CHAPTER III

CHARACTERIZATION OF THE ARC

A topological space is called an <u>arc</u> if it is homeomorphic to the unit interval subspace, [0, 1], of $\mathbb{R}$.

3.1 <u>Theorem</u>. If A is an arc, then A is a compact, connected, metrizable space with exactly two non-cut points.

<u>Proof</u>. By theorem 2.25, [0, 1] is a connected subspace of $\mathbb{R}$; by theorem 2.12, [0, 1] is a metrizable space, and by theorem 2.42, [0, 1] is a compact space. Thus, since connectedness, compactness and metrizability are topological properties, any arc is connected, compact and metrizable. By theorem 2.26, the open interval $\{x \mid 0 < x < 1\}$ is connected. By theorem 2.20, any set between a connected set and its closure is connected. Hence, $\{x \mid 0 \leq x < 1\}$ and $\{x \mid 0 < x \leq 1\}$ are connected. This means that 0 and 1 are non-cut points of [0, 1]. Now, let $0 < p < 1$. If $M = \{x \mid 0 \leq x < p\}$ and $N = \{x \mid p < x \leq 1\}$, then $[0, 1] - \{p\} = N \cup M$ separation and p is a cut point of [0, 1]. Since the property of being either cut point or non-cut point is a topological property, any arc has exactly two non-cut points. #

Let p and q be points of a connected space S. A point x in S is said to <u>separate p and q in S</u> if there exists a separation $P_x \cup Q_x$ of $S - \{x\}$ such that $p \in P_x$ and $q \in Q_x$. Thus, if x separates p and q in S, x must be a cut point of S. Define E(p, q) to be $\{x \mid x = p, \text{ or } x = q, \text{ or } x \text{ separates } p \text{ and } q \text{ in } S\}$.

3.2 Lemma. Let S be a connected space; let p and q be two distinct points of S; let a and b be two distinct points of E(p, q) - {p, q}. Let $P_a \cup Q_a$ be any separation of S - {a} such that $p \in P_a$ and $q \in Q_a$, let $P_b \cup Q_b$ be any separation of S - {b} such that $p \in P_b$ and $q \in Q_b$. Then $P_a \subset P_b$ or $P_b \subset P_a$, where $\subset$ denotes the proper subset relation.

Proof. Since $S - \{a\} = P_a \cup Q_a$ and $P_a \cap Q_a = \phi$ and $b \neq a$, $b \in P_a$ or $b \in Q_a$, but not both (a) Let $b \in P_a$. By corollary 2.32, $Q_a \cup \{a\}$ is a connected subset of S. Furthermore, since $b \neq a$ and $b \in P_a$, $b \notin Q_a \cup \{a\}$. Hence $Q_a \cup \{a\} \subseteq P_b \cup Q_b$. Since $q \in Q_a \cup \{a\}$ and $q \in Q_b$, the connected set $Q_a \cup \{a\} \subseteq Q_b$, by theorem 2.18. Hence $(Q_a \cup \{a\}) \cap (P_b \cup \{b\}) = \phi$. Since $Q_a \cup \{a\} \cup P_a = S$, $P_b \cup \{b\} \subseteq P_a$. This means, since $b \notin P_b$, that $P_b \subset P_a$. (b) Let $b \in Q_a$. Since $Q_a \cap P_a = \phi$ and $b \neq a$, $b \notin P_a \cup \{a\}$. Hence, since $S - \{b\} = P_b \cup Q_b$, $P_a \cup \{a\} \subseteq P_b \cup Q_b$. As above, since $P_a \cup \{a\}$ is connected and $p \in P_b \cap (P_a \cup \{a\})$, $P_a \cup \{a\} \subsetneq P_b$. Since $a \notin P_a$, $P_a \subset P_b$. #

3.3 Corollary. If a and b are any two distinct points of E(p, q) - {p, q}, then $P_a \subset P_b$ is equivalent to $a \in P_b$, where P_a and P_b are as defined in the lemma 3.2.

