CHAPTER IV

HYPERTREES

4.1 r-tree

An r-tree is a connected acyclic r-graph. In [1] there exists a theorem about 2-tree the theorem state that " If H is 2-graph then the following statements are equivalent.

(1) H is a 2-tree.

(2) Every two vertices of H are joined by a unique path.

(3) H is connected and $\sum_{i=1}^{2} (-1)^{i+1} P_i = 1$.

(4) H is acyclic and $\sum_{i=1}^{2} (-1)^{i+1} P_i = 1$. "

In this chapter we will prove the theorem about characterization of r-tree.

Lemma 4.1.1 Let H be an r-tree with P_1 vertices such that $P_1 > r$. Then every two vertices of H are joined by a unique path.

Proof Since H is an r-tree, thus H is connected. Let P' and P" be distinct path joining u and v in H. Clearly there exists a vertex which is on both P' and P".

Case 1. Assume that as we traverse from u to v there exists a vertex on both P' and P" such that its successor on P' is not on P". Let w be the first vertex with this property. Let x be the next point which is on both P' and P", then the segments of P' and P" which are between w and x together from a cycle of H.

Case 2. Assume that as we traverse from u to v every vertex on P' is also on P". Hence we may assume that P' is the sequence $x_0, E'_1, x_1, E'_2, \ldots, x_q, E'_q, x_{q+1}$ and P" is the sequence $x_0, E''_1, x_2, E''_2, \ldots$ $\ldots, x_q, E''_q, x_{q+1}$. Since P' and P" are distinct, hence there must exist an i such that $E'_{i+1} \neq E''_{i+1}$. This implies that $x_i, E'_{i+1}, x_{i+1}, E''_{i+1}, x_i$ is a cycle of H.

We see that in any case we get a contradiction. Thus there is at most one path joining every pair of vertices.

Lemma 4.1.2 Let H be an r-graph with P_1 vertices such that $P_1 > r$. Let E be any edges of H. If every two vertices of H are joined by a unique path then

4.1.2.1 $H \setminus E$ is not connected

4.1.2.2 $H \setminus E$ has exactly r components.

Proof Let u, v be any two vertices of H such that E is on the path P from u to v. Since P is an only one path from u to v, thus in $H \setminus E$ we can not find a path from u to v. Hence $H \setminus E$ is not connected. So we get 4.1.2.1. Suppose $H \setminus E$ has $m < r$ components. Assume $E = \{v_1, v_2, \ldots, v_r\}$. Since $H \setminus E$ has $m < r$ components. Thus there exists v_i and v_j in E which is in the same components, say $H^{i,j}$. Since $H^{i,j}$ is connected, thus there is a path P from v_i to v_j. And P is also a path in H from v_i to v_j. In H, v_i, E, v_j is also a path from v_i to v_j. Thus there are two path from v_i to v_j in H. Hence we get a contradiction. And we observe that $m \nless r$. Hence $m = r$. Thus $H \setminus E$ has exactly r components.

Lemma 4.1.3 Let H be an r-graph with P_1 vertices such that $P_1 > r$. Let P_i be the number of i-edge in K(H). If every two vertices of H are joined by a unique path. Then H is connected and $P_i = \binom{r}{i} P_r$, $i = 2,\ldots, r-1$ and $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$.

Proof Clearly H is connected.

Assume $P_i = \binom{r}{i} P_r$, $i = 2,\ldots, r-1$ and $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$ is true for all r-graphs with P_1' vertices, where $r < P_1' < P_1$, and every vertices are joined by a unique path. Let E be any edge of H. Thus by lemma 4.1.2 $H \setminus E$ is not connected and $H \setminus E$ has exactly r components, say $H^1, H^2, \ldots, H^r$. Let P_i^j be the number of i-edge in $K(H^j)$.

