CHAPTER III

SEMILATTICES OF INVERSE SEMIGROUPS

From any given semilattice Y of inverse semigroups S_α, we can construct a semilattice Y of groups G_α such that for each $\alpha \in Y$, G_α is the maximum group homomorphic image of S_α. The latter semigroup so constructed is a homomorphic image of the given one.

Let $\bar{S}$ denote the semilattice of groups which we construct from a semilattice of inverse semigroups, S, as mentioned above. In this chapter, we show that if S is proper, then $\bar{S}$ is proper. Moreover, if S is F-inverse, then so is $\bar{S}$. The converses of both cases are true under some additional properties of S.

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. For each $\alpha \in Y$, let σ_α denote the minimum group congruence of S_α. Set

$$\bar{S} = \bigcup_{\alpha\in Y} S_\alpha/\sigma_\alpha$$

which is a disjoint union of groups. Define an operation * on $\bar{S}$ as follows :

$$(a\sigma_\alpha)*(b\sigma_\beta) = (ab)\sigma_{\alpha\beta} \qquad (\alpha,\ \beta \in Y,\ a \in S_\alpha,\ b \in S_\beta).$$

Then we have the following proposition :

3.1 <u>Proposition</u> ([7]). The operation * is well-defined, and $(\bar{S},*)$ becomes a semilattice Y of groups S_α/σ_α.

<u>Proof</u>. To show * is well-defined, let $a, a' \in S_\alpha$, $b, b' \in S_\beta$ such that $a\sigma_\alpha = a'\sigma_\alpha$ and $b\sigma_\beta = b'\sigma_\beta$. Then there exist $e \in E(S_\alpha)$,

$f \in E(S_\beta)$ such that $ea = ea'$, $bf = b'f$ [Introduction page 5]. From $ea = ea'$, we have $feab' = fea'b'$ and belongs to $S_{\alpha\beta}$, and then $(feab')\sigma_{\alpha\beta} = (fea'b')\sigma_{\alpha\beta}$. Since fe, ab' and $a'b' \in S_{\alpha\beta}$, we get $(fe)\sigma_{\alpha\beta} * (ab')\sigma_{\alpha\beta} = (fe)\sigma_{\alpha\beta} * (a'b')\sigma_{\alpha\beta}$. Because fe is an idempotent of $S_{\alpha\beta}$, $(fe)\sigma_{\alpha\beta}$ is an identity of $S_{\alpha\beta}/\sigma_{\alpha\beta}$ [Lemma 1.3]. Then we have $(ab')\sigma_{\alpha\beta} = (a'b')\sigma_{\alpha\beta}$. That is $ab'\sigma_{\alpha\beta}a'b'$. Similarly, we can prove that $ab\sigma_{\alpha\beta}ab'$. Hence $ab\sigma_{\alpha\beta}a'b'$ so that $(ab)\sigma_{\alpha\beta} = (a'b')\sigma_{\alpha\beta}$.

Let $a\sigma_\alpha \in S_\alpha/\sigma_\alpha$, $b\sigma_\beta \in S_\beta/\sigma_\beta$, $c\sigma_\gamma \in S_\gamma/\sigma_\gamma$. Then

$$((a\sigma_\alpha)*(b\sigma_\beta))*(c\sigma_\gamma) = ((ab)\sigma_{\alpha\beta})*(c\sigma_\gamma) = ((ab)c)\sigma_{(\alpha\beta)\gamma}.$$

Since S and Y are associative, we get

$$\begin{aligned} ((a\sigma_\alpha)*(b\sigma_\beta))*(c\sigma_\gamma) &= (a(bc))\sigma_{\alpha(\beta\gamma)} \\ &= (a\sigma_\alpha)*((bc)\sigma_{\beta\gamma}) \\ &= (a\sigma_\alpha)*((b\sigma_\beta)*(c\sigma_\gamma)). \end{aligned}$$

