_2 + q_2^n),$$

$$= \frac{1}{2}(q_1^n + q_2^n).$$

Next, assume that f, g are functions on S into F of the form (3.9.1). For any n_1, n_2 in $\mathbb{N}$, we have

$$g(n_1a)f(n_2a)+ g(n_2a)f(n_1a) = \frac{\beta}{2}(q_1^{n_1}+ q_2^{n_1})(q_1^{n_2}-q_2^{n_2})$$

$$+ \frac{\beta}{2} (q_1^{n_2}+ q_2^{n_2})(q_1^{n_1} - q_2^{n_1}),$$

$$= \frac{\beta}{2}(q_1^{n_1+n_2} - q_1^{n_1}q_2^{n_2} + q_1^{n_2}q_2^{n_1} - q_2^{n_1+n_2}$$

$$+ q_1^{n_1+n_2} - q_1^{n_2}q_2^{n_1} + q_1^{n_1}q_2^{n_2} - q_2^{n_1+n_2}),$$

$$= \beta(q_1^{n_1+n_2} - q_2^{n_1+n_2}),$$

$$= f(n_1a + n_2a).$$

From (3.9.1) we see that

(3.9.9) $f(0) = \beta(q_1^0 - q_2^0) = 0$

and

$$f(a) = \beta(q_1 - q_2) .$$

Since $\beta \neq 0$ and $q_1 \neq q_2$, hence

(3.9.10) $f(a) \neq 0$.

Furthermore,

$$2g(a)^2 - g(2a) = 2. \frac{1}{4}(q_1 + q_2)^2 - \frac{1}{2}(q_1^2 + q_2^2),$$

$$= \frac{1}{2}(q_1^2 + 2q_1q_2 + q_2^2) - \frac{1}{2}(q_1^2 + q_2^2),$$

$$= q_1q_2.$$

Since $q_1, q_2 \in F^*$, hence we have

$$(3.9.10) \qquad 2g(a)^2 - g(2a) \neq 0 .$$

If $g(a)^2 = g(2a)$ then

$$\frac{1}{4}(q_1 + q_2)^2 = \frac{1}{2}(q_1^2 + q_2^2),$$

$$\frac{1}{4}(q_1^2 + q_2^2 - 2q_1q_2) = 0,$$

$$\frac{1}{4}(q_1 - q_2)^2 = 0,$$

$$q_1 = q_2$$

which is contrary to the condition $q_1 \neq q_2$. Therefore,

$$(3.9.11) \quad g(a)^2 \neq g(2a).$$

Hence (f,g) is a type I solution of $(*)$.

<u>LEMMA 3.10</u> Let S, F be as in lemma 3.9. Then (f,g) is a type II solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into F if and only if f,g are functions on S into F of the form

$$(3.10.1) \quad \begin{cases} f(na) = n\beta q^n , \\ g(na) = q^n \end{cases}$$

for all n in $\mathbb{N}$, where $\beta, q \in F^*$.

PROOF Let (f,g) be a type II solution of (*) on S into F, hence we have

(3.10.2) $$2g(a)^2 - g(2a) \neq 0$$

and

(3.10.3) $$g(a)^2 = g(2a) .$$

Observe that

$$\begin{aligned} g(a)^2 &= g(a)^2 + \left[g(a)^2 - g(2a)\right], \\ &= 2g(a)^2 - g(2a), \\ &\neq 0. \end{aligned}$$

Hence

(3.10.4) $g(a) \neq 0.$

Let $q = g(a)$. Then

(3.10.5) $q \in F^*$.

Let $\beta = \frac{f(a)}{g(a)}$. Since $f(a) \neq 0$ and $g(a) \neq 0$, hence

(3.10.6) $\beta \in F^*$.

Observe that (3.10.1) holds for the case $n = 0,1$. Let $k \in \mathbb{N}$. Assume that

(3.10.7) $$\begin{cases} f(ka) = k\beta q^k, \\ g(ka) = q^k . \end{cases}$$

From (*), we have

$$f((k + 1)a) = g(a)f(ka) + g(ka)f(a).$$

By the assumption (3.10.7), it follows that

$$(3.10.8)\quad f((k+1)a) = q.\ k\ \beta q^{k} + q^{k}.\ \beta q,$$

$$= (k+1)\beta q^{k+1},$$

Since $g(a)^2 = g(2a)$, hence, from lemma 3.7 we have

$$g((k+1)a) = g(a)g(ka).$$

By the assumption (3.10.7), it follows that

$$(3.10.9)\quad g((k+1)a) = q.q^{k},$$

$$= q^{k+1}.$$

Therefore, (3.10.1) holds for all n in $\mathbb{N}$.

