CHAPTER III

SEMILATTICES OF INVERSE SEMIGROUPS

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. In this chapter, the properties of being fundamental of S, S_α's, some kinds of ideals and some kinds of Rees quotient semigroups of S are studied, and many relations among them are given.

The first proposition of this chapter shows that a semilattice Y of inverse semigroups S_α is fundamental if S_α is fundamental for all $\alpha \in Y$. The following lemma is required :

3.1 Lemma. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α . For each $\alpha \in Y$, let $\mathcal{H}_\alpha$ denote the Green's relation $\mathcal{H}$ on S_α . Then for $a,b \in S$, if $a \mathcal{H} b$, then $a,b \in S_\alpha$ for some $\alpha \in Y$ and $a \mathcal{H}_\alpha b$.

Proof : Let $a,b \in S$ such that $a \mathcal{H} b$. Then $a \mathcal{L} b$ and $a \mathcal{R} b$. Since $a,b \in S = \bigcup_{\alpha \in Y} S_\alpha$, $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. Since $a \mathcal{L} b$, $Sa = Sb$, so $a = xb$ and $b = ya$ for some $x,y \in S$, say $x \in S_\gamma$, $y \in S_{\gamma'}$. Then $a = xb \in S_\gamma S_\beta \subseteq S_{\gamma\beta}$ and $b = ya \in S_{\gamma'} S_\alpha \subseteq S_{\gamma'\alpha}$. But $a \in S_\alpha$, $b \in S_\beta$, we have $\alpha = \gamma\beta$ and $\beta = \gamma'\alpha$. Hence $\alpha \leq \beta$ and $\beta \leq \alpha$ [Introduction, page 3], it then follows that $\alpha = \beta$. Therefore $a,b \in S_\alpha$. Next we show that $a \mathcal{H}_\alpha b$. From $\alpha = \gamma\beta$ and $\beta = \gamma'\alpha$, we have $\alpha \leq \gamma$ and $\alpha = \beta \leq \gamma'$, so $xbb^{-1} \in S_{\gamma\alpha} = S_\alpha$ and $yaa^{-1} \in S_{\gamma'\alpha} = S_\alpha$. Then $a = xb = (xbb^{-1})b$ and $b = (yaa^{-1})a$. This proves $a \mathcal{L}_\alpha b$ where $\mathcal{L}_\alpha$ denotes the Green's relation $\mathcal{L}$ on S_α . Dually, $a \mathcal{R} b$ implies $a \mathcal{R}_\alpha b$ where $\mathcal{R}_\alpha$ denotes the Green's relation $\mathcal{R}$ on S_α . Hence $a \mathcal{H}_\alpha b$. #

3.2 <u>Proposition</u>. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. If S_α is fundamental for all $\alpha \in Y$, then S is fundamental.

<u>Proof</u> : By Introduction, page 8 , S is an inverse semigroup. To show S is fundamental, let $(a,b) \in \mu(S)$. By Lemma 3.1, there exists $\alpha \in Y$ such that $a,b \in S_\alpha$ and $a \mathcal{H}_\alpha b$. From Lemma 2.1, we have $\mu(S) \cap (S_\alpha \times S_\alpha) \subseteq \mu(S_\alpha)$. Then $(a,b) \in \mu(S_\alpha)$. Since S_α is fundamental, a = b.

Hence $\mu(S)$ is the identity congruence, so S is fundamental as desired. #

The converse of Proposition 3.2 is not true in general as shown in the following example :

<u>Example</u>. Let $S = \{I, K, E_{11}, E_{12}, E_{21}, E_{22}, O\}$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. As shown in the example of Chapter II, under the usual matrix multiplication, S is a fundamental inverse semigroup. Let $Y = \{\alpha, \beta\}$ be a semilattice with its Hasse diagram :

Let $S_\alpha = \{ I, K \}$ and $S_\beta = \{O, E_{11}, E_{12}, E_{21}, E_{22}\}$. Then S_α and S_β

are inverse subsemigroups of S and $S = S_\alpha \cup S_\beta$ is a disjoint union. Moreover, from the table of multiplication, $S_\alpha S_\beta = S_\beta$. Hence S is a semilattice Y of inverse semigroups S_α and S_β. We can easily see that S_α is a nontrivial subgroup of S. Hence, S_α is not fundamental.

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. For each $\alpha \in Y$, let

$$A_\alpha = \bigcup_{\beta \leq \alpha} S_\beta \quad ;$$

equivalently,

$$A_\alpha = \bigcup_{\beta \in \alpha Y} S_\beta \quad .$$

Then for every $\alpha \in Y$, A_α is an ideal of S. To prove this, let $a \in A_\alpha$ and $x \in S$. Then $a \in S_\beta$ for some $\beta \leq \alpha$ and $x \in S_\gamma$ for some $\gamma \in Y$. Since $\beta \leq \alpha$, $\beta = \beta\alpha$ and so $\beta\gamma = \beta\alpha\gamma = (\beta\gamma)\alpha$. Then $\beta\gamma \leq \alpha$. Hence $S_{\beta\gamma} \subseteq A_\alpha$. Because $ax, xa \in S_{\beta\gamma} = S_{\gamma\beta}$, ax and $xa \in A_\alpha$. Hence A_α is an ideal of S. Since αY is a semilattice, A_α is a semilattice αY of inverse semigroups S_β .

