CHAPTER VI

SOLUTIONS OF $g(x-y) = g(x)g(y) + f(x)f(y)$ AND $g(\frac{x}{y}) = g(x)g(y) + f(x)f(y)$.

In this chapter we shall determine all continuous solution of

$$(A_*) \qquad g(x-y) = g(x)g(y) + f(x)f(y)$$

on $\mathbb{R}^n$ to $\mathbb{R}$, and all continuous solutions of

$$(A^*) \qquad g(\frac{x}{y}) = g(x)g(y) + f(x)f(y)$$

on $\mathbb{R}^*$ to $\mathbb{C}$. Furthermore we shall show that discontinuous solutions of (A_*) and (A^*) also exist.

6.1 Continuous Solution of $g(x-y) = g(x)g(y) + f(x)f(y)$ on $\mathbb{R}^n$

Let $f,g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous functions such that f,g satisfy

$$(A_*) \qquad g(x-y) = g(x)g(y) + f(x)f(y),$$

for all x,y in $\mathbb{R}^n$. We shall characterize all such functions f,g. To do this, we may consider f,g as functions on the topological group $(\mathbb{R}^n,+)$ into the topological field $(\mathbb{R},+,.)$. Since $\mathbb{R}^n$ is also the topological vector space over $\mathbb{R}$, hence by Corollary 4.9, the continuous solutions of (A_*) must be those and only those (f,g) of the forms :

(6.1.1) $f(x) = b$, $g(x) = a$ for all x in $\mathbb{R}^n$, where $a,b \in [0, 1)$ and $b = \sqrt{a - a^2}$; or

(6.1.2) $$f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a continuous homomorphism from $\mathbb{R}^n$ to Δ. By Theorem 5.3.6, $h(x)$ must be of the form $h(x) = e^{i(k_1x_1+\dots+k_nx_n)}$ where $x = (x_1,\dots, x_n)$ and k_i's are real numbers. Hence f,g in (6.1.2) must be of the form

$$f(x) = \frac{e^{i(k_1x_1+\dots+k_nx_n)} - e^{-i(k_1x_1+\dots+k_nx_n)}}{2i}$$

$$= \sin(k_1x_1 + \dots + k_nx_n)$$

and

$$g(x) = \frac{e^{i(k_1x_1+\dots+k_nx_n} + e^{-i(k_1x_1+\dots+k_nx_n)}}{2}$$

$$= \cos(k_1x_1 + \dots + k_nx_n)$$

where $x = (x_1, \dots, x_n)$ and k_i's are real numbers.

In particular, when $n = 1$, the continuous solutions of (A_*) are those and only those (f,g) of the forms :

(6.1.3) $f(x) = b$, $g(x) = a$ for all x in $\mathbb{R}$, where $a,b \in [0,1)$ and $b = \sqrt{a - a^2}$; or

(6.1.4) $f(x) = \sin(kx)$, $g(x) = \cos(kx)$

where k is a real number.

6.2 Continuous Solution of $g(\frac{x}{y}) = g(x)g(y) + g(x)f(y)$ on $\mathbb{R}^*$

Let $f, g : \mathbb{R}^* \longrightarrow \mathbb{C}$ be continuous functions such that f, g satisfy

$$(A^*) \qquad g(\frac{x}{y}) = g(x)g(y) + f(x)f(y) ,$$

for all x, y in $\mathbb{R}^*$. We shall characterize all such functions f, g. To do this we may consider f, g as functions on the topological group $(\mathbb{R}^*, .)$ into the topological field $(\mathbb{C}, +, .)$. It can be shown that $(\mathbb{R}^+, .)$ is the only subgroup of index 2 in $\mathbb{R}^*$, and that $(\mathbb{R}^*, .)$ has no subgroup of index 4. Hence by Theorem 4.5, the continuous solutions of (A^*) must be those and only those (f, g) of the forms :

(6.2.1) $\quad f(x) = b, \quad g(x) = a \quad$ for all x in $\mathbb{R}^*$ where a, b are elements of $\mathbb{C}$ such that $a \neq 1$, $a - a^2 = b^2$; or

$$(6.2.2) \qquad f(x) = \begin{cases} b , & x \in \mathbb{R}^+ \\ -b , & x \in \mathbb{R}^- \end{cases} , \quad g(x) = \begin{cases} a , & x \in \mathbb{R}^+ \\ -a , & x \in \mathbb{R}^- \end{cases}$$

where a, b are elements of $\mathbb{C}$ such that $a \neq 1$, $a - a^2 = b^2$; or

$$(6.2.3) \qquad f(x) = \begin{cases} 0 , & x \in \mathbb{R}^+ \\ d , & x \in \mathbb{R}^- \end{cases} , \quad g(x) = \begin{cases} 1 , & x \in \mathbb{R}^+ \\ c , & x \in R^- \end{cases}$$

where c, d are elements of $\mathbb{C}$ such that $c \neq 1$, $c^2 + d^2 = 1$; or

$$(6.2.4) \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a continuous homomorphism from $\mathbb{R}^*$ to $\mathbb{C}^*$.

