CHAPTER I

SOME PROPERTIES OF SEMILATTICES OF GROUPS

In this chapter, we study the maximum group homomorphic image of a semilattice of groups with semilattice having a zero element, and the equivalent condition of a semilattice of groups to be proper in terms of its corresponding homomorphisms. Finally, we introduce some suitable sufficient conditions of semilattices of groups to be F-inverse.

1.1 <u>Proposition</u>. Let Y be a semilattice with a zero element 0, and let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α. Then G_0 is the maximum group homomorphic image of S (up to isomorphism).

<u>Proof</u>. By Introduction page 5, S is an inverse semigroup. Then the minimum group congruence σ on S exists. For any $\alpha \in Y$, $a \in G_\alpha$, we have $ae_0 \in G_\alpha G_0 \subseteq G_{\alpha 0} = G_0$. Define a map $\theta : S/\sigma \longrightarrow G_0$ by

$$(a\sigma)\theta = ae_0 \qquad (a \in S)$$

If $a\sigma = b\sigma$, then $ae_\alpha = be_\alpha$ for some $\alpha \in Y$ [Introduction page 5] so that, by Introduction page 5,

$$ae_0 = ae_{\alpha 0} = (ae_\alpha)e_0 = (be_\alpha)\, e_0 = be_{\alpha 0} = be_0.$$

Therefore θ is well-defined. For any $a \in G_0$, $a = ae_0 = (a\sigma)\theta$, so θ is onto. To show θ is a homomorphism, let $a\sigma$, $b\sigma \in S/\sigma$. Then, by Introduction page 3,

$$((a\sigma)(b\sigma))\theta = ((ab)\sigma)\theta = abe_0 = (ae_0)(be_0) = ((a\sigma)\theta)((b\sigma)\theta).$$

If $(a\sigma)\theta = (b\sigma)\theta$, then $ae_0 = be_0$ so that $a\sigma = b\sigma$. Hence θ is one-to-one.

This show that $S/\sigma \cong G_0$, and hence G_0 is the maximum group homomorphic image of S (up to isomorphism). #

From this proposition, we then get the following corollary :

1.2 Corollary. Let Y be a finite semilattice and $S = \bigcup_{\alpha\in Y} G_\alpha$ be a semilattice Y of groups G_α. Then there exists η in Y such that G_η is the maximum group homomorphic image of S.

Proof. Assume $Y = \{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$. Let $\eta = \alpha_1 \alpha_2 \ldots\ldots \alpha_n$. Then $\eta\in Y$ and $\eta\leq\alpha_i$ for all $i\in\{1,2,\ldots\ldots n\}$. Hence η is the zero element of Y. By Proposition 1.1, G_η is the maximum group homomorphic image of S. #

1.3 Lemma. Let ρ be a group congruence on a semigroup S. Then E(S) is contained in the ρ-class which represents the identity of the group S/ρ.

Proof. Let $e\in E(S)$. Then

$$e\rho = (ee)\rho = (e\rho)(e\rho).$$

Since S/ρ is a group, the group inverse of $e\rho$, $(e\rho)^{-1}$, exists so that

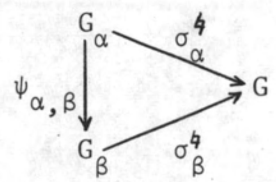
$$(e\rho)^{-1}(e\rho) = (e\rho)^{-1}(e\rho)(e\rho) = e\rho$$

Because $(e\rho)^{-1}(e\rho)$ is the identity of S/ρ, $e\rho$ is the identity of S/ρ and hence e belongs to the ρ-class which represents the identity of S/ρ. #

Lemma 1.3 can be restated as follows : If ρ is a group congruence of a semigroup S, then for any $e \in E(S)$, $e\rho$ is the identity of the group S/ρ.

The following proposition shows that in any semilattice of groups, its corresponding homomorphisms and its maximum group homomorphic image give commutative diagrams in a natural way.