Proof. In the proof of lemma 3.2, it was established that

$b \in P_a$ implied that $a \in Q_b$. It follows from this then that if $a \notin Q_b$, then $b \notin P_a$. Hence, if $a \notin Q_b$, $b \in Q_a$. It was also shown that $b \in Q_a$ implied that $P_a \subset P_b$. Thus it follows that if $a \in P_b$, then $a \notin Q_b$, and so $P_a \subset P_b$. Conversely, let $P_a \subset P_b$. Then $P_b \not\subset P_a$. From the proof lemma 3.2, this means that $b \notin P_a$. Thus, $b \in Q_a$. This was proved to imply that $a \in P_b$. #

By a <u>linearly</u> <u>ordered set</u> we mean a pair (L, <*), where L is a set, and <* is an order relation on L which is (1) x <* x for no x in L, (2) x <* y implies y ≮* x, (3) x <* y and y <* z implies x <* z, and (4) x <* y or y <* x for any two distinct points x and y in L. We write x ≤* y if x = y or x <* y. Let S be a connected space. Let p and q be distinct points of S. A relation <* on E(p, q) is defined as follows. $<^* = \{(x, y) \mid (x = p \text{ and } y \neq p)$ or $(y = q \text{ and } x \neq q)$ or $(P_x \subset P_y$, for $p \neq x \neq q$ and $p \neq y \neq q)\}$ where of course $P_x \cup Q_x$ is any separation of $S - \{x\}$ such that $p \in P_x$ and $q \in Q_x$ and $P_y \cup Q_y$ is any separation of $S - \{y\}$ such that $p \in P_y$ and $q \in Q_y$.

The following lemma follows from lemma 3.2 and the definition of <* on E(p, q).

3.4 <u>Lemma</u>. <* is a linear order relation on E(p, q) in the connected space S. #

Thus, in any connected space S, each two-point set {p, q} determines a linearly ordered subset (E(p, q), <*).

3.5 Lemma. Let S be a compact, connected T_1-space with exactly two non-cut points, p and q. Then, S = E(p, q).

Proof. Let $x \in S$ and $x \notin E(p, q)$. Then $p \neq x \neq q$. Hence x is a cut point of S. Since x does not separate p and q in S, there exists a separation $A \cup B$ of S - x such that p and q are both in A. Since S has no non-cut points but p and q, it is a contradiction by corollary 2.41. Thus, every point of S - {p, q} separates p and q in S, and so S = E(p, q). #

The lemma 3.5 established then that any compact, connected T_1-space with exactly two non-cut points is linearly ordered by <*.

Let S be a set and < any linear order relation on S. Let the topology on S be the topology having as a subbase $\{A \mid (A \subseteq S)$ and $(A = \{x \mid x < b\}$ or $A = \{x \mid a < x\}$ for a and b in S$)\}$. This topology is called the <u>order topology</u> for S determined by <. The following lemma is clear.

3.6 Lemma. Let S be any non-empty set and < any linear order relation on S. The order topology determined by < is Hausdorff. #

3.7 Theorem. Let $(S, \mathcal{T})$ be a compact, connected, Hausdorff space with exactly two non-cut points. Then the original topology $\mathcal{T}$ on S is an order topology for S.

Proof. By lemma 3.4 and 3.5, S can be linearly ordered by the relation <*. S = E(p, q), where p and q are the two non-cut points of S. Let $0 - \mathcal{T}$ denote the order topology for $E(p, q)$ determined by $<^*$. Let B be a base element determined by the defining subbase for the order topology. $B \subseteq S = E(p, q)$, $B = \{x \mid x <^* b$ for $b \in S\}$, $B = \{x \mid a <^* x$ for $a \in S\}$ or $B = \{x \mid a <^* x <^* b$ for $a, b \in S\}$. Now $S \in \mathcal{T}$. If $p \neq a \neq q$ and $p \neq b \neq q$, then by the definition of $<^*$ and by corollary 3.3, $\{x \mid x <^* b\} = P_b$, $\{x \mid a <^* x <^* b\} = P_b \cap Q_a$ and $\{x \mid a <^* x\} = Q_a$. If $s \in S$ such that $p \neq s \neq q$, $S - \{s\} = P_s \cup Q_s$ separation. Since $(S, \mathcal{T})$ is a T_1-space, P_s and Q_s are open in $(S, \mathcal{T})$. It follows now that if $a = p$ or $a = q$ or $b = p$ or $b = q$, $B \in \mathcal{T}$. Thus, each B in the base for the order topology, determined by the defining subbase, is open in $(S, \mathcal{T})$. Thus $0 - \mathcal{T} \subseteq \mathcal{T}$. Since $(S, \mathcal{T})$ is a compact, Hausdorff space, by hypothesis, and since, by lemma 3.6, $0 - \mathcal{T}$ is Hausdorff, $0 - \mathcal{T} = \mathcal{T}$ by theorem 2.47. #