Hence * is associative. Therefore $(\bar{S}, *)$ is a semigroup. It follows from the definition of * that $S_\alpha/\sigma_\alpha * S_\beta/\sigma_\beta \subseteq S_{\alpha\beta}/\sigma_{\alpha\beta}$ for all $\alpha, \beta \in Y$. Hence, under the operation *, $\bar{S}$ is a semilattice Y of groups S_α/σ_α. #

From Lemma 1.3, for any $\alpha \in Y$, $e \in E(S_\alpha)$, we have $e\sigma_\alpha$ is the identity of the group S_α/σ_α. Hence, we immediately get

$$E(\bar{S}) = \{e\sigma_\alpha / \alpha \in Y,\ e \in E(S_\alpha)\}.$$

The semilattice of groups constructed from a semilattice of inverse semigroups as above gives the following significant properties :

3.2 Theorem ([3]). Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Then $(\bar{S}, *)$ defined as above is a homomorphic image of S. Moreover, S and $(\bar{S}, *)$ have the same maximum group homomorphic image (up to isomorphism).

Proof. To show $(\bar{S}, *)$ is a homomorphic image of S, let a map $\psi : \bigcup_{\alpha \in Y} S_\alpha \longrightarrow \bigcup_{\alpha \in Y} S_\alpha/\sigma_\alpha$ be defined by

$$a\psi = a\sigma_\alpha$$

for all $\alpha \in Y$, $a \in S_\alpha$. Clearly, ψ is onto. Let $a \in S_\alpha$, $b \in S_\beta$. Then $ab \in S_{\alpha\beta}$ and

$$(ab)\psi = (ab)\sigma_{\alpha\beta} = (a\sigma_\alpha)*(b\sigma_\beta) = (a\psi)*(b\psi).$$

Hence ψ is a homomorphism.

Next, we show that S and $(\bar{S}, *)$ have the same maximum group homomorphic image. Since $(\bar{S}, *)$ is a semilattice of groups, $\bar{S}$ is an inverse semigroup. Hence $\bar{S}$ has the minimum group congruence. Let σ and $\bar{\sigma}$ denote the minimum group congruences of S and $\bar{S}$; respectively. We will show that there exists an isomorphism from S/σ onto $\bar{S}/\bar{\sigma}$. Define the mapping $\theta : S/\sigma \longrightarrow \bar{S}/\bar{\sigma}$ by

$$(a\sigma)\theta = (a\sigma_\alpha)\bar{\sigma}$$

for all $\alpha \in Y$, $a \in S_\alpha$. To show θ is well-defined, let $a \in S_\alpha$ and $b \in S_\beta$ such that $a\sigma = b\sigma$. Therefore $ae = be$ for some $e \in E(S)$, say $e \in S_\gamma$. Pick $f \in E(S_{\alpha\beta})$ (such f always exists because $S_{\alpha\beta}$ is an inverse semigroup). Then $aef = bef$ and it belongs to $S_{\alpha\beta\gamma}$. Hence $(aef)\sigma_{\alpha\beta\gamma} = (bef)\sigma_{\alpha\beta\gamma}$ which implies $(aef)\sigma_{\alpha(\gamma\alpha\beta)} = (bef)\sigma_{\beta(\gamma\alpha\beta)}$ so that $(a\sigma_\alpha)*(ef)\sigma_{\gamma\alpha\beta} = (b\sigma_\beta)*(ef)\sigma_{\gamma\alpha\beta}$. Since $ef \in E(S_{\gamma\alpha\beta})$, $(ef)\sigma_{\gamma\alpha\beta}$ is an idempotent of $\bar{S}$ and hence $(a\sigma_\alpha)\bar{\sigma} = (b\sigma_\beta)\bar{\sigma}$.

