CHAPTER II

A THEOREM ON MEAN VALUE ITERATIONS

2.1 Introduction. In this chapter we consider a mapping T which continuously maps the closed interval $E = [0,1]$ into itself. We prove that a certain mean value iterative scheme always converges to a fixed point of T on E. This result was proved in [7], where T was required to have a unique fixed point in the interval. In this chapter we show that this restriction is unnecessary, convergence is proved by considering only the continuity of $T: E \longrightarrow E$.

2.2 A convergent iterative scheme. A well-known theorem of Brouwer asserts: Every continuous mapping T maps the closed N-cell (= N-disk) into itself has at least one fixed point, i.e. a point p such that $Tp = p$.

Consider a mapping T with the following properties.

(i) T is continuous on $E = [0,1]$.

(ii) T maps E into itself.

From Brouwer's fixed point theorem, T has at least one fixed point on this interval. We will now show that the iterative scheme

(1) $x_{n+1} = Tv_n$, (2) $v_n = \frac{1}{n}(x_1 + \ldots + x_n)$, $n = 1,2,3,\ldots$, (3) $v_1 = x_1 \in E = [0,1]$

converges to a fixed point of T.

2.3 Theorem. Let E be the closed unit interval and let $T: E \longrightarrow E$ be continuous. Then the iterative scheme

$$(1) \qquad x_{n+1} = Tv_n,$$

$$(2) \qquad v_n = \frac{1}{n}(x_1 + \dots + x_n),\ n = 1, 2, 3, \dots,$$

$$(3) \qquad v_1 = x_1 \in E$$

converges to a fixed point of T on E.

Before proving the above theorem, we establish the following theorem.

2.4 Theorem. If either of the sequences $\{x_n\}$ and $\{v_n\}$ converges, then the other also converges to the same point, and their common limit is a fixed point of T.

To prove this theorem we need a lemma.

2.5 Lemma. If $x_n \longrightarrow p$, and if

$$v_n = \frac{1}{n}(x_1 + \dots + x_n),$$

then $v_n \longrightarrow p$.

Proof. We have at once

$$v_n - p = \frac{1}{n}\left\{(x_1-p) + (x_2-p) + \dots + (x_n-p)\right\}.$$

Since every convergent sequence is bounded, the sequence $x_n - p \longrightarrow 0$, and so, is bounded. Hence there exists K such that $|x_n - p| < K$ for all n, and for any given $\varepsilon > 0$ there exists N such that $|x_n - p| < \frac{1}{2}\varepsilon$ when $n \geq N$.

Take a definite such value of N (>1) and let $n \geq N$. Then

$$|v_n - p| < \frac{(N-1)K}{n} + \frac{(n-N+1)}{2n}\varepsilon$$
$$< \frac{(N-1)K}{n} + \frac{1}{2}\varepsilon .$$

But N, K are fixed, and we can make the first term of the last expression less than $\frac{1}{2}\varepsilon$ by taking $n > 2(N-1)K\varepsilon^{-1}$. Hence for any given $\varepsilon > 0$, there exists N_1, $N_1 = \max \{ N, [2(N-1)K\varepsilon^{-1} + 1] \}$, such that $|v_n - p| < \varepsilon$ when $n \geq N_1$. Therefore $v_n \longrightarrow p$.

Q.E.D.

Proof. (of Theorem 2.4) Let $\lim_{n\to\infty} x_n = p$. Then by Lemma 2.5, $\lim_{n\to\infty} v_n = p$. Since T is continuous, $\lim_{n\to\infty} Tv_n = Tp$. But $Tv_n = x_{n+1}$, so that $Tp = p$.

If now we assume that $\lim_{n\to\infty} v_n = q$, then $\lim_{n\to\infty} x_{n+1} = Tq$ and by Lemma 2.5, $\lim_{n\to\infty} v_n = Tq$. Hence, $Tq = q$.

Q.E.D.

Proof. (of Theorem 2.3) We first show that $v_n \in E$ by induction on n. Since $v_1 = x_1 \in E$, we have finished the first step.

Assume the statement holds for lesser values of n. Then

$$\begin{aligned} v_n &= \frac{1}{n}(x_1 + x_2 + \dots + x_n) \\ &= \frac{1}{n}\left((n-1)\frac{(x_1 + \dots + x_{n-1})}{n-1} + Tv_{n-1}\right) \\ &= \frac{1}{n}\left((n-1)v_{n-1} + Tv_{n-1}\right) \\ &= \left(1 - \frac{1}{n}\right)v_{n-1} + \frac{1}{n}Tv_{n-1} \in E \end{aligned}$$

by induction hypothesis. (Recall that if x_1, x_2 are points in $\mathbb{R}$ then the point

$$x(t) = tx_2 + (1-t)x_1 \ , \quad 0 \leq t \leq 1,$$

lies on the straight line segment joining x_1 and x_2 with $x(0) = x_1$, $x(1) = x_2$.)

Hence by induction, $v_n \in E$ for all positive integral values of n.