3.8 Theorem. If $(D, <^*)$ is a countable, linearly ordered set and if $<^*$ has the following two properties : (1) $<^*$ determines no first and no last element in D, and (2) for any two distinct elements a and b in D such that $a <^* b$, there is at least one other element c such that $a <^* c <^* b$; then there exists a one-one order-preserving function from D onto the rational numbers in the open interval (0, 1).

Proof. Since both D and the set Q of rational in (0, 1) are countable, D can be represented as $\{d_1, d_2, \ldots\}$, where $d_i \neq d_j$

for $i \neq j$, and Q can be represented as $\{r_1, r_2, \ldots\}$, where $r_i \neq r_j$ for $i \neq j$. The given linear ordering on D will be denoted, as in the hypothesis, by $<^*$. The natural ordering on the set of all natural numbers and on the set Q of rational numbers in (0, 1) will both be denoted by $<$.

A. A one-one function f from D into Q is defined. Let $f(d_1) = r_1$. Let $f(d_2) = r_{i_2}$, where i_2 is the smallest natural number in the set $\{n \mid r_n < r_1$, if $d_2 <^* d_1$; and $r_1 < r_n$ if $d_1 <^* d_2\}$. This set is not empty since the natural ordering on Q does not determine a first or a last element. Thus, if $d_1 <^* d_2$, $f(d_1) < f(d_2)$; and if $d_2 <^* d_1$, $f(d_2) < f(d_1)$. Assume, now, that $f(d_1)$, $f(d_2)$, ..., $f(d_{k-1})$, where $k \geq 3$, have all been defined in Q so that $f(d_i) < f(d_j)$, if $d_i <^* d_j$ for $1 \leq i, j \leq k-1$. Consider d_k. Since D is linearly ordered by $<^*$, the subset $\{d_1, d_2, \ldots, d_{k-1}\}$ is also linearly ordered by $<^*$. Let $d_{j_1} <^* d_{j_2} <^* \ldots <^* d_{j_{k-1}}$ be the linear ordering on $\{d_1, d_2, \ldots d_{k-1}\}$ induced by $<^*$ on D. If $d_k <^* d_{j_i}$ for all i = 1, 2, ..., k-1; then, since Q has no first element, there is a rational number r in (0, 1) such that r $r < f(d_{j_1}) < f(d_{j_2}) < \ldots < f(d_{j_k})$. Hence, if $A = \{t \mid r_t < f(d_{j_1})\}$, then $A \neq \phi$. Let q be the smallest natural number in A. Define $f(d_k) = r_q$. Similarly, if $d_{j_{k-1}} <^* d_k$, then $d_{j_i} <^* d_k$ for i = 1, 2, ..., k-1. Since Q has no last number, $\{s \mid f(d_{j_{k-1}}) < r_s \neq \phi$. Let m be the smallest natural number in this set. Define $f(d_k) = r_m$. Then $f(d_{j_1}) < \ldots < f(d_k)$. Lastly, for some one

$i = 1, 2, \ldots, k-2$; let $d_{j_i} <^* d_k <^* d_{j_{i+1}}$. Since between any two rationals lies a third rational, $\{n \mid f(d_j) < r_n < f(d_{j_{i+1}})\} \neq \phi$. Let p be the smallest natural number in this set. Define $f(d_k) = r_p$. Thus f has been defined by induction on all of D in such way that (1) if $d_s <^* d_t$, then $f(d_s) < f(d_t)$; and (2) for every natural number s, i_s (where $f(d_s) = r_{i_s}$) is the smallest subscript in the given listing $\{r_1, r_2, \ldots\}$ such that f preserves order.