Hence we get

$$P_i^j = \binom{r}{i} P_r^j \;; \quad i = 2,3,\ldots,r-1; \; j = 1,2,\ldots,r \qquad (4.1.3.1),$$

and

$$P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j = 1 , \qquad j = 1,2,\ldots,r \qquad (4.1.3.2),$$

we observe that

$$P_r = \sum_{j=1}^{r} P_r^j + 1 \qquad (4.1.3.3),$$

and

$$\sum_{j=1}^{r} P_i^j = P_i - \binom{r}{i} , \qquad i = 2,3,\ldots,r-1, \quad (4.1.3.4),$$

thus

$$P_i = \sum_{j=1}^{r} P_i^j + \binom{r}{i} \qquad (4.1.3.5),$$

from (4.1.3.1) and (4.1.3.5), we get

$$P_i = \sum_{j=1}^{r} \binom{r}{i} P_r^j + \binom{r}{i} ,$$

$$P_i = \binom{r}{i} \{ \sum_{j=1}^{r} P_r^j + 1 \}, \quad \ldots\ldots\ldots\ldots\ldots \quad (4.1.3.6)$$

from (4.1.3.3) and (4.1.3.6), we get

$$P_i = \binom{r}{i} P_r ,$$

from (4.1.3.2) $P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j = 1$, $j = 1,2,\ldots, r$,

thus $\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j = r \quad \ldots\ldots\ldots \quad (4.1.3.7)$,

observe $\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = \sum_{j=1}^{r} P_1^j + \sum_{j=1}^{r} \sum_{i=2}^{r} (-1)^{i+1} P_i^j$

$\ldots\ldots \quad (4.1.3.8)$,

and we observe $\sum_{j=1}^{r} P_1^j = P_1 \quad \ldots\ldots\ldots \quad (4.1.3.9)$

from (4.1.3.9) and (4.1.3.8), we get

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = P_1 + \sum_{j=1}^{r} \sum_{i=2}^{r} (-1)^{i+1} P_i^j ,$$

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = P_1 + \sum_{i=2}^{r} (-1)^{i+1} \sum_{j=1}^{r} P_i^j \quad \ldots\ldots \quad (4.1.3.10)$$

from (4.1.3.4) and (4.1.3.10), we get

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = P_1 + \sum_{i=2}^{r} (-1)^{i+1} \{ P_i - \binom{r}{1} \},$$

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = P_1 + \sum_{i=2}^{r} (-1)^{i+1} P_i - \sum_{i=2}^{r} (-1)^{i+1} \binom{r}{i},$$

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = P_1 + \sum_{i=2}^{r} (-1)^{i+1} P_i + \sum_{i=2}^{r} (-1)^i \binom{r}{i},$$

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = \sum_{i=1}^{r} (-1)^{i+1} P_i + \sum_{i=2}^{r} (-1)^i \binom{r}{i},$$

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = \sum_{i=1}^{r} (-1)^{i+1} P_i + \{ \sum_{i=0}^{r} (-1)^i \binom{r}{i} - (-1)^0 \binom{r}{0} - (-1)^1 \binom{r}{1} \} \quad .....(4.1.3.11).$$

By Binomial theorem

$$\sum_{i=0}^{r} (-1)^i \binom{r}{i} = (1-1)^r \quad ....(4.1.3.12)$$

from (4.1.3.12) and (4.1.3.11), we get

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = \sum_{i=1}^{r} (-1)^{i+1} P_i + \{ (1-1)^r - 1 + r \}$$

$$\sum_{j=1}^{r} (P_1^j + \sum_{i=2}^{r} (-1)^{i+1} P_i^j) = \sum_{i=1}^{r} (-1)^{i+1} P_i + r - 1 \quad ....(4.1.3.13)$$