From the above proof and Theorem 2.3, we have the following :

3.3 Proposition. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Then for each $\alpha \in Y$, A_α is an ideal of S, and it is fundamental if S is fundamental.

Since A_α is a semilattice αY of inverse semigroups S_β , the following proposition follows from Proposition 3.2 :

3.4 Proposition. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Let $\alpha \in Y$. If for each $\beta \leq \alpha$, S_β is fundamental, then A_α is fundamental.

The following corollary follows directly from Proposition 3.4:

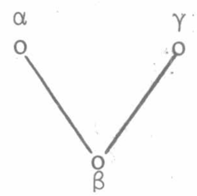
3.5 Corollary. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. If S_α is fundamental for all $\alpha \in Y$, then A_α is fundamental for all $\alpha \in Y$.

The next example shows that the converse of Corollary 3.5 is not true in general.

Example. Let $\overline{S}$ be the semigroup $S = \{I, K, E_{11}, E_{12}, E_{21}, E_{22}, 0\}$ (as in the first example of this chapter) adjoined the new element $\overline{0}$ and define the operation $*$ on $\overline{S}$ by

$$x * y = \begin{cases} xy & \text{if } x, y \in S, \\ \overline{0} & \text{if } x = y = \overline{0}, \\ 0 & \text{otherwise}. \end{cases}$$

Let Y be a semilattice with the Hasse diagram



and let $S_\alpha = \{I, K\}$, $S_\beta = \{0, E_{11}, E_{12}, E_{21}, E_{22}\}$ and $S_\gamma = \{\overline{0}\}$. Then S_α, S_β and S_γ are inverse subsemigroups of $\overline{S}$ and $\overline{S} = S_\alpha \cup S_\beta \cup S_\gamma$ is a disjoint union. Moreover, $S_\alpha S_\beta \subseteq S_\beta = S_{\alpha\beta}$, $S_\beta S_\alpha \subseteq S_\beta = S_{\beta\alpha}$, $S_\gamma S_\beta = S_\beta S_\gamma = \{0\} \subseteq S_\beta = S_{\beta\gamma} = S_{\gamma\beta}$, $S_\alpha S_\gamma = S_\gamma S_\alpha = \{0\} \subseteq S_\beta = S_{\alpha\gamma} = S_{\gamma\alpha}$.

Hence $\overline{S}$ is a semilattice Y of inverse semigroups S_α, S_β and S_γ . We have $A_\alpha = S_\alpha \cup S_\beta = S$, $A_\gamma = S_\gamma \cup S_\beta$ and $A_\beta = S_\beta$. We have shown that $A_\alpha = S$ is fundamental. A_β is an ideal of S, so it is fundamental. Since S_γ and S_β are fundamental and $A_\gamma = S_\gamma \cup S_\beta$ is a semilattice γY of inverse semigroups S_γ and S_β , it follows from Proposition 3.2, A_γ is fundamental.

Hence A_α, A_β and A_γ are fundamental. But S_α which is a nontrivial subgroup of S is not fundamental.

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. The following proposition shows that a sufficient condition for S to be fundamental is that A_α is fundamental for all $\alpha \in Y$.

3.6 Proposition. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Then A_α is fundamental for all $\alpha \in Y$ if and only if S is fundamental.

Proof : For each $\alpha \in Y$, A_α is an ideal of S. Then, if S is fundamental, then A_α is fundamental for all $\alpha \in Y$ [Theorem 2.3].

Conversely, assume A_α is fundamental for all $\alpha \in Y$. To show S is fundamental, let $(a,b) \in \mu(S)$. Then by Lemma 3.1, $a,b \in S_\lambda$ for some $\lambda \in Y$. Then $a,b \in A_\lambda$ since $A_\lambda = \bigcup_{\beta \leq \lambda} S_\beta$. Since A_λ is an ideal of S, by Lemma 2.2, we have $\mu(A_\lambda) = \mu(S) \cap (A_\lambda \times A_\lambda)$. Then $(a,b) \in \mu(A_\lambda)$. Since A_λ is fundamental by assumption, $\mu(A_\lambda)$ is the identity congruence on A_λ, so $a = b$. Hence $\mu(S)$ is the identity congruence on S, which implies S is fundamental. #

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α and A be an ideal of S. Let ρ_A be the Rees congruence on S induced by A. Then we have

3.7 Lemma. Let $a,b \in S$. If $(a\rho_A, b\rho_A) \in \mu(S/A)$, then either $a,b \in A$ or $a,b \in S_\alpha \setminus A$ for some $\alpha \in Y$.