Next, assume that f, g are functions on S into F of the form (3.10.1). For any n_1, n_2 in $\mathbb{N}$, we have

$$(3.10.10)\quad g(n_1a)f(n_2a)+g(n_2a)f(n_1a) = q^{n_1}.n_2\beta q^{n_2} + q^{n_2}.n_1\beta q^{n_1},$$

$$= (n_1 + n_2)\beta q^{n_1+n_2},$$

$$= f(n_1a + n_2a).$$

From (3.10.1), we have

$$(3.10.11)\quad f(0) = 0,$$

$$(3.10.12)\quad f(a) = \beta q \neq 0,$$

$$(3.10.13)\quad 2g(a)^2 - g(2a) = 2q^2 - q^2,$$

$$= q^2,$$

$$\neq 0,$$

and

$$(3.10.14)\quad g(a)^2 = q^2 ,$$

$$= g(2a).$$

Therefore, (f,g) is a type II solution of (*) on S.

LEMMA 3.11 Let S, F be as in lemma 3.9, Then (f,g) is a type III solution of

$$(*)\quad f(x+y) = g(x)f(y) + g(y)f(x)$$

on S into F if and only if f, g are functions on S into F of the form

$$(3.11.1)\quad \begin{cases} f(na) = \begin{cases} 0 & \text{if } n = 0 , \\ \beta q^n & \text{otherwise} , \end{cases} \\ g(na) = \begin{cases} 1 & \text{if } n = 0 , \\ \frac{1}{2} q^n & \text{otherwise} \end{cases} \end{cases}$$

for all n in $\mathbb{N}$, where $\beta, q \in F^*$.

PROOF Let (f,g) be a type III solution of (*) on S into F, hence

$$(3.11.2)\quad 2g(a)^2 - g(2a) = 0$$

and

$$(3.11.3)\quad g(a) \neq 0 .$$

Let

$$(3.11.4)\quad \beta = \frac{f(a)}{2g(a)} , \quad q = 2g(a).$$

Since $f(a) \neq 0$ and $g(a) \neq 0$, hence

(3.11.5) $\beta, q \in F^*$.

It can be verified that (3.11.1) hold for the case $n = 0, 1, 2$.

Let $n \geq 3$. Assume that

$$(3.11.6) \quad \begin{cases} f(ka) = \begin{cases} 0 & \text{if } k = 0, \\ \beta q^k & \text{otherwise}, \end{cases} \\ \\ g(ka) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{1}{2} q^k & \text{otherwise} \end{cases} \end{cases}$$

for all $k < n$. Observe that

$$(3.11.7) \quad f(na) = g(a)\, f((n-1)a) + g((n-1)a)f(a),$$

$$= \frac{1}{2}q \cdot \beta q^{n-1} + \frac{1}{2}\, q^{n-1} \cdot \beta q,$$

$$= \beta q^n.$$

From lemma 3.7, we have

$$(3.11.8) \quad g(na) = g(a)g((n-1)a) + \left[g(2a)-g(a)^2\right]\left[g((n-2)a) + \sum_{i=1}^{n-2} g(a)^i g((n-2-i)a)\right],$$

$$= \frac{1}{2}q \cdot \frac{1}{2}\, q^{n-1} + \left[\frac{1}{2}q^2 - \frac{1}{4}q^2\right]\left[\frac{1}{2}\, q^{n-2} + \sum_{i=1}^{n-3} \frac{1}{2^i}q^i \cdot \frac{1}{2}\, q^{n-2-i} + \frac{1}{2^{n-2}}\, q^{n-2}\right],$$

$$= \frac{1}{4}q^n + \frac{1}{4}q^2\left[\frac{1}{2}q^{n-2} + \sum_{i=1}^{n-3}\frac{1}{2^{i+1}}q^{n-2} + \frac{1}{2^{n-2}}q^{n-2}\right],$$

$$= \frac{1}{4}q^n + \frac{1}{4}q^n\left[\frac{1}{2} + \sum_{i=1}^{n-3}\frac{1}{2^{i+1}} + \frac{1}{2^{n-2}}\right],$$

$$= \frac{1}{4}q^n + \frac{1}{4}q^n\left[\sum_{i=1}^{n-2}\frac{1}{2^i} + \frac{1}{2^{n-2}}\right],$$

$$= \frac{1}{4}q^n + \frac{1}{4}q^n\left[\frac{\frac{1}{2}(1 - \frac{1}{2^{n-2}})}{1-\frac{1}{2}} + \frac{1}{2^{n-2}}\right],$$

$$= \frac{1}{4}q^n + \frac{1}{4}q^n,$$

$$= \frac{1}{2}q^n.$$

Next, assume that f, g are functions on S into F of the form (3.11.1). For any n_1, n_2 in $\mathbb{P}$, we have

(3.11.9) $$g(n_1a)f(n_2a) + g(n_2a)f(n_1a) = \frac{1}{2}q^{n_1}.\beta q^{n_2} + \frac{1}{2}q^{n_2}.\beta q^{n_1},$$

$$= \beta q^{n_1+n_2},$$

$$= f(n_1a + n_2a).$$

It can be verified from (3.11.1) that (3.11.9) also hold for the cases where $n_1 = 0$ or $n_2 = 0$.