1.4 Proposition. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ be a semilattice Y of groups G_α with corresponding homomorphisms $\psi_{\alpha,\beta}$ ($\alpha \geq \beta$). Let σ be the minimum group congruence on S. Then for any $\alpha, \beta \in Y$ with $\alpha \geq \beta$, the following diagram commutes :



where $G = S/\sigma$, and $\sigma^{\natural}_{\lambda} = \sigma^{\natural}/G_\lambda$ for all $\lambda \in Y$.

Proof. Let $\alpha, \beta \in Y$, $\alpha \geq \beta$ and $a \in G_\alpha$. Then

$$a(\psi_{\alpha,\beta}\sigma^{\natural}_{\beta}) = (a\psi_{\alpha,\beta})\sigma^{\natural}_{\beta} = (ae_\beta)\sigma^{\natural}_{\beta} = (ae_\beta)\sigma = (a\sigma)(e_\beta\sigma).$$

Since e_β is an idempotent and σ is a group congruence on S, by Lemma 1.3, $e_\beta\sigma$ is an identity of S/σ. Therefore

$$a(\psi_{\alpha,\beta}\sigma^{\natural}_{\beta}) = a\sigma = a\sigma^{\natural} = a\sigma^{\natural}_{\alpha}. \quad \#$$

Next, we study semilattices of groups which are proper. An equivalent condition of a semilattice of groups to be proper is that all of its corresponding homomorphisms are one-to-one [McAlister, 4].

1.5 Proposition. Let $S = \bigcup_{\alpha\in Y} G_\alpha$ be a semilattice Y of groups G_α and $\psi_{\alpha,\beta}$ $(\alpha \geq \beta)$ be its corresponding homomorphisms. Then the following are equivalent :

(i) S is proper.

(ii) All homomorphisms $\psi_{\alpha,\beta}$ are one-to-one.

Proof. Assume S is proper. Let $\alpha \geq \beta$. To show that $\psi_{\alpha,\beta}$ is one-to-one, let $a, b \in G_\alpha$ such that $a\psi_{\alpha,\beta} = b\psi_{\alpha,\beta}$. Then $ae_\beta = be_\beta$ and so $b^{-1}ae_\beta = b^{-1}be_\beta = e_\alpha e_\beta = e_{\alpha\beta}$ which implies $b^{-1}ae_\beta = e_\beta$. Since S is proper, $b^{-1}a \in E(S)$. But $b^{-1}a \in G_\alpha$. Then $b^{-1}a = e_\alpha$ and hence $a = b$.

Conversely, assume that all the homomorphisms $\psi_{\alpha,\beta}$ are one-to-one. Let $a \in S$, $e \in E(S)$ such that $ae = e$. Assume $a \in G_\alpha$ and $e \in G_\beta$. Then $e_\beta = ae_\beta \in G_{\alpha\beta}$ and so $\beta = \alpha\beta$ which implies $\alpha \geq \beta$. Hence $\psi_{\alpha,\beta}$ is defined. From $\beta = \alpha\beta$, we have $e_\beta = e_{\alpha\beta} = e_\alpha e_\beta$. Therefore $ae_\beta = e_\beta = e_\alpha e_\beta$ which implies $a\psi_{\alpha,\beta} = e_\alpha\psi_{\alpha,\beta}$. Since $\psi_{\alpha,\beta}$ is one-to-one, we have $a = e_\alpha$. Hence $a \in E(S)$. This shows that S is proper. #

Recall that an inverse semigroup S is called F-inverse if each $\sigma(S)$-class has a maximum element.

Any F-inverse semigroup is proper and has an identity. The converse is not necessarily true. An additional condition for a semilattice of groups to be F-inverse is given in the next proposition and then in its corollary.

1.6 Proposition. Let $S = \bigcup_{\alpha\in Y} G_\alpha$ be a semilattice Y of groups G_α satisfying the following conditions :

(i) Y has an identity 1,

(ii) S is proper, and

(iii) $S/\sigma = \{a\sigma / a \in G_1\}$.

Then S is F-inverse.