from (4.1.3.7) and (4.1.3.13), we get

$$\sum_{i=1}^{r} (-1)^{i+1} P_i + r - 1 = r ,$$

therefore $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$

Lemma 4.1.4 If $P_i = \binom{r}{i}P_r$; $i = 2,3,\ldots, r-1$, then

$$\sum_{i=1}^{r} (-1)^{i+1}P_i = P_1 - (r-1)P_r.$$

Proof
$$\sum_{i=2}^{r} (-1)^{i+1} P_i = -\sum_{i=2}^{r} (-1)^i P_i ,$$

$$= -\sum_{i=2}^{r} (-1)^i \binom{r}{i} P_r,$$

$$= -P_r \sum_{i=2}^{r} \binom{r}{i}(-1)^i,$$

$$= -P_r\{ \sum_{i=0}^{r} \binom{r}{i}(-1)^i - \binom{r}{0}(-1)^0 - \binom{r}{1}(-1)^1\}$$

$$= -P_r \{(1-1)^r - 1 + r\},$$

$$= -(r-1)P_r .$$

Hence $\sum_{i=1}^{r} (-1)^{i+1} P_i = P_1 - (r-1)P_r$.

Lemma 4.1.5 Let H be an r-graph with P_1 vertices, and P_r edges, such that $P_1 > r$. If H is connected and H has a cycle then $P_1 - (r-1) P_r \leq 0$.

Proof Assume that H is connected and H has a cycle thus by remark 2.4.1 H has a simple cycle say P.
Assume P is $x_1, E_1, x_2, E_2, \ldots, E_n, x_{n+1}$, thus P is the cycle of length n. Let A be the set of edge on the cycle, thus $A = \{E_1, E_2, \ldots, E_n\}$. Let B be the set of edge not on the cycle,

thus $|B| = P_r - |A| = P_r - n$.

Let $C = \{v/\ v \varepsilon E \text{ and } E \varepsilon A\}$, thus $|C| = |\bigcup_{E_i \varepsilon A} E_i|$,

since $|\bigcup_{E_i \varepsilon A} E_i| = (|E_1|+|E_2|+\ldots+|E_n|) - |E_1 \cap E_2| - |E_2 \cap E_3| \ldots$

$$\ldots\ldots |E_{n-1} \cap E_n| - |E_n \cap E_1|,$$

since we have $|E_n \cap E_1| = 1$ and $|E_i \cap E_{i+1}| = 1$ for all $i = 1,\ldots,n-1$.

Thus $$|C| = |\bigcup_{E_i \varepsilon A} E_i| = nr - n,$$

$$= n(r-1).$$

Let $D = \{v/\ v \varepsilon X \text{ and } v \notin C\}$, since $|C \cup D| = P_1$ and $C \cap D = \phi$,

thus $$|D| = |C \cup D| - |C|,$$

$$|D| = P_1 - |C|,$$

$$= P_1 - n(r-1).$$

For any vertex in D, there corresponds an incident edge on a geodesic to a vertex of the cycle. Through this correspondence, each E in B corresponds to at most r-1 vertices in D. Hence there are at most $(r-1)|B|$ vertices in D. So that

$$|D| \leq (r-1)|B|,$$

$$P_1 - n(r-1) \leq (r-1)(P_r - n),$$

$$P_1 - n(r-1) \leq (r-1)P_r - n(r-1),$$

$$P_1 \leq (r-1)P_r,$$

hence $$P_1 - (r-1)P_r \leq 0.$$

<u>Theorem 4.1.6</u> Let H be an r-graph with P_1 vertices such that $P_1 > r$. Let P_i be the number of i-edge in K(H). The following statements are equivalent.

(4.1.6.1) H is an r-tree.

(4.1.6.2) Every two distinct vertices of H are joined by a unique path.

(4.1.6.3) H is connected and $P_i = \binom{r}{i} P_r$; $i = 2,3,..,r-1$ and $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$.

(4.1.6.4) H is acyclic and $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$ and $P_i = \binom{r}{i} P_r$; $i = 2,3,\ldots, r-1$.

