CHAPTER III

DEGREE SEQUENCE

## 3.1 Degree Sequence

Let $(d_1, d_2, \ldots, d_p)$ be a sequence of non-negative integer. If there exists a hypergraph H with vertices $v_1, v_2, \ldots, v_p$ such that $d_H(v_i) = d_i (d_H^r(v_i) = d_i)$ we say that $\pi = (d_1, d_2, \ldots, d_p)$ is the degree sequence (r-degree sequence) of H. If $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_p$ we say that $\pi = (d_1, d_2, \ldots, d_p)$ is non-increasing degree sequence (non-increasing r-degree sequence) of H.

Havel and Hakimi [4], as cited in [1], found that if we let $\pi = (d_1, d_2, \ldots, d_p)$ be a sequence of non-negative integer in which $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_p$ then $\pi$ is a degree sequence of 2-graph if and only if $\pi' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p)$ is a degree sequence of a 2-graph.

The question " which sequence of non-negative integers are r-degree sequence ? " is answered by A.K.DEWDNEY [3]. We state and prove this result in theorem 3.1.2. First we prove the following lemma.

Lemma 3.1.1 Let $\pi = (d_1, d_2, \ldots, d_p)$ be a non-increasing r-degree sequence. Then there exists an r-graph $H = (X, \mathcal{E})$ with vertices $v_1, v_2, \ldots, v_p$ such that

(3.1.1.1) $H$ has degree sequence $\pi$ and

(3.1.1.2) $H' = (X', \mathcal{E}')$, where $X' = X \setminus \{v_1\}, \mathcal{E}' = \{E \setminus \{v_1\} / v_1 \varepsilon E$ and $E \varepsilon \mathcal{E}\}$, has a non-increasing $(r-1)$-degree sequence.

<u>Proof</u> Let $H^* = (X^*, \mathcal{E}^*)$ be an r-graph with vertices $v_1, v_2, \ldots, v_p$ such that

$$d^r_{H^*}(v_i) = d_i, \quad i = 1,2,\ldots,p$$

Let $H^{*'} = (X^{*'}, \mathcal{E}^{*'})$, where $X^{*'} = X^* - \{v_1\}, \mathcal{E}^{*'} = \{E^* \setminus \{v_1\} / v_1 \varepsilon E^*$ and $E^* \varepsilon \mathcal{E}\}$, Thus $H^{*'}$ is an $(r-1)$-graph, whose $(r-1)$-degree sequence is $(d^{r-1}_{H^{*'}}(v_2), \ldots, d^{r-1}_{H^{*'}}(v_p))$. If $(d^{r-1}_{H^{*'}}(v_2), d^{r-1}_{H^{*'}}(v_3), \ldots, d^{r-1}_{H^{*'}}(v_p))$ is not a non-increasing sequence, let $k^*$ be any integer for which

$$d^{r-1}_{H^{*'}}(v_{k^*}) < d^{r-1}_{H^{*'}}(v_{k^*+1}) \qquad \ldots\ldots\ldots\ldots(3.1.1.3)$$

hence there exists $(r-2)$ subset $S_1$ of $X^{*'} - \{v_{k^*}, v_{k^*+1}\}$ such that $S_1 \cup \{v_{k^*+1}\} \varepsilon \mathcal{E}^{*'}$ and $S_1 \cup \{v_{k^*}\} \notin \mathcal{E}^{*'}$. Let $T_1 = S_1 \cup \{v_1\}$. Since $S_1 \cup \{v_{k^*+1}\} \varepsilon \mathcal{E}^{*'}$; therefore $E^*_{k^*+1} = S_1 \cup \{v_{k^*+1}\} \cup \{v_1\}$

$= T_1 \cup \{v_{k^*+1}\} \varepsilon \mathcal{E}^*$, while $E_{k^*} = T_1 \cup \{v_{k^*}\} = S_1 \cup \{v_{k^*}\} \cup \{v_1\} \notin \mathcal{E}^*$.