For $a \in S_\alpha$, $b \in S_\beta$, we have

$$\begin{aligned} ((a\sigma)(b\sigma))\theta &= ((ab)\sigma)\theta \\ &= ((ab)\sigma_{\alpha\beta})\bar{\sigma} \\ &= ((a\sigma_\alpha)*(b\sigma_\beta))\bar{\sigma} \\ &= ((a\sigma_\alpha)\bar{\sigma})\ ((b\sigma_\beta)\bar{\sigma}) \\ &= ((a\sigma)\theta)\ ((b\sigma)\theta). \end{aligned}$$

Hence θ is a homomorphism. To show θ is one-to-one, let $\alpha, \beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$ such that $(a\sigma_\alpha)\bar{\sigma} = (b\sigma_\beta)\bar{\sigma}$. Therefore $(a\sigma_\alpha)*(e\sigma_\gamma) = (b\sigma_\beta)*(e\sigma_\gamma)$ for some $\gamma \in Y$, $e \in E(S_\gamma)$ so that $(ae)\sigma_{\alpha\gamma} = (be)\sigma_{\beta\gamma}$. Let $f \in E(S_{\alpha\beta})$. Then $(ae)\sigma_{\alpha\gamma}*(f\sigma_{\alpha\beta}) = (be)\sigma_{\beta\gamma}*(f\sigma_{\alpha\beta})$ which implies $(aef)\sigma_{\alpha\beta\gamma} = (bef)\sigma_{\alpha\beta\gamma}$ and so $aefg = befg$ for some $g \in E(S_{\alpha\beta\gamma})$. Hence $a\sigma = b\sigma$ because $efg \in E(S)$. Clearly θ is onto.

This completes the proof that $S/\sigma \cong \bar{S}/\bar{\sigma}$, that is, S and $\bar{S}$ have the same maximum group homomorphic image. #

Theorem 3.2 gives the following two corollaries :

3.3 <u>Corollary</u>. Let Y be a semilattice with a zero element 0, and let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Then S and S_0 have the same maximum group homomorphic image.

<u>Proof</u>. Let $\bar{S} = \bigcup_{\alpha\in Y} S_\alpha/\sigma_\alpha$ and the operation $*$ on $\bar{S}$ be as the beginning of the chapter. Then $(\bar{S}, *)$ is a semilattice Y of groups S_α/σ_α and Y has the zero element 0. By Proposition 1.1, S_0/σ_0 is the maximum group homomorphic image of $(\bar{S}, *)$, that is, $\bar{S}$ and S_0 have the same maximum group homomorphic image. Hence, by Theorem 3.2,

we have S and S_0 have the same maximum group homomorphic image. #

3.4 <u>Corollary</u>. Let Y be a finite semilattice, and let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Then there exists η in Y such that S and S_η have the same maximum group homomorphic image.

<u>Proof</u>. Since Y is a finite semilattice, Y has the zero element (which is the product of all elements in Y), say η. Hence, by Corollary 3.3, S and S_η have the same maximum group homomorphic image. #

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Let $\bar{S} = \bigcup_{\alpha \in Y} S_\alpha/\sigma_\alpha$ and the operation * be as before. The next theorem shows that being proper of S and of $\bar{S}$ are related very closely.

3.5 <u>Theorem</u> ([3]). S is proper if and only if $\bar{S}$ is proper and S_α is proper for all $\alpha \in Y$.

<u>Proof</u>. Assume S is proper. Then S_α is proper for all $\alpha \in Y$ because for each $\alpha \in Y$, S_α is an inverse subsemigroup of S. We will prove that $\bar{S}$ is proper by Proposition 1.5. Let $\bar{\psi}_{\alpha,\beta}$ $(\alpha \geq \beta)$ be the corresponding homomorphisms of $\bar{S}$. Assume $\alpha \geq \beta$, $a\sigma_\alpha \in S_\alpha/\sigma_\alpha$ and $e \in E(S_\beta)$ such that $(a\sigma_\alpha)\bar{\psi}_{\alpha,\beta} = e\sigma_\beta$. Therefore $(a\sigma_\alpha)*(e\sigma_\beta) = e\sigma_\beta$ which implies $(ae)\sigma_{\alpha\beta} = e\sigma_\beta$. Since $\alpha\beta = \beta$, we have $ae\sigma_\beta = e\sigma_\beta$ and so $aef = ef$ for some $f \in E(S_\beta)$. Since S is proper and $ef \in E(S)$, $a \in E(S)$ so that $a \in E(S_\alpha)$. Hence $a\sigma_\alpha$ is the identity of S_α/σ_α. This proves $\bar{\psi}_{\alpha,\beta}$ is one-to-one. Thus $\bar{S}$ is proper.