<u>Proof</u> By lemma 4.1.1, we see that (4.1.6.1) implies (4.1.6.2)

By lemma 4.1.3, we see that (4.1.6.2) implies (4.1.6.3)

Next we shall show that (4.1.6.3) implies (4.1.6.4). Assume that H satisfies (4.1.6.3) and suppose that H has a cycle, thus by lemma 4.1.5

$\sum_{i=1}^{r} (-1)^{i+1} P_i \leqslant 0$, which is contary to the hypothesis $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$ of (4.1.6.3). Thus H is acyclic.

Next we shall show that (4.1.6.4) implies (4.1.6.1). Assume that H satisfies (4.1.6.4). Hence H is acyclic, therefore each component of H is an r-tree. Assume that H have k components, say $H^1, H^2, \ldots, H^k$. Let P_i^j be the number of i-edge in $K(H^j)$.

For each component H^j we have

$$\sum_{i=1}^{r} (-1)^{i+1} P_i^j = 1, \quad j = 1,2,\ldots, k ,$$

thus $$\sum_{j=1}^{k} \sum_{i=1}^{r} (-1)^{i+1} P_i^j = k \quad \ldots\ldots\ldots\ldots\ (1),$$

Observe that $$\sum_{j=1}^{k} \sum_{i=1}^{r} (-1)^{i+1} P_i^j = \sum_{i=1}^{r} (-1)^{i+1} \sum_{j=1}^{k} P_i^j \quad \ldots\ldots\ (2),$$

since H have k components thus,

$$P_i = \sum_{j=1}^{k} P_i^j \ , \quad i = 1,2,\ldots,r \quad \ldots\ldots\ldots\ (3),$$

from (2) and (3), $\sum_{j=1}^{k} \sum_{i=1}^{r} (-1)^{i+1} P_i^j = \sum_{i=1}^{r} (-1)^{i+1} P_i$.

Since H satisfies (4.1.6.4), hence

$$\sum_{i=1}^{r} (-1)^{i+1} P_i = 1 .$$

Therefore $$\sum_{j=1}^{k} \sum_{i=1}^{r} (-1)^{i+1} P_i^j = 1 \quad \ldots\ldots\ldots\ldots\ldots\ (4)$$

from (1) and (4), k = 1. Thus H is connected, therefore H is an r-tree.

<u>Lemma 4.1.7</u> Let $P_1 > r$, $r \geqslant 2$. Let $\pi = (d_1, d_2, \ldots, d_{P_1})$ where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{P_1}$, be given such that $d_i \geqslant 1$ for all i and $\frac{\Sigma d_i}{r}$

is a positive integer and $\frac{\Sigma d_i}{r} = \frac{P_1 - 1}{r-1}$. Then there exists j such that $d_j \geqslant 2$.

Proof Suppose for every j $d_j < 2$

i.e. for every j, $d_j = 1$.

Then $$\sum_{j=1}^{P_1} d_j = P_1 .$$

Since $$\frac{\Sigma d_i}{r} = \frac{P_1 - 1}{r-1},$$

$$\frac{P_1}{r} = \frac{P_1 - 1}{r-1},$$

$$P_1 r - P_1 = P_1 r - r .$$

Thus $P_1 = r$. So we get a contradiction. Hence there exists j such that $d_j \geqslant 2$. Hence the lemma is proved.

Lemma 4.1.8 Let $P_1 > r$, $r \geqslant 2$. Let $\pi = (d_1, d_2, \ldots, d_{P_1})$, where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_{P_1}$, be given such that $d_i \geqslant 1$ for all i and $\frac{\Sigma d_i}{r}$ is a positive integer and $\frac{\Sigma d_i}{r} = \frac{P_1 - 1}{r-1}$, then there exists at least r-1 d_i's such that $d_i = 1$.