From (3.11.1) we see that

(3.11.10) $f(0) = 0$,

(3.11.11) $f(a) = \beta q \neq 0$,

$$(3.11.12) \quad 2g(a)^2 - g(2a) = 2 \cdot \frac{1}{4} q^2 - \frac{1}{2} q^2 ,$$

$$= 0,$$

and

$$(3.11.13) \quad g(a) = \frac{1}{2} q,$$

$$\neq 0.$$

Therefore, (f,g) is a type III solution of (*) on S.

LEMMA 3.12 Let S, F be as in lemma 3.9. Then (f,g) is a type IV solution of

$$(*) \quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into F if and only if f, g are functions on S into F of the form

$$(3.12.1) \quad \begin{cases} f(na) = \begin{cases} \beta & \text{if } n = 1 , \\ 0 & \text{otherwise} , \end{cases} \\ g(na) = \begin{cases} 1 & \text{it } n = 0 , \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

for all n in $\mathbb{N}$, where $\beta \in F^*$.

PROOF Let (f,g) be a type IV solution of (*) on S into F, hence

$$(3.12.2) \quad 2g(a)^2 - g(2a) = 0$$

and

$$(3.12.3) \quad g(a) = 0 ,$$

Let

(3.12.4) $\beta = f(a)$.

Since $f(a) \neq 0$, hence

(3.12.5) $\beta \notin F^*$.

From (3.12.2) and (3.12.3), we have

(3.12.6) $g(2a) = 0$.

From (*) and (3.12.3) respectively, we have

(3.12.7) $f(2a) = 2g(a)f(a),$

$$= 0.$$

Observe that for any n in $\mathbb{P}$ we have

(3.12.8) $f(2na) = g(2a)f(2(n-1)a) + g(2(n-1)a)\ f(2a),$

$$= 0,$$

We also have

(3.12.9) $f((2n+1)a) = g(2a)f((2n-1)a) + g((2n-1)a)f(2a),$

$$= 0,$$

Therefore

(3.12.10) $f(na) = 0$

for all n in $\mathbb{P}$ such that $n \neq 1$. For any n in $\mathbb{P}$, it follows from (*) that

$$f((n+1)a) = g(a)f(na) + g(na)f(a).$$

From (3.12.10), it follows that

(3.12.11) $g(na) = 0$

for all n in $\mathbb{P}$. Therefore, (f,g) are of the form (3.12.1).

Next, assume that f, g are functions on S into F of the form (3.12.1). For any n_1, n_2 in $\mathbb{P}$, we have

(3.12.12) $g(n_1a)f(n_2a) + g(n_2a)f(n_1a) = 0,$

$= f(n_1a + n_2a).$

It can be verified from (3.12.1) that (3.12.12) also hold for the cases where $n_1 = 0$ or $n_2 = 0$.

We see from (3.12.1) that

(3.12.13) $f(0) = 0,$

(3.12.14) $f(a) = \beta \neq 0,$

(3.12.15) $2g(a)^2 - g(2a) = 0,$

and

(3.12.16) $g(a) = 0.$

Therefore, (f,g) is a type IV solution of (*) on S.

We may now summarize the results obtained in corolary 3.3 and lemma 3.9 - lemma 3.12 in the following theorem.

<u>THEOREM 3.13</u> Let S be a cyclic monoid with generator a, F be an algebraically closed field of characteristic different from 2. Then (f,g) is a non-trivial solution of

$$(*)\quad f(x + y) = g(x)f(y) + g(y)f(x)$$

on S into F if and only if f, g are functions of the form

$$(3.13.1)\quad \begin{cases} f(na) = \beta q^n, \\ g(na) = \frac{1}{2} q^n \end{cases}$$

for all n in $\mathbb{N}$, where $\beta, q \in F^*$, or

$$(3.13.2)\quad \begin{cases} f(na) = \beta(q_1^n - q_2^n), \\ g(na) = \frac{1}{2}(q_1^n + q_2^n) \end{cases}$$

for all n in $\mathbb{N}$, where $\beta, q_1, q_2 \in F^*$ such that $q_1 \neq q_2$, or

$$(3.13.3)\quad \begin{cases} f(na) = n\beta q^n, \\ g(na) = q^n \end{cases}$$

for all n in $\mathbb{N}$, where $\beta, q \in F^*$, or

$$(3.13.4)\quad \begin{cases} f(na) = \begin{cases} 0 & \text{if } n = 0, \\ \beta q^n & \text{otherwise}, \end{cases} \\ g(na) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{1}{2} q^n & \text{otherwise} \end{cases} \end{cases}$$

for all n in $\mathbb{N}$, where $\beta, q \in F^*$, or

$$(3.13.5)\quad \begin{cases} f(na) = \begin{cases} \beta & \text{if } n = 1, \\ 0 & \text{otherwise}, \end{cases} \\ g(na) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

for all n in $\mathbb{N}$, where $\beta \in F^*$.