Moreover, for each n, we have

$$v_{n+1} = \frac{Tv_n - v_n}{n+1} + v_n. \tag{4}$$

Since both v_n and Tv_n are in E, we obtain

$$\left| v_{n+1} - v_n \right| \leq \frac{1}{n+1} \ , \ n = 1,2,3,\dots \ . \tag{5}$$

This means the step size becomes arbitrarily small as n increases. The proof can now be accomplished in two steps.

1. We first show that $\{v_n\}$ converges. The sequence $\{v_n\}$ is contained in E, a compact set, so it has at least one cluster point, by Theorem 1.13. We will prove that $\{v_n\}$ converges to a unique cluster point.

Assume, on the contrary, that p_1 and p_2 are two distinct cluster points of $\{v_n\}$ and $p_1 < p_2$.

a) We will show that a consequence of this assumption is that $Tx = x$ for every x in (p_1, p_2).

Suppose there exists $x^* \in (p_1, p_2)$ such that $Tx^* \neq x^*$. Then either $Tx^* > x^*$ or $Tx^* < x^*$.

We assert that if $Tx^* > x^*$, then there is a $\delta \in (0, (x^*-p_1)/2)$ such that $Tx > x$ for all x satisfying $|x-x^*| < \delta$.

To prove this assertion, consider the function

$$g(x) = Tx - x.$$

Since $Tx^* > x^*$, therefore

$$g(x^*) = Tx^* - x^* > 0.$$

Let $0 < \varepsilon' < g(x^*)$. Since $x^* > p_1$, a number ε'' can be chosen such that

$$\varepsilon'' = \min \left\{ (x^*-p_1)/2, \ \varepsilon'/2 \right\}.$$

Since T is continuous, there exists a $\delta > 0$, $\delta < \varepsilon''$ say, such that $|Tx - Tx^*| < \varepsilon''$ whenever $|x-x^*| < \delta$. Now

$$\begin{aligned} |g(x) - g(x^*)| &= |Tx - x - Tx^* + x^*| \\ &\leq |Tx - Tx^*| + |x^* - x| \\ &< \varepsilon'' + \delta \\ &< \varepsilon'' + \varepsilon'' = 2\varepsilon'' \leq \varepsilon'. \end{aligned}$$

Consequently

$$g(x) > g(x^*) - \varepsilon' > 0.$$

That is

$$Tx - x > 0, \text{ or } Tx > x,$$

whenever $|x - x^*| < \delta$ and

$$\delta \in (0, (x^*-p_1)/2).$$

Then, by (4),

$$v_{n+1} - v_n = \frac{Tv_n - v_n}{n+1} > 0$$

when $|v_n - x^*| < \delta$.

I.e.,

(6) $$|v_n - x^*| < \delta \text{ implies } v_{n+1} > v_n .$$

Now (5) implies there is a number $M' > 0$ such that

(7) $$|v_{n+1} - v_n| < \delta , \quad n = M', M'+1, \ldots .$$

Let $0 < \varepsilon < p_2 - x^*$ and note that $p_2 > x^*$. Since p_2 is a cluster point of $\{v_n\}$, a number N can be chosen such that $N > M'$ and $|v_N - p_2| < \varepsilon$. Hence

(8) $$v_N > p_2 - \varepsilon > x^*$$

We now show by induction on n that if $v_N > x^*$, then $v_{N+n} > x^*$ for all positive integers n.

From (8), it follows that either $v_N < x^* + \delta$ or $v_N \geq x^* + \delta$.

If $v_N < x^* + \delta$, then

$$|v_N - x^*| < \delta$$

and then, by (6),

$$v_{N+1} > v_N > x^*.$$

If $v_N \geq x^* + \delta$, then

$$v_{N+1} > v_N - \delta \geq x^*$$

since $|v_{N+1} - v_N| < \delta$, by (7).

Thus, we have finished the first step.

Assume the statement holds for $n = k$, k a positive integer, i.e. assume that $v_{N+k} > x^*$. We want to show that it is true for $n = k+1$.

By assumption, we see that either $x^* < v_{N+k} < x^* + \delta$ or $v_{N+k} \geqq x^* + \delta$.

If $x^* < v_{N+k} < x^* + \delta$, then

$$| v_{N+k} - x^* | < \delta$$

and then, by (6),

$$v_{N+k+1} > v_{N+k} > x^*.$$

If $v_{N+k} \geqq x^* + \delta$, then

$$v_{N+k+1} > v_{N+k} - \delta \geqq x^*$$

since $|v_{N+k+1} - v_{N+k}| < \delta$, by (7).

Hence by induction, $v_{N+n} > x^*$ for all positive integral values of n. Therefore

$$v_n > x^* > x^* - \delta > p_1$$

i.e.,

$$v_n - p_1 \geqq \delta, \quad n = N+1, N+2, \ldots,$$

which means p_1 is not a cluster point of $\{v_n\}$, contrary to our assumption.