Proof Suppose there exists at most r-2 d_i's such that $d_i = 1$. Thus there exists at least $P_1 - r + 2$ d_i's such that $d_i \geqslant 2$.

$$\begin{aligned}
\text{Hence} \qquad \Sigma d_i &\geq 2(P_1 - r + 2) + r - 2, \\
&= 2P_1 - 2r + 4 + r - 2, \\
&= 2P_1 - r + 2, \\
&> 2P_1 - r, \\
&= P_1 + (P_1 - r),
\end{aligned}$$

$$\begin{aligned}
\text{thus} \qquad \Sigma d_i &> P_1 + (P_1 - r), \\
(r-1)\,\Sigma d_i &> (r-1)P_1 + (r-1)(P_1 - r), \\
&\geq (r-1)P_1 + (P_1 - r), \\
&= rP_1 - r .
\end{aligned}$$

$$\text{Therefore} \qquad \frac{\Sigma d_i}{r} > \frac{P_1 - 1}{r-1} .$$

Thus we get a contradiction. Hence there exist at least $r-1$ d_i's such that $d_i = 1$.

<u>Theorem 4.1.9</u> Let $r \geq 2$, $P_1 > r$. Let $\pi = (d_1, d_2, \ldots, d_{P_1})$ where $d_1 \geq d_2 \geq \ldots \geq d_{P_1}$. Then π is a degree sequence of a non-trivial r-tree iff

(4.1.9.1) $\dfrac{\Sigma d_i}{r}$ is a positive integer,

(4.1.9.2) $d_i \geq 1$ for all i,

(4.1.9.3) $\dfrac{\Sigma d_i}{r} = \dfrac{P_1 - 1}{r-1}$.

Proof   First we proved the necessary part. Assume π is a degree sequence of some r-tree. Let H be an r-tree with degree sequence π. From remark 2.3.1

$$\Sigma d_i = rP_r ,$$

thus

$$\frac{\Sigma d_i}{r} = P_r ,$$

since P_r is a positive, therefore $\frac{\Sigma d_i}{r}$ is also a positive integer. Hence we get (4.1.9.1). Let u be any vertex of H. Since $r \geqslant 2$ and $P_1 > r$, thus there is another vertex v distinct from u. Since H is connected, then there is a path P from u to v say $P = u, E_1, x_1, E_2, x_2, \ldots, E_k, u$. Since $u \in E_1$, thus $d_H(u) \geqslant 1$. Hence every vertices of H has degree at least 1, so we get (4.1.9.2). Since H is an r-tree, thus from theorem 4.1.6, $\sum_{i=1}^{r} (-1)^{i+1} P_i = 1$ and from lemma 4.1.4, $\sum_{i=1}^{r} (-1)^{i+1} P_i = P_1 - (r-1)P_r$, thus

$$P_1 - (r-1)P_r = 1 ,$$

$$P_1 - 1 = (r-1)P_r ,$$

$$\frac{P_1 - 1}{r-1} = P_r ,$$

since $$P_r = \frac{\Sigma d_i}{r} ,$$

thus $$\frac{\Sigma d_i}{r} = \frac{P_1 - 1}{r-1} ,$$

so we get (4.1.9.3).

Next we prove the sufficiency part. This will be done by induction. Assume that if $\pi = (d_1, d_2, d_3, \ldots, d_{P_1})$ where $d_1 \geq d_2 \geq \ldots \geq d_{P_1}$ satisfies the condition $\frac{\Sigma d_i}{r}$ is a positive integer, $d_i \geq 1$ for all i and $\frac{\Sigma d_i}{r} = \frac{P_1 - 1}{r-1}$ such that $\frac{\Sigma d_i}{r} < n$, then π is a degree sequence of a nontrivial r-tree. Let $\pi^* = (d_1^*, d_2^*, d_3^*, \ldots, d_{P_1}^*)$ where $d_1^* \geq d_2^* \geq \ldots \geq d_{P_1}^*$ be such that (4.1.9.1),(4.1.9.2), and (4.1.9.3) hold, and $\frac{\Sigma d_1^*}{r} = n$ is a positive integer. We shall show that π^* is also a degree sequence of a non-trivial r-tree. Since π^* satisfy the condition in lemma 4.1.9, thus there exists j such that $d_j^* \geq 2$. Let j be the largest integer such that $d_j^* \geq 2$.