Let $X^{*''} = X^* \setminus \{v_1\}$ and $\mathcal{E}^{*''} = \{E^* / E^* \varepsilon \mathcal{E}^*$ and $v_1 \notin E^*\}$. Hence $H^{*''} = (X^{*''}, \mathcal{E}^{*''})$ is an r-graph whose degree sequence is $(d^r_{H^{*''}}(v_2), d^r_{H^{*''}}(v_3), \ldots, d^*_{H^{*''}}(v_p))$. Observe that for any vertex $v_i$, $i = 2,3,\ldots,p$ we have

$$d^r_{H^*}(v_i) = d^{r-1}_{H^{*\prime}}(v_i) + d^r_{H^{*\prime\prime}}(v_i) \qquad .......(3.1.1.4).$$

In particular, for the case $i = k^*$ and $i = k^*+1$, we have

$$d^r_{H^*}(v_{k^*}) = d^{r-1}_{H^{*\prime}}(v_{k^*}) + d^r_{H^{*\prime\prime}}(v_{k^*}) \qquad .....(3.1.1.5)$$

$$d^r_{H^*}(v_{k^*+1}) = d^{r-1}_{H^{*\prime}}(v_{k^*+1}) + d^r_{H^{*\prime\prime}}(v_{k^*+1}) \quad ...(3.1.1.6)$$

From the assumption that $(d_1, d_2, \ldots, d_p)$ is non-increasing we have

$$d^r_{H^*}(v_{k^*}) \geqslant d^r_{H^*}(v_{k^*+1}) \qquad ...........(3.1.1.7)$$

From (3.1.1.5), (3.1.1.6), and (3.1.1.7) we have

$$d^{r-1}_{H^{*\prime}}(v_{k^*}) + d^r_{H^{*\prime\prime}}(v_{k^*}) \geq d^{r-1}_{H^{*\prime}}(v_{k^*+1}) + d^r_{H^{*\prime\prime}}(v_{k^*+1}) \ldots(3.1.1.8)$$

From (3.1.1.3) and (3.1.1.8), we get

$$d^r_{H^{*\prime\prime}}(v_{k^*}) > d^r_{H^{*\prime\prime}}(v_{k^*+1}),$$

thus there exists an (r-1)- subset of $T_2$ of $X'' - \{v_{k^*}, v_{k^*+1}\}$ such that $T_2 \cup \{v_{k^*}\} \in \mathcal{E}^{*\prime\prime}$ and $T_2 \cup \{v_{k^*+1}\} \notin \mathcal{E}^{*\prime\prime}$. Let $E^*_k = T_2 \cup \{v_{k^*}\}$ and $E_{k^*+1} = T_2 \cup \{v_{k^*+1}\}$. Hence $E^*_k \in \mathcal{E}^*$ and $E_{k^*+1} \notin \mathcal{E}^*$.

Construct a new r-graph $H = (X, \mathcal{E})$ where $X = X^*$ and $\mathcal{E} =$

$$= (\mathcal{E}^* \setminus \{E^*_{k^*+1}, E^*_{k^*}\}) \cup \{E_{k^*}, E_{k^*+1}\}.$$

Hence H is a r-graph with degree sequence $(d_1, d_2, \ldots, d_p)$. And construct a new (r-1)-graph $H' = (X', \mathcal{E}')$ where $X' = X \setminus \{v_1\}$ and $\mathcal{E}' = \{E \setminus \{v_1\} / v_1 \varepsilon\ E$ and $E \varepsilon \mathcal{E}\}$. From definition of $\mathcal{E}$ we see that $S_1 \cup \{v_{k^*}\} \varepsilon\ \mathcal{E}'$ and $S_1 \cup \{v_{k^*+1}\} \notin \mathcal{E}'$. So H' is a (r-1)-graph in which all vertices have the same (r-1)-degree excepts two vertices $v_{k^*}$ and $v_{k^*+1}$. i.e. $d_{H'}^{r-1}(v_{k^*})$ has increased by unity, while $d_{H}^{r-1}(v_{k^*+1})$ has decreased by unity. Inspecting the formula