If $Tx^* < x^*$, similar reasoning contradicts the assumption that p_2 is a cluster point. Therefore $Tx^* = x^*$ for all $x^* \in (p_1, p_2)$.

b) We will now show that p_1 and p_2 are not both cluster points. Observe that

(9) $$v_n \notin (p_1,p_2) \quad \text{for all } n = 1,2,3,\ldots,$$

since if there exists $v_{n_0} \in (p_1,p_2)$ then by (a), we have just proved,

$$Tv_{n_0} = v_{n_0}$$

and then, by (4),

$$v_m = v_{n_0} \quad \text{for all } m > n_0.$$

Now let $\varepsilon' < v_{n_0} - p_1$ and $\varepsilon'' < p_2 - v_{n_0}$, then

$$v_m - p_1 > \varepsilon' \quad \text{for all } m > n_0,$$

and

$$p_2 - v_m > \varepsilon'' \quad \text{for all } m > n_0.$$

Therefore, neither p_1 nor p_2 could be cluster points, contrary to our assumption.

By assumption $p_2 > p_1$, let $\varepsilon = (p_2-p_1)/2$. Then there exists a number M such that $\frac{1}{M} < \varepsilon$.

It follows from (9) that either $v_M \leqq p_1$ or $v_M \geqq p_2$.

We again prove by induction on n that if $v_M \geqq p_2$, then $v_{M+n} \geqq p_2$ for all positive integers n.

For n = 1; we have, by (5),

$$|v_{M+1} - v_M| \leqq \frac{1}{M+1} < \frac{1}{M} < \varepsilon$$

thus

$$v_{M+1} > v_M - \varepsilon \geqq p_2 - \tfrac{1}{2}(p_2 - p_1) = \tfrac{1}{2}(p_2+p_1) > \tfrac{1}{2}(p_1+p_1) = p_1.$$

Since $v_{M+1} \notin (p_1,p_2)$, by (9), therefore

$$v_{M+1} \geqslant p_2.$$

Hence, we have finished the first step.

Assume the statement holds for $n = k$, i.e., $v_{M+k} \geq p_2$. We want to show that it is true for $n = k+1$.

By (5),

$$|v_{M+k+1} - v_{M+k}| \leq \frac{1}{M+k+1} < \frac{1}{M} < \varepsilon ,$$

and by induction hypothesis, we obtain

$$v_{M+k+1} > v_{M+k} - \varepsilon \geq p_2 - \tfrac{1}{2}(p_2 - p_1) > p_1.$$

Since $v_{M+k+1} \notin (p_1,p_2)$, by (9), therefore

$$v_{M+k+1} \geq p_2.$$

Hence by induction, $v_{M+n} \geq p_2$ for all positive integral values of n. I.e.,

$$v_n \geq p_2 > p_1 \quad \text{for all } n \geq M.$$

Therefore

$$v_n - p_1 \geq p_2 - p_1 = 2\varepsilon > \varepsilon \quad \text{for all } n \geq M,$$

and p_1 is not a cluster point.

Similarly, if $v_M \leqslant p_1$ then $v_n \leq p_1 < p_2$ for all $n \geq M$ and p_2 is not a cluster point.

Either way, $\{v_n\}$ cannot have two distinct cluster points. Therefore $\{v_n\}$ converges to its unique cluster point p.

The point p is also a limit of $\{v_n\}$, by Theorem 1.14.

To finish the proof we have to show that p is a fixed point of T.

2. To show that p is a fixed point of T, i.e. to show $Tp = p$, assume $Tp > p$. Let $\varepsilon = \frac{1}{2}(Tp - p) > 0$. Consider the function

$$g(x) = Tx - x.$$

Since T is continuous, g is also continuous and

$$g(p) = Tp - p.$$

Since $v_n \longrightarrow p$, therefore, by the continuity of g,

$$g(v_n) \longrightarrow g(p).$$

Then there is a number $N > 0$ such that

$$|g(v_n) - g(p)| < \varepsilon$$

when $n \geq N$. Consequently

$$g(v_n) > g(p) - \varepsilon = (Tp - p) - \tfrac{1}{2}(Tp - p) = \tfrac{1}{2}(Tp - p) = \varepsilon,$$

i.e., $$Tv_n - v_n > \varepsilon \quad \text{for all } n \geq N.$$

By (4)

$$v_{n+1} - v_n = \frac{Tv_n - v_n}{n+1} > \frac{\varepsilon}{n+1} \quad \text{for all } n \geq N.$$

Therefore

$$v_{N+m} - v_N = \sum_{n=N}^{N+m-1} (v_{n+1} - v_n)$$

$$\geq \sum_{n=N}^{N+m-1} \frac{\varepsilon}{n+1} \longrightarrow \infty \text{ as } m \longrightarrow \infty$$

Therefore $v_n \longrightarrow \infty$, contradicting the fact that $v_n \in [0,1]$ for all n.

Similarly, assuming $Tp < p$ implies $v_n \longrightarrow -\infty$. Therefore $Tp = p$.

Q.E.D.