B. Next it is shown that f is onto. r_1 is in the range of f since $f(d_1) = r_1$. Let $r_1, r_2, \ldots, r_{k-1}$ be in the range of f for some natural number $k \geq 2$. Then $\{m \mid \{r_1, r_2, \ldots, r_{k-1}\} \subseteq \{f(d_1), f(d_2), \ldots, f(d_m)\}\} \neq \phi$. Let q be the smallest natural number in this set. $\{r_1, r_2, \ldots, r_{k-1}\} \subseteq \{f(d_1), f(d_2), \ldots, f(d_q)\}$. If $r_k \in \{f(d_1), f(d_2), \ldots, f(d_q)\}$ then, of course, r_k is in the range of f. Assume, then, that $r_k \notin \{f(d_1), f(d_2), \ldots, f(d_q)\}$. There are three cases to consider : (1) $r_k < f(d_j)$ for all $j = 1, 2, \ldots, q$; (2) $r_k > f(d_j)$ for all $j = 1, 2, \ldots, q$; and (3) $f(d_i) < r_k < f(d_j)$ for some i, j such that $1 \leq i, j \leq q$.

Case (1) : Let $r_k < f(d_j)$ for all $j = 1, 2, \ldots, q$. Since D has no first element, $\{m \mid d_{q+m} <^* d_j$ for all $j = 1, 2, \ldots, q\} \neq \phi$. Let m* be the smallest natural number in this set. Then d $d_{q+m^*} <^* d_j$ for all $j = 1, 2, \ldots, q$. If $m^* = 1$, then it is clear that $f(d_{q+m^*}) = r_k$ and r_k is in the range of f. Assume $m^* > 1$. Then for each natural number t such that $1 \leq t < m^*$, there is at least one natural number j such that $d_j <^* d_{q+t}$ for $1 \leq j \leq q$. By assumption, $f(d_j) \neq r_k$ for all $j = 1, 2, \ldots, q$. Furthermore, $f(d_{q+t}) \neq r_k$ for $1 \leq t < m^*$ since f preserves order. Now, the domain of f is D; hence, $f(d_{q+m^*})$ is defined as a rational number. Let $f(d_{q+m^*}) = r_p$. Since f is order-preserving, f is a one-one function. Furthermore, by the definition of q, $\{r_1, r_2, \ldots, r_{k-1}\} \subseteq \{f(d_1), \ldots, f(d_q)\}$. Hence, $k-1 < p$. If $k < p$, then the definition of f is contradicted. Hence, $f(d_{q+m^*}) = r_k$ and so r_k is in the range of f.

The proof of case (2) is exactly analogous but depends on the fact that D has no last element instead of the fact that D has no first element.

Case (3) : Let $\{d_1, d_2, \ldots, d_q\}$ and k be as defined in the proof of case (1). Let $f(d_i) < r_k < f(d_j)$ for some natural numbers i and j such that $1 \leq i, j \leq q$. Let $\{\rho_1, \rho_2, \ldots, \rho_q\}$ denote the set $\{f(d_1), f(d_2), \ldots, f(d_q)\}$, where $\rho_1 < \rho_2 < \ldots < \rho_q$. There is then a smallest natural number $t^* \neq 1$ such that $r_k < \rho_{t^*}$. Hence $\rho_{t^*-1} < r_k < \rho_{t^*}$. By hypothesis, $\{n \mid f^{-1}(\rho_{t^*-1}) <^* d_n <^* f^{-1}(\rho_{t^*})\} \neq \phi$. Let n^* be the smallest natural number in this set. Since $r_k \notin \{f(d_1), f(d_2), \ldots, f(d_q)\}$, if $j \leq q$, then $f(d_j) \neq r_k$. Furthermore, by the definition of n^*, if $q < j < n^*$, $f(d_j) \neq r_k$ since f preserves order. Now $f(d_{n^*})$ is defined since the domain of f is D. Let $f(d_{n^*}) = r_p$. Since $\{r_1, r_2, \ldots, r_{k-1}\} \subseteq \{f(d_1), f(d_2), \ldots, f(d_q)\}$, $p \geq k$. If $p > k$, then the definition of f is again contradicted as it was in the proof of case (1). Thus, if $r_1, r_2, \ldots, r_{k-1}$ are in the range of f, r_k is also. Hence the range of f is Q. Thus a one-one order preserving function from D onto Q has been established. #

In any linearly ordered set (L, <), cuts are defined: A cut is an order pair (A, B) of subsets of L such that (1) $L = A \cup B$, (2) $A \neq \phi \neq B$, and (3) for any x in A and any y in B, $x < y$. There are three types of cuts. A cut (A, B) is jump if A has a last element and B has a first element. A cut is called a gap if A has no last element and B has no first element, A cut is called a filled cut if A has a last element and B has no first element, or A has no last element and B has a first element. A linearly ordered set (L, <) with no jumps is called order-dense in itself, and a linearly ordered set with neither jumps nor gaps is called order-connected. The following lemma follows immediately from the definition.