$$\sum_{i=2}^{p} i \cdot d_{H'}^{r-1}(v_i) = \sum_{i=2}^{k^*-1} i \cdot d_{H'}^{r-1}(v_i) + k^* \cdot d_{H'}^{r-1}(v_k) + (k^*+1) d_{H'}^{r-1}(v_{k+1})$$

$$+ \sum_{i=k^*+2}^{p} i \cdot d_{H}^{r-1}(v_i),$$

the term $k^* \cdot d_{H'}^{r-1}(v_{k^*})$ has increased by $k^*$, while the next term $(k^*+1) \cdot d_{H'}^{r-1}(v_{k^*+1})$ has decreased by $k^*+1$. Hence $\sum_{i=2}^{p} i \cdot d_{H'}^{r-1}(v_i)$ has decreased by 1. If the new sequence $(d_{H'}^{r-1}(v_2), d_{H'}^{r-1}(v_3), \ldots, d_{H'}^{r-1}(v_p))$ in H' is not non-increasing then we can find a new integer k such that

$$d_{H'}^{r-1}(v_k) < d_{H'}^{r-1}(v_{k+1}),$$

and continuing the same process on H'. Each time we work the $\sum_{i=1}^{p} i \cdot d_{H'}^{r-1}(v_i)$ has decreased by 1. Since $\sum_{i=2}^{p} i \cdot d_{H'}^{r-1}(v_1)$ is a positive integer. So $\sum_{i=2}^{p} i \cdot d_{H'}^{r-1}(v_i)$ has decreased until we arrive at an (r-1)-graph in which the sequence $(d_{H'}^{r-1}(v_2), d_{H'}^{r-1}(v_3), \ldots, d_{H'}^{r-1}(v_p))$ is non-increasing.

<u>Theorem 3.1.2</u>

A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_p)$ is an r-degree sequence if and only if there is a non-increasing (r-1)-degree sequence $(d'_1, d'_2, \ldots, d'_q)$ for which the following conditions hold :

(3.1.2.1) $\sum_{i=1}^{q} d'_i = (r-1)d_1$

(3.1.2.2) $(d_2 - d'_1, d_3 - d'_2, \ldots, d_{q+1} - d'_q, d_{q+2}, \ldots, d_p)$ is a r-degree sequence.

<u>Proof</u> we prove the sufficiency part.

Let $H' = (X', \mathcal{E}')$ be an (r-1)-graph with vertices $v_2, \ldots, v_{q+1}$ such that

$$d^{r-1}_{H'}(v_i) = d'_{i-1}, \quad i = 2, 3, \ldots, q+1.$$

Let $H'' = (X'', \mathcal{E}'')$ be an r-graph with vertices $v_2, v_3, \ldots, v_p$ such that

$$\begin{aligned} d^r_{H''}(v_i) &= d_i - d'_{i-1}, \quad i = 2, \ldots, q+1 \\ &= d_i, \quad i = q+2, \ldots, p. \end{aligned}$$

Let $v_1$ be a vertex not in $X'$. Define $\mathcal{E}''' = \{\{v_1\} \cup E' \mid E' \in \mathcal{E}'\}$ so $|E| = r$ for every $E \in \mathcal{E}'''$.

Let $H = (X, \mathcal{E})$, where $X = \{v_1\} \cup X''$, $\mathcal{E} = \mathcal{E}'' \cup \mathcal{E}'''$. Since $|E| = r$ for every $E \in \mathcal{E}$, then H is an r-graph. Observe that $d^r_H(v_1) = |\mathcal{E}'| = |\mathcal{E}'''|$. By remark 2.3.1, the number of (r-1)-edges in H' is