By Theorem 5.4.2, we see that

$$h(x) = |x|^c \qquad \text{for all } x \text{ in } \mathbb{R}^*,$$

or

$$h(x) = \begin{cases} |x|^c & \text{if } x > 0 \\ -|x|^c & \text{if } x < 0, \end{cases}$$

where c is a complex number. Hence f,g in (6.2.4) must be of the forms :

$$(6.2.4.a) \qquad f(x) = \frac{|x|^c - |x|^{-c}}{2i}, \quad g(x) = \frac{|x|^c + |x|^{-c}}{2}$$

for all x in $\mathbb{R}^*$,

or

$$(6.2.4.b) \qquad f(x) = \begin{cases} \dfrac{|x|^c - |x|^{-c}}{2i} & \text{if } x > 0 \\[2ex] \dfrac{|x|^{-c} - |x|^c}{2i} & \text{if } x < 0 \end{cases}$$

and

$$g(x) = \begin{cases} \dfrac{|x|^c + |x|^{-c}}{2} & \text{if } x > 0 \\[2ex] -\dfrac{|x|^c + |x|^{-c}}{2} & \text{if } x < 0 \end{cases}$$

where c is a complex number.

6.3 Existence of Discontinuous Solution of $g(x-y) = g(x)g(y)+f(x)f(y)$

Let $f,g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be functions such that f,g satisfy

$$(A_*) \qquad g(x-y) = g(x)g(y) + f(x)f(y)$$

for all x,y in $\mathbb{R}^n$. Since $\mathbb{R}^n$ has no subgroup of index 2, hence, by Remark 3.31, any solution (f,g) of (A_*) on $\mathbb{R}^n$ to $\mathbb{R}$ must be of the forms :

(6.3.1) $f(x) = b$, $g(x) = a$ for all x in $\mathbb{R}^n$, where $a,b \in [0,1)$ and $b = \sqrt{a - a^2}$; or

$$(6.3.2) \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i}, \quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from $\mathbb{R}^n$ to $\triangle$. Observe that each solution of the form (6.3.1) is continuous. Hence any discontinuous solution of (A_*), if exists, must be of the form (6.3.2). By Theorem 4.4, the solution of the form (6.3.2) is continuous if and only if h is continuous. Hence any discontinuous $h : \mathbb{R}^n \longrightarrow \triangle$ will provide a discontinuous solution (f,g) by (6.3.2). We have seen from Section 5.5 that such discontinuous homomorphism exists. Hence a discontinuous solution (f,g) of (A_*) on $\mathbb{R}^n$ to $\mathbb{R}$ exists.

6.4 Existence of Discontinuous Solution of $g(\frac{x}{y}) = g(x)g(y) + f(x)f(y)$

Let $f,g : \mathbb{R}^* \longrightarrow \mathbb{C}$ be functions such that f,g satisfy

$$(A^*) \qquad g(\frac{x}{y}) = g(x)g(y) + f(x)f(y)$$

for all x,y in $\mathbb{R}^*$. Since $\mathbb{R}^+$ is the only subgroup of index 2 of $\mathbb{R}^*$, and $\mathbb{R}^*$ has no subgroup of index 4, hence by Theorem 3.30 any solution of (A^*) on $\mathbb{R}^*$ to $\mathbb{C}$ must be of the forms :

(6.4.1) $\qquad f(x) = b \,,\; g(x) = a \qquad$ for all x in $\mathbb{R}^*$, where a,b are elements of $\mathbb{C}$ such that $a \neq 1$, $a-a^2 = b^2$; or

$$(6.4.2) \qquad f(x) = \begin{cases} b \,, & x \in \mathbb{R}^+ \\ -b \,, & x \in \mathbb{R}^- \end{cases} \,,\quad g(x) = \begin{cases} a \,, & x \in \mathbb{R}^+ \\ -a \,, & x \in \mathbb{R}^- \end{cases}$$

where a,b are elements of $\mathbb{C}$ such that $a \neq 1$, $a-a^2 = b^2$; or

$$(6.4.3) \qquad f(x) = \begin{cases} 0 \,, & x \in \mathbb{R}^+ \\ d \,, & x \in \mathbb{R}^- \end{cases} \,,\quad g(x) = \begin{cases} 1 \,, & x \in \mathbb{R}^+ \\ c \,, & x \in \mathbb{R}^- \end{cases}$$

where c,d are elements of $\mathbb{C}$ such that $c \neq 1$, $c^2 + d^2 = 1$; or

$$(6.4.4) \qquad f(x) = \frac{h(x) - h(x^{-1})}{2i} \,,\quad g(x) = \frac{h(x) + h(x^{-1})}{2}$$

where h is a homomorphism from $\mathbb{R}^*$ into $\mathbb{C}^*$.

Observe that each solution of the form (6.4.1) is continuous. Since $\mathbb{R}^+$ is an open subgroup of $\mathbb{R}^*$, hence by Corollary 4.3 each solution of the forms (6.4.2) and (6.4.3) is continuous. Hence any discontinuous solution of (A^*), if exist, must be of the form (6.4.4). By Theorem 4.4, the solution of the form (6.4.4) is continuous if and only if h is continuous. Hence any discontinuous $h : \mathbb{R}^* \longrightarrow \mathbb{C}^*$ will provide a discontinuous solution (f,g) by (6.4.4). The existence of discontinuous of such h is already discussed in Theorem 5.6.4. Hence a discontinuous solution (f,g) of (A^*) on $\mathbb{R}^*$ to $\mathbb{C}$ exists.