Conversely, assume $\bar{S}$ is proper and S_α is proper for all $\alpha \in Y$.

Let $a \in S$, $e \in E(S)$, say $a \in S_\alpha$, $e \in S_\beta$, such that $ae = e$. Since $ae \in S_{\alpha\beta}$ and $e \in S_\beta$, we have $\alpha\beta = \beta$ and so $(ae)\sigma_{\alpha\beta} = e\sigma_\beta$ which implies $(a\sigma_\alpha)*(e\sigma_\beta) = e\sigma_\beta$. Hence $a\sigma_\alpha$ is the identity of S_α/σ_α since $\bar{S}$ is proper. Let $f \in E(S_\alpha)$. Then $a\sigma_\alpha = f\sigma_\alpha$. Since S_α is a proper inverse semigroup, $f\sigma_\alpha = E(S_\alpha)$ [Introduction page 6] and hence $a \in E(S_\alpha) \subseteq E(S)$. Thus, S is proper. #

Next we will show that $\bar{S}$ is F-inverse if S is. To prove this, we need the following Lemma :

3.6 Lemma. From the above notation, for $\alpha, \beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$ we get the following relation :

$$a\sigma b \text{ if and only if } (a\sigma_\alpha)\bar{\sigma}(b\sigma_\beta).$$

Proof. Let $\alpha, \beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$. Assume $a\sigma b$. Then $ae = be$ for some $e \in E(S)$, say $e \in S_\gamma$. Hence $\alpha\gamma = \beta\gamma$, so $(ae)\sigma_{\alpha\gamma} = (be)\sigma_{\beta\gamma}$ which implies $(a\sigma_\alpha)*(e\sigma_\gamma) = (b\sigma_\beta)*(e\sigma_\gamma)$. Since $e\sigma_\gamma \in E(\bar{S})$, we have $(a\sigma_\alpha)\bar{\sigma}(b\sigma_\beta)$.

To prove the converse, assume $(a\sigma_\alpha)\bar{\sigma}(b\sigma_\beta)$. Therefore $(a\sigma_\alpha)*(e\sigma_\gamma) = (b\sigma_\beta)*(e\sigma_\gamma)$ for some $\gamma \in Y$, $e \in E(S_\gamma)$ which implies $(ae)\sigma_{\alpha\gamma} = (be)\sigma_{\beta\gamma}$. Let $f \in E(S_{\alpha\beta})$. Then $(ae)\sigma_{\alpha\gamma}*f\sigma_{\alpha\beta} = (be)\sigma_{\beta\gamma}*f\sigma_{\alpha\beta}$ so that $(aef)\sigma_{\alpha\beta\gamma} = (bef)\sigma_{\alpha\beta\gamma}$. Hence $aefg = befg$ for some $g \in E(S_{\alpha\beta\gamma})$, and so $a\sigma b$ because $efg \in E(S)$. #

3.7 Theorem. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α, and $\bar{S}$ and the operation * be defined as before. Then, if S is F-inverse, then $\bar{S}$ is F-inverse.

Proof. Let σ and $\bar{\sigma}$ be the minimum group congruences of S and $\bar{S}$; respectively and let $(a\sigma_\alpha)\bar{\sigma}$ be a $\bar{\sigma}$-class of $\bar{S}$. Since $a\sigma \in S/\sigma$ and S is F-inverse, $a\sigma$ has a maximum element m, say $m \in S_\beta$. Claim that $(m\sigma_\beta)$ is the maximum element of $(a\sigma_\alpha)\bar{\sigma}$ in $\bar{S}$. Let $x\sigma_\gamma \in (a\sigma_\alpha)\bar{\sigma}$. Then, from Lemma 3.6, we get $x \in a\sigma$. Therefore $x \leq m$ and so $x = me$ for some $e \in E(S)$ say $e \in S_\lambda$. Hence $\gamma = \beta\lambda$ so that $x\sigma_\gamma = (me)\sigma_{\beta\lambda}$ which implies $x\sigma_\gamma = (m\sigma_\beta)*(e\sigma_\lambda)$. Because $e\sigma_\lambda \in E(\bar{S})$, we obtain $x\sigma_\gamma \leq m\sigma_\beta$ in $\bar{S}$. Since $m \in a\sigma$, by Lemma 3.6, $m\sigma_\beta \in (a\sigma_\alpha)\bar{\sigma}$. Hence $m\sigma_\beta$ is the maximum element of $(a\sigma_\alpha)\bar{\sigma}$. #