$$\frac{\sum_{i=2}^{q+1} d^{r-1}_{H'}(v_i)}{r-1}$$

Thus $$d_H^r(v_1) = \frac{\sum_{i=2}^{q+1} d_{H'}^{r-1}(v_i)}{r-1},$$

$$= \frac{\sum_{i=2}^{q+1} d'_{i-1}}{r-1},$$

$$= \frac{\sum_{i=1}^{q} d'_i}{r-1},$$

from condition (3.1.2.1) $\sum_{i=1}^{q} d'_i = (r-1)d_1$,

thus $$d_H^r(v_1) = \frac{(r-1)\,d_1}{r-1}.$$

Hence $$d_H^r(v_1) = d_1 .$$

For each $i = 2,3,\ldots, q+1$, observe that

$$d_H^r(v_i) = d^r_{H''}(v_i) + d^{r-1}_{H'}(v_i),$$

$$= d_i - d'_{i-1} + d'_{i-1},$$

$$= d_i .$$

While for each $i = q+2, q+3,\ldots, p$,

$d_H^r(v_i) = d^r_{H''}(v_i) = d_i$ . Therefore $d_1, d_2,\ldots, d_p$

is an r-degree sequence.

Next we prove the necessary part.

By lemma 3.1.1 there exists an r-graph $H = (X, \mathcal{E})$ such that H has degree sequence $\pi$ with vertices $v_1, v_2, \ldots, v_p$, and $H' = (X', \mathcal{E}')$ where $X' = X \setminus \{v_1\}$ and $\mathcal{E}' = \{E \setminus \{v_1\} \mid v_1 \in E \text{ and } E \in \mathcal{E}\}$ has a non-increasing (r-1)-degree sequence $(d_{H'}^{r-1}(v_2), d_{H'}^{r-1}(v_3), \ldots, d_{H'}^{r-1}(v_p))$. By remark 2.3.1 the number of (r-1)-edges in H' is $\dfrac{\sum_{i=2}^{p} d_{H'}^{r-1}(v_i)}{r-1}$. Let q be the largest integer satisfying $d_{H'}^{r-1}(v_{q+1}) > 0$ thus



$$d_{H'}^{r-1}(v_i) = 0 \qquad \text{for } i > q+1$$

thus

$$\frac{\sum_{i=2}^{p} d_{H'}^{r-1}(v_i)}{r-1} = \frac{\sum_{i=2}^{q+1} d_{H'}^{r-1}(v_i)}{r-1}$$

By definition of $\mathcal{E}'$, we observe that the number of (r-1)-edges in H' is $d_H^r(v_1)$, therefore

$$\frac{\sum_{i=2}^{q+1} d_{H'}^{r-1}(v_i)}{r-1} = d_H^r(v_1) \;,$$

$$\sum_{i=2}^{q+1} d_{H'}^{r-1}(v_i) = (r-1) d_H^r(v_1),$$

$$= (r-1)\, d_1 \;.$$

Define $\qquad d_i' = d_{H'}^{r-1}(v_{i+1}) \;, \qquad i = 1, 2, \ldots, q,$

therefore $(d'_1, d'_2, \ldots, d'_q)$ is a (r-1) non-increasing degree

degree sequence of H' and $\sum_{i=1}^{q} d'_i = \sum_{i=1}^{q} d_{H'}^{r-1} (v_{i+1})$ ,

$$= \sum_{i=2}^{q+1} d_{H'}^{r-1} (v_i) ,$$

$$= (r-1)\, d_1 ,$$

thus the condition 3.1.2.1 is satisfies.

Let $X'' = X \smallsetminus \{v_1\}$ and let $\mathcal{E}'' = \{E \mid E \in \mathcal{E}$ and $v_1 \notin E\}$.

Hence $H'' = (X'', \mathcal{E}'')$ is an r-graph.

Observe that for any vertex $v_i$, $i = 2,3,\ldots, p$.

$$d_H^r (v_i) = d_{H'}^{r-1} (v_i) + d_{H''}^r (v_i) ,$$

$$d_{H''}^r (v_i) = d_H^r (v_i) - d_{H'}^{r-1} (v_i) ,$$

since $\quad d_H^{r-1} (v_i) = 0 \quad$ for $i > q+1$

thus $\quad d_{H''}^r (v_i) = d_i - d'_{i-1}, \quad i = 2,\ldots, q+1$

$$= d_i \; ; \qquad i > q+1$$

So the sequence $(d_2 - d'_1, d_3 - d'_2, \ldots, d_{q+1} - d'_q, d_{q+2}, \ldots, d_p)$ is an r-degree sequence.