Proof. Let $x\sigma \in S/\sigma$. Then there exists $a \in G_1$ such that $x\sigma = a\sigma$. Let $y \in a\sigma$. Then $y\sigma = a\sigma$ and therefore $ye_\beta = ae_\beta$ for some e_β which implies $y^{-1}ye_\beta = y^{-1}ae_\beta$. Assume $y \in G_\gamma$. Then $y^{-1}y = e_\gamma$ and so $e_\gamma e_\beta = y^{-1}e_\gamma ae_\beta$. By Introduction page 6, we have $e_\gamma e_\beta = y^{-1}ae_\gamma e_\beta$ so that $e_{\gamma\beta} = y^{-1}ae_{\gamma\beta}$. Since S is proper, $y^{-1}a \in E(S)$. Then $y^{-1}a = e_\gamma$ because $y^{-1}a \in G_{\gamma 1} = G_\gamma$. It follows that $yy^{-1}a = ye_\gamma$ which implies $e_\gamma a = y$ and so $y \leq a$. Hence a is the maximum element of $x\sigma$. #

1.7 Corollary. Let $S = \bigcup_{\alpha\in Y} G_\alpha$ be a semilattice Y of groups G_α and $\psi_{\alpha,\beta}$ $(\alpha \geq \beta)$ be its corresponding homomorphisms such that

(i) Y has an identity 1,

(ii) S is proper, and

(iii) $\psi_{1,\alpha}$ is onto for all $\alpha \in Y$.

Then S is F-inverse and $S \cong G_1 \times Y \cong G_1 \times E(S)$.

Proof. By Theorem 1.6, to show S is F-inverse, it suffices to show $S/\sigma = \{a\sigma / a \in G_1\}$. Let $x\sigma \in S/\sigma$, say $x \in G_\alpha$. Since $\psi_{1,\alpha}$ is onto, there exists $a \in G_1$ such that $a\psi_{1,\alpha} = x$. Then $ae_\alpha = x$ which implies $ae_\alpha = xe_\alpha$. Hence $a\sigma = x\sigma$. This proves $S/\sigma = \{a\sigma / a \in G_1\}$.

Now we show $S \cong G_1 \times Y$. Let a map $\theta : G_1 \times Y \rightarrow S$ be defined by

$$(a,\alpha)\theta = a\psi_{1,\alpha} \qquad (a \in G_1,\ \alpha \in Y).$$

To show θ is a homomorphism, let $(a,\alpha), (b,\beta) \in G_1 \times Y$. Therefore

$$\begin{aligned} ((a,\alpha)(b,\beta))\theta &= (ab,\alpha\beta)\theta \\ &= (ab)\psi_{1,\alpha\beta} \\ &= (ab)e_{\alpha\beta} \\ &= abe_{\alpha}e_{\beta} \\ &= (ae_{\alpha})(be_{\beta}) \\ &= (a\psi_{1,\alpha})(b\psi_{1,\beta}) \\ &= ((a,\alpha)\theta)((b,\beta)\theta). \end{aligned}$$

Next, assume $(a,\alpha)\theta = (b,\beta)\theta$. Then $a\psi_{1,\alpha} = b\psi_{1,\beta}$ which implies $ae_{\alpha} = be_{\beta}$. Since $ae_{\alpha} \in G_{1\alpha} = G_{\alpha}$ and $be_{\beta} \in G_{1\beta} = G_{\beta}$, it follows that $\alpha = \beta$ and hence $ae_{\alpha} = be_{\alpha}$. Then $a\psi_{1,\alpha} = b\psi_{1,\alpha}$. By Proposition 1.5, $\psi_{1,\alpha}$ is one-to-one, so $a = b$ and hence $(a,\alpha) = (b,\beta)$. Therefore θ is one-to-one.

To show θ is onto, let $x \in S$. Then there exist $\alpha \in Y$ and $a \in G_1$ such that $a\psi_{1,\alpha} = x$ because $\psi_{1,\alpha}$ is onto. Hence $(a,\alpha)\theta = a\psi_{1,\alpha} = x$.

This shows that $S \cong G_1 \times Y$. But $Y \cong E(S)$, so $S \cong G_1 \times Y \cong G_1 \times E(S)$ as required. #