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. To each $\alpha \in Y$, let $A_\alpha = \bigcup_{\beta \leq \alpha} S_\beta$. Then A_α is an ideal of S for all $\alpha \in Y$. To prove this, let $\alpha \in Y$, $a \in A_\alpha$ and $b \in S$. Then there exists $\beta \leq \alpha$ such that $a \in S_\beta$ and there exists $\gamma \in Y$ such that $b \in S_\gamma$. It then follows that $ab, ba \in S_{\beta\gamma}$. Since $\beta \leq \alpha$, $\beta\gamma \leq \alpha$ so that $ab, ba \in A_\alpha$.

In general, if S is F-inverse, A_α is not necessarily F-inverse, but it is if it has an identity. To prove this fact, we introduce the following Lemma first :

3.8 Lemma. Let A be an ideal of an inverse semigroup S. Then A is also inverse semigroup and $a\sigma(A) = a\sigma(S) \cap A$ for all $a \in A$.

Proof. Let $a \in A$. Then $a^{-1} = a^{-1}aa^{-1} \in A$. Hence A is an inverse semigroup. It is clear that $a\sigma(A) \subseteq a\sigma(S) \cap A$. To show the converse, let $x \in a\sigma(S) \cap A$. Then $x \in a\sigma(S)$ and $x \in A$. Since $x \in a\sigma(S)$, $xe = ae$ for some $e \in E(S)$. Let $f \in E(A)$. Therefore $xef = aef$ which implies $x\sigma(A)a$ because $ef \in E(A)$. Hence $x \in a\sigma(A)$. #

3.9 Proposition. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α, let $A_\alpha (\alpha \in Y)$ be defined as above. Let S be F-inverse. For $\alpha \in Y$, if A_α has an identity, then A_α is also F-inverse.

Proof. Let $\alpha \in Y$, σ_α^A be the minimum group congruence on A_α and $a\sigma_\alpha^A \in A_\alpha/\sigma_\alpha^A$. Therefore, by Lemma 3.8, we have $a\sigma_\alpha^A = a\sigma \cap A_\alpha$. Since S is F-inverse, $a\sigma$ has a maximum element say m so that $a\sigma m$. Assume A_α has the identity 1_α. Because A_α is an ideal, $m1_\alpha \in A_\alpha$. Since $m\sigma m1_\alpha$, by the transivity of σ we have $a\sigma m1_\alpha$. Thus $m1_\alpha \in a\sigma_\alpha^A$ because $m1_\alpha \in a\sigma$ and $m1_\alpha \in A_\alpha$. We claim that $m1_\alpha$ is the maximum element of $a\sigma_\alpha^A$. Let $x \in a\sigma_\alpha^A$. Then $x \in a\sigma$ and $x \in A_\alpha$. Because $x \in a\sigma$, we get $x \leq m$ which implies $x1_\alpha \leq m1_\alpha$. Since $x \in A_\alpha$ and 1_α is the identity of A_α, $x \leq m1_\alpha$. #

The last theorem shows that if we replace the words "proper" by "F-inverse" and "S_α" by "A_α" in Theorem 3.5, we still get the converse of the theorem.

3.10 Theorem. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α, and $(\bar{S}, *)$ be defined from S as before. If $\bar{S}$ is F-inverse and A_α is F-inverse for all $\alpha \in Y$, then S is F-inverse.