Let $\pi' = (d_1', d_2', \ldots, d_{P_1-r+1}')$ where

$$d_i' = d_i^* \text{ for all } i = 1,2,3,\ldots,j-1,j+1,\ldots,P_1 - r+1;$$

$$d_j' = d_j^* - 1 .$$

Since $d_j^* \geq 2$, thus $d_j^* - 1 \geq 1$ and therefore $d_i' \geq 1$; $i = 1,2,\ldots,P_1-r+1$. Hence π' satisfies condition (4.1.9.2) since $\pi^* = (d_1^*, d_2^*, \ldots, d_{P_1}^*)$, where $d_1^* \geq d_2^* \geq d_3^* \ldots \geq d_{P_1}^*$ and $d_i^* > 1$ for all i and $\frac{\Sigma d_i^*}{r} = n$ is a positive integer and $\frac{\Sigma d_i^*}{r} = \frac{P_1 - 1}{r-1}$, thus by lemma 4.1.8 there exist at least r-1 d_i^*'s such that $d_i^* = 1$.

Thus $$\sum_{i=P_1-r+2}^{P_1} d_i^* = r - 1.$$

Since $\sum_{i=1}^{P} d_i^* / r = n$, thus $\sum_{i=1}^{P_1} d_i^* = nr$,

$$\sum_{i=1}^{P_1-r+1} d_i^* + \sum_{i=P_1-r+2}^{P_1} d_i^* = nr ,$$

$$\sum_{i=1}^{P_1-r+1} d_i^* = nr - \sum_{i=P_1-r+2}^{P_1} d_i^* ,$$

$$\sum_{i=1}^{P_1-r+1} d_i^* = nr - (r-1).$$

Since $d_i' = d_i^*$ for all $i = 1,2,\ldots,j-1,j+1,\ldots,P_1-r+1$,

and $d_j' = d_j^* - 1$,

thus $$\sum_{i=1}^{P_1-r+1} d_i' = \sum_{i=1}^{P_1-r+1} d_i^* - 1 ,$$

$$\sum_{i=1}^{P_1-r+1} d_i' = nr - (r-1) - 1 ,$$

$$\sum_{i=1}^{P_1-r+1} d_i' = nr - r ,$$

$$\sum_{i=1}^{P_1-r+1} d_i' < nr .$$

Hence $$\frac{\Sigma d_i'}{r} < n .$$

And from $\Sigma d_i' = nr - r$,

$\Sigma d_i' = r(n-1)$,

thus $\frac{\Sigma d_i'}{r} = n - 1$.

Hence $\frac{\Sigma d_i'}{r}$ is a positive integer. Thus π' satisfies condition (4.1.9.1) and condition (4.1.9.3).

Thus π' satisfies the hypothesis, thus π' is a degree sequence of a nontrivial r-tree say $H' = (X', \mathcal{E}')$ where $X' = \{x_1, x_2, \ldots, x_j, x_{j+1}, \ldots, x_{P_1-r+1}\}$ such that $d_{H'}(x_i) = d_i'$, $i = 1, 2, \ldots, P_1 - r+1$. Let $H = (X, \mathcal{E})$ where $X = X' \cup \{x_{P_1-r}, x_{P_1-r+2}, x_{P_1-r+3} \ldots, x_{P_1}\}$ and $\mathcal{E}' = \mathcal{E}' \cup \{\{x_j, x_{P_1-r+2}, x_{P_1-r+3}, \ldots, x_{P_1}\}\}$, therefore H is an r-tree with degree sequence π^*.