3.9 Lemma. A linearly ordered set (L, <) is order-dense in itself if and only if for each two elements α and β in L, there exists at least one element γ in L such that $\alpha < \gamma < \beta$. #

Let (L, <) be a linearly ordered set and let D be a subset of L. D is called order-dense in (L, <) if for each two elements α and β in L, there exists at least one element δ in D such that $\alpha < \delta < \beta$.

3.10 Theorem. Let (L, <) be a linearly ordered set and let $0-\mathfrak{T}$ be the order topology for L. If $(L, 0-\mathfrak{T})$ is a connected space, then (L, <) is order-connected.

Proof. Let $(L, 0-\mathfrak{T})$ be connected and let (A, B) be a cut in (L, <). Assume first (A, B) is a gap. By the definition of the order topology, $\{x \mid x < z \text{ for } z \text{ in } L\}$ is open in $(L, 0-\mathfrak{T})$. Let $\alpha \in A$. Since A has no last element, then there exists a z in A such that $\alpha < z$. Hence, every point of A is an interior point of A and A is open. A similar procedure establishes that B is open. Hence $(L, 0-\mathfrak{T})$ is not connected. This is a contradiction, so L has no gap. Assume next that (A, B) is a jump. Let γ be the last point in A and β be the first point in B. By the definition of the order topology, $\{x \mid x < \beta\}$ and $\{x \mid \alpha < x\}$ are both open in $(L, 0-\mathfrak{T})$. Since $A = \{x \mid x < \beta\}$ and $B = \{x \mid \alpha < x\}$, $(L, 0-\mathfrak{T})$ is not connected. This is a contradiction; hence, (L, <) has no jumps. #

3.11 Theorem. Any compact, connected, metrizable space $(S, \mathfrak{T})$ with exactly two non-cut points is an order-connected set relative to the ordering which determines the topology. Furthermore, there exists a denumerable subset D which is order-dense in S.

Proof. By lemma 3.5, $(S, \mathfrak{T})$ is an E(p, q) where p and q are the two non-cut points. Hence $\mathfrak{T}$ is the order topology defined from the linear order relation <*. By lemma 3.10, (S, <*) is order-connected. Furthermore, by theorem 2.48 and 2.49, $(S, \mathfrak{T})$ is separable. Hence, $(S, \mathfrak{T})$ has a countable subset D which is dense in the space $(S, \mathfrak{T})$. Since $(S, <^*)$ is order-connected, there are no jumps. Hence, by the definition of the order topology, between any two distinct points of S lies a point of D. D is then order-dense in $(S, <^*)$. #

3.12 <u>Theorem</u>. If $(L, <^*)$ is a non-empty linearly ordered set with the following properties : (1) $<^*$ determines no first and no last element on L, (2) $(L, <^*)$ is order-connected, and (3) there exists a countable subset D which is order-dense in $(L, <^*)$; then $(L, <^*)$ is order-isomorphic to $\{x \mid x \text{ is a real and } 0 < x < 1\}$, i.e., to $(0, 1)$, with the usual ordering.

<u>Proof</u>. A. First a function f^* from $(L, <^*)$ into $(0, 1)$ is defined. By lemma 3.8, there is a one-one, order-preserving function f from the order-dense subset $(D, <^*)$ of $(L, <^*)$ onto the set Q of rational numbers in $(0, 1)$, with the usual ordering. (Note: $<^*$ will be used ambiguously to denote the given order relation on L and also its restriction to D.) Thus, for every element d in D, $f(d)$ is a unique rational number in $(0, 1)$. Now, let $\alpha \in L - D$, let $B'_\alpha = \{y \text{ in } L \mid \alpha \leq^* y\}$. Let $H^\alpha = \{t \text{ in } (0, 1) \mid r \in f(B'_\alpha \cap D)$ implies $t < r\}$ and let $M^\alpha = (0, 1) - H^\alpha$. Since f is order-preserving and D is order-dense, $H^\alpha \neq \phi \neq M^\alpha$. Now, let $t_1 \in H^\alpha$ and $t_2 \in M^\alpha$. Since $t_2 \notin H^\alpha$, there exists r^* such that $r^* \in f(B'_\alpha \cap D)$ and $r^* \leq t_2$. Since $t_1 \in H^\alpha$, $t_1 < r^*$ and hence $t_1 < t_2$. Also for any t in $(0, 1)$,