Proof. Let σ and $\bar{\sigma}$ be the minimum group congruences of S and $\bar{S}$; respectively. To show S is F-inverse, let $a\sigma \in S/\sigma$ say $a \in S_\alpha$. Since $(a\sigma_\alpha)\bar{\sigma} \in \bar{S}/\bar{\sigma}$ and $\bar{S}$ is F-inverse, $(a\sigma_\alpha)\bar{\sigma}$ has the maximum element $m\sigma_\beta$. Let $x \in a\sigma$. Then $x \in S_\gamma$ for some $\gamma \in Y$. Hence, by Lemma 3.6, we get $(x\sigma_\gamma) \in (a\sigma_\alpha)\bar{\sigma}$ and so $x\sigma_\gamma \leq m\sigma_\beta$. By the equivalent definition of the natural partial order $\leq$, it follows that $(x\sigma_\gamma)_*^{-1}(x\sigma_\gamma) = (x\sigma_\gamma)_*^{-1}(m\sigma_\beta)$

and so by Introduction page 4, $(x^{-1}x)\sigma_{\gamma} = (x^{-1}m)\sigma_{\gamma\beta}$ which implies $\gamma = \gamma\beta$ so that $\gamma \leq \beta$. Then $S_{\gamma} \subseteq A_{\beta}$ and hence $x \in A_{\beta}$. Thus $a\sigma \subseteq A_{\beta}$. Let σ_{β}^{A} denote the minimum group congruence on A_{β}. Then, by Lemma 3.8, $a\sigma_{\beta}^{A} = a\sigma \cap A_{\beta} = a\sigma$ because $a\sigma \subseteq A_{\beta}$. Hence $a\sigma$ has a maximum element because A_{β} is F-inverse. #

APPENDIX

EXAMPLES OF INVERSE SEMIGROUPS

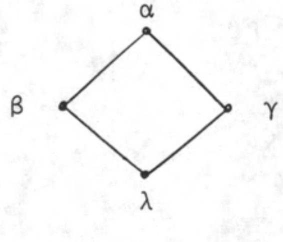
A.1 An example of an inverse semigroup which is not proper.

Let $S = \{a, e, 1\}$, where, for all $x, y \in S$, $xy = yx$; 1 is the identity of S; $a^2 = 1$, $e^2 = e$, $ea = e$. Then S is a regular semigroup. Since S is commutative, S is an inverse semigroup [Introduction page 1]. Because $ae = e$, $a \notin E(S)$ and $e \in E(S)$, it follows that S is not a proper inverse semigroup, and hence S is not F-inverse.

We note that any semilattice without identity is a proper inverse semigroup but not an F-inverse semigroup, and any semilattice with identity is an F-inverse semigroup.

A.2 An example of an inverse semigroup which is proper and has an identity but is not F-inverse.

Let $Y = \{\alpha, \beta, \gamma, \lambda\}$ be a semilattice with Hasse diagram



Let $\mathbb{Z}$ denote the set of all integers,

$$G_\alpha = 12\mathbb{Z}\times\{\alpha\} = \{(12n, \alpha) \mid n \in \mathbb{Z}\},$$

$$G_\beta = 6\mathbb{Z}\times\{\beta\},$$

$$G_\gamma \quad = \quad 2\mathbb{Z}\times\{\gamma\}$$

and
$$G_\lambda \quad = \quad \mathbb{Z}\times\{\lambda\}.$$

Then for each $\delta \in Y$, G_δ is a group under the operation defined by

$$(k_1, \delta)(k_2, \delta) \quad = \quad (k_1 + k_2, \delta).$$

Set $S = G_\alpha \cup G_\beta \cup G_\gamma \cup G_\lambda$, and define the operation on S by

$$(n_1, \delta_1)(n_2, \delta_2) \quad = \quad (n_1 + n_2, \delta_1\delta_2)$$

for all $\delta_1, \delta_2 \in Y$, $(n_1, \delta_1) \in G_{\delta_1}$, $(n_2, \delta_2) \in G_{\delta_2}$. It then follows clearly that S is a semilattice of groups, and hence S is an inverse semigroup. Moreover,