$t \in H^\alpha$ or $t \in M^\alpha$ and so (H^α, M^α) is a cut in (0, 1). Since (0, 1) is order-connected, H^α has a last element or M^α has a first element. Now, let m be any element in M^α. By the definition of M^α, there exists an r such that $r \in f(B'_\alpha \cap D)$ and $r \leq m$. Consider $f^{-1}(r)$ in $B'_\alpha \cap D$. By the definition of B'_α, $\alpha <^* f^{-1}(r)$. Since D is order-dense in $(L, <^*)$, there exists a d in D such that $\alpha <^* d <^* f^{-1}(r)$. By the definition of B'_α, $d \in B'_\alpha \cap D$. Hence $f(d) \in f(B'_\alpha \cap D)$ and $f(d) < r$, by the definition of f. This means that $f(d) < m$. Since $f(d) \in f(B'_\alpha \cap D)$, $f(d) \notin H^\alpha$. Hence, $f(d) \in M^\alpha$ and m is not the first element of M^α. Thus H^α has a last element β and $H^\alpha = \{t \text{ in } (0, 1) \mid t \leq \beta\}$. Define $f^*(\alpha) = \beta$ and for d in D let $f^*(d) = f(d)$.

B. Secondly, it is shown that f* is order-preserving, and hence it is one-one. First, let $d \in D$ and let $\alpha \in L - D$ be such that $\alpha <^* d$. $d \in B'_\alpha \cap D$ by the above definition of B'_α. Hence $f(d) \in f(B'_\alpha \cap D)$, and so $f(d) \in M^\alpha$ which was defined above in part A. Now by definition of f*, $f^*(\alpha)$ is the last element of H^α. Hence, $f^*(\alpha) < f(d) = f^*(d)$. Similarly, if $d \in D$, $\alpha \in L - D$ and $d <^* \alpha$, then $f(d) < r$ for all r in $f(B'_\alpha \cap D)$ by definition of B'_α and f. Hence, $f(d) \in H^\alpha$, by definition of H^α. Since $f^*(\alpha)$ is the last element in H^α, $f^*(d) = f(d) < f^*(\alpha)$. Lastly, let α_1, α_2 be both in L - D and let $\alpha_1 <^* \alpha_2$. Since D is order-dense in $(L, <^*)$, there exists $d \in D$ such that $\alpha_1 <^* d <^* \alpha_2$. Thus, by what was just proved above, $f^*(\alpha_1) < f^*(d) < f^*(\alpha_2)$. Therefore,

$f^*(\alpha_1) < f^*(\alpha_2)$ and f^* is order-preserving and hence it is one-one.