$$E(S) \quad = \quad \{(0, \alpha), (0, \beta), (0, \gamma), (0, \lambda)\}$$

and for any $(n, \delta) \in S$,

$$(n, \delta)^{-1} \quad = \quad (-n, \delta).$$

Since α is the identity of Y, $(0, \alpha)$ is the identity of S. To show that S is proper, let $(n, \delta_1) \in S$ and $(0, \delta_2) \in E(S)$ such that $(n, \delta_1)(0, \delta_2) = (0, \delta_2)$. Then $(n, \delta_1\delta_2) = (0, \delta_2)$ so that $n = 0$ and hence $(n, \delta_1) = (0, \delta_1) \in E(S)$.

For (n, δ_1) and $(m, \delta_2) \in S$, if $(n, \delta_1)\sigma(m, \delta_2)$, then

$$(n, \delta_1)(0, \delta_3) \quad = \quad (m, \delta_2)(0, \delta_3)$$

for some $\delta_3 \in Y$ so that $n = m$. If $(n, \delta_1), (n, \delta_2) \in S$, then

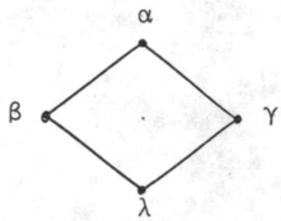
$$(n, \delta_1)(0, \delta_1\delta_2) \quad = \quad (n, \delta_2)(0, \delta_1\delta_2)$$

and so $(n, \delta_1)\sigma(n, \delta_2)$. Therefore for any $(n, \delta_0) \in S$, we have

$$(n, \delta_0)\sigma \quad = \quad \{(n, \delta) \mid \delta \in Y, (n, \delta) \in S\}.$$

Hence $(6, \beta)\sigma = \{(6, \beta), (6, \gamma), (6, \lambda)\}$. Since $(6, \lambda) \leq (6, \beta)$, $(6, \lambda) \leq (6, \gamma)$ and $(6, \beta)$ and $(6, \gamma)$ are not related, $(6, \beta)\sigma$ has no maximum element. Therefore S is not an F-inverse semigroup.

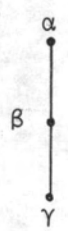
An inverse subsemigroup of an F-inverse semigroup need not be F-inverse. For example, let $S = \{\alpha, \beta, \gamma, \lambda\}$ be a semilattice with the Hasse diagram



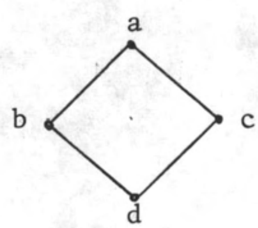
Because α is the identity of S, S is an F-inverse semigroup. Let $T = \{\beta, \gamma, \lambda\}$. Then T is an inverse subsemigroup of S but T is not F-inverse. The inverse subsemigroup T of S is also an ideal of S. This also shows that an ideal of an F-inverse semigroup is not necessarily F-inverse.

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. We recall that for each $\alpha \in Y$, $A_\alpha = \bigcup_{\beta\le\alpha} S_\beta$ is an ideal of S. The following example shows that A_α need not be F-inverse if S is F-inverse.

A.3 Example. Let $Y = \{\alpha, \beta, \gamma\}$ and $S = \{a, b, c, d\}$ be semilattices with Hasse diagrams



and



;

respectively. Then S is an F-inverse semigroup.

Let $S_\alpha = \{a\}$, $S_\beta = \{c\}$ and $S_\gamma = \{b, d\}$. Then $S = S_\alpha \cup S_\beta \cup S_\gamma$ is a semilattice Y of inverse semigroups S_α, S_β and S_γ. Since $A_\beta = \bigcup_{\substack{\delta \in Y \\ \delta \leq \beta}} S_\delta$, $A_\beta = S_\beta \cup S_\gamma = \{b, c, d\}$. Because A_β is a semilattice without identity, A_β is not F-inverse.