C. Lastly, it is shown that f^* is onto. Let t^* be any real number in (0, 1). If t^* is rational, then t^* is in the range of f and hence in the range of f^*. If t^* is irrational, let $A_{t^*} = \{x \text{ in } (0, 1) \mid x \leq t^*\}$ and let $B_{t^*} = \{x \text{ in } (0, 1) \mid t^* < x\}$. Then $Q = (A_{t^*} \cap Q) \cup (B_{t^*} \cap Q)$, where Q is the set of rationals in (0, 1). Hence $D = f^{-1}(A_{t^*} \cap Q) \cup f^{-1}(B_{t^*} \cap Q)$ in L. Let $G_\gamma = \{y \text{ in } L \mid d \in f^{-1}(A_{t^*} \cap Q) \text{ implies } d <^* y\}$ and let $F_\gamma = L - G_\gamma$. Since D is order-dense and f is order-preserving, $F_\gamma \neq \phi \neq G_\gamma$. If $y_1 \in F_\gamma$, then $y_1 \notin G_\gamma$. Hence, there exists a $\bar{d}$ in D such that $\bar{d} \in f^{-1}(A_{t^*} \cap Q)$ and $y_1 \leq^* \bar{d}$. Now, let $y_2 \in G_\gamma$. By the definition of G_γ, $\bar{d} <^* y_2$. Hence $y_1 <^* y_2$, and (F_γ, G_γ) is a cut in $(L, <^*)$. Since $(L, <^*)$ is order-connected, F_γ has a last element, or G_γ has a first element but not both. Denote this element by γ. Assume first that $f^*(\gamma) < t^*$. Since Q is order-dense in (0, 1), there exists an r in Q such that $f^*(\gamma) < r < t^*$. Hence $r \in A_{t^*} \cap Q$ and $f^{-1}(r) \in f^{-1}(A_{t^*} \cap Q)$ and $\gamma <^* f^{-1}(r)$. Therefore, $\gamma \notin G_\gamma$, and so $\gamma \in F_\gamma$. Furthermore, since $f^{-1}(r) \in f^{-1}(A_{t^*} \cap Q)$, $f^{-1}(r) \notin G_\gamma$ and so $f^{-1}(r) \in F_\gamma$. Thus, γ is not the last element of F_γ. This contradicts the definition of γ. Hence, $t^* \leq f^*(\gamma)$. Assume, next, that $t^* < f^*(\gamma)$. Again, since Q is order-dense in (0, 1), there exists an r in Q such that $t^* < r < f^*(\gamma)$. This means that if $d \in f^{-1}(A_{t^*} \cap D)$, $f(d) \in A_{t^*}$, and hence $f(d) \leq t^*$. Thus, $f(d) < r < f^*(\gamma)$, and so $d <^* f^{-1}(r) <^* \gamma$. This means that $f^{-1}(r)$ and γ are both in G_γ. Thus γ is in G_γ but is not the first element of G_γ. This is a contradiction. Hence, $f^*(\gamma) = t^*$ and f^* is onto. #

3.13 Theorem. Any compact, connected metrizable space $(S, \mathcal{T})$ with exactly two non-cut points is an arc.

Proof. By lemma 3.4 and 3.5, S is E(p, q) where p and q are the two non-cut points. The natural linear ordering $<^*$ on E(p, q) determines the topology $\mathcal{T}$ for S by theorem 3.7, and determines p as the first element and q as the last element. Furthermore, by theorem 3.11, $(S, <^*)$ is order-connected and there exists a countable order-dense subset D in $(S, <^*)$. Now, let $x \in S$ and $x \neq p$. Since D is order-dense $(S, <^*)$, there exists $d \in D$ such that $p <^* d <^* x$. Hence $S - \{p\} = E(p, q) - \{p\}$ has no first element. Similarly $S - \{q\} = E(p, q) - \{q\}$ has no last element. Hence, $S - \{p, q\}$ is linearly ordered by the restriction $<^*$ to $S - \{p, q\}$ and $<^*$ determines no first element and no last element on $S - \{p, q\}$. Let $L = S - \{p, q\}$. Then $(L, <^*)$ is a linearly ordered set with no first and no last element. Furthermore, $D \cap L$ is denumerable and order-dense in $(L, <^*)$ since D is order-dense in $(S, <^*)$. Also, $(L, <^*)$ is order-connected because any jump or gap in $(L, <^*)$ would yield a jump or gap in $(S, <^*)$ which is impossible. Thus, $(L, <^*)$ is a linearly ordered, order-connected set with no first element, no last element and with a denumerable order-dense subset. Hence, by theorem 3.12, $(L, <^*)$ is order-isomorphic to the open interval (0, 1) with the usual ordering. Let h be a one-one function from $(L, <^*)$ onto $(0, 1)$ which preserves order. Define $h^* : (S, <^*) \xrightarrow{\text{onto}} [0, 1]$ as follows :

$h^*(p) = 0$, $h^*(q) = 1$ and $h^*(x) = h(x)$ for $p \neq x \neq q$. By definition of the ordering on $(S, <^*)$ and $[0, 1]$, h^* is an order-isomorphism from $(S, <^*)$ onto $[0, 1]$. By definition of the order topology, h^* is a homeomorphism from $(S, \bar{\mathcal{T}})$ onto $[0, 1]$, and $(S, \mathcal{T})$ is, then, an arc. #