Proof : Assume $(a\rho_A, b\rho_A) \in \mu(S/A)$. Since

$$E(S/A) = E(S/\rho_A) = \{ e\rho_A \mid e \in E(S) \} ,$$

it follows that $(a\rho_A)(e\rho_A)(a\rho_A)^{-1} = (b\rho_A)(e\rho_A)(b\rho_A)^{-1}$ for all $e \in E(S)$, so

$$(aea^{-1})\rho_A = (beb^{-1})\rho_A \qquad \text{for all } e \in E(S).$$

Suppose that $a \in A$ and $b \in S \setminus A$. Then $b^{-1}b \in E(S)$ and so

$$(ab^{-1}ba^{-1})\rho_A = (bb^{-1}bb^{-1})\rho_A = (b^{-1}b)\rho_A .$$

Because A is an ideal of S and $a \in A$, $ab^{-1}ba^{-1} \in A$ which implies $b^{-1}b \in A$. Hence $b = bb^{-1}b \in A$, a contradiction. Similarly, the case $b \in A$ and $a \in S \setminus A$ cannot occur. Hence we have either $a,b \in A$ or $a,b \in S \setminus A$.

Assume $a,b \in S \setminus A$. Then there exist $\alpha, \beta \in Y$ such that $a \in S_\alpha$ and $b \in S_\beta$. Then $a^{-1}a \in S \setminus A$, $a^{-1}a \in S_\alpha$ and $a^{-1}a \in E(S)$,

$$(aa^{-1})\rho_A = (a(a^{-1}a)a^{-1})\rho_A = (b(a^{-1}a)b^{-1})\rho_A .$$

Hence $b(a^{-1}a)b^{-1} = aa^{-1} \in S_\alpha$. But $ba^{-1}ab^{-1} \in S_{\alpha\beta}$, so $\alpha = \alpha\beta$. Because $b \in S \setminus A$, $b^{-1}b \in S \setminus A$. Since $b^{-1}b \in E(S)$ and $b^{-1}b \in S \setminus A$,

$$(a(b^{-1}b)a^{-1})\rho_A = (b(b^{-1}b)b^{-1})\rho_A = (b^{-1}b)\rho_A = \{b^{-1}b\} ,$$

and hence $a(b^{-1}b)a^{-1} = b^{-1}b \in S_\beta$. But $ab^{-1}ba^{-1} \in S_{\alpha\beta}$, so $\beta = \alpha\beta$.
Therefore $\alpha = \beta$ and then $a,b \in S_\alpha$. #

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Let I be an ideal of Y. Then $A_I = \bigcup_{\alpha \in I} S_\alpha$ is an ideal of S because for $\alpha \in I$, $\beta \in Y$, $a \in S_\alpha$, $b \in S_\beta$, we have $\alpha\beta = \beta\alpha \in I$ and ab, ba $\in S_\alpha S_\beta \subseteq S_{\alpha\beta} \subseteq A_I$. Since I is a semilattice, A_I is a semilattice I of inverse semigroups S_α.

From Lemma 3.7, we have the following proposition:

3.8 Proposition. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α, and let I be an ideal of Y. If $a,b \in S$ and $(a\rho_{A_I}, b\rho_{A_I}) \in \mu(S/A_I)$, then either $a,b \in A_I$ or $a,b \in S_\beta$ and $(a,b) \in \mu(S_\beta)$ for some $\beta \in Y \setminus I$.

Proof : Let $a,b \in S$ such that $(a\rho_{A_I}, b\rho_{A_I}) \in \mu(S/A_I)$. By Lemma 3.7, we have either $a,b \in A_I$ or $a,b \in S_\beta \setminus A_I$ for some $\beta \in Y$.

Assume $a,b \in S_\beta \setminus A_I$. Since $A_I = \bigcup_{\alpha \in I} S_\alpha$ and $S_\beta \not\subseteq A_I$, it follows that $\beta \notin I$, so $S_\beta \cap A_I = \phi$. Because $(a\rho_{A_I}, b\rho_{A_I}) \in \mu(S/A_I)$ and $E(S/A_I) = \{e\rho_{A_I} \mid e \in E(S)\}$, we have

$$(a\rho_{A_I})(e\rho_{A_I})(a\rho_{A_I})^{-1} = (b\rho_{A_I})(e\rho_{A_I})(b\rho_{A_I})^{-1}$$

for all $e \in E(S)$ and hence

$$(aea^{-1})\rho_{A_I} = (beb^{-1})\rho_{A_I}$$

for all $e \in E(S)$. Let $f \in E(S_\beta)$. Then $(afa^{-1})\rho_{A_I} = (bfb^{-1})\rho_{A_I}$. But afa^{-1} and $bfb^{-1} \in S_\beta$, so afa^{-1} and $bfb^{-1} \notin A_I$. Thus $afa^{-1} = bfb^{-1}$.

This proves $afa^{-1} = bfb^{-1}$ for all $f \in E(S_\beta)$. Therefore $(a,b) \in \mu(S_\beta)$. #

Because for each $\alpha \in Y$, $A_\alpha = \bigcup_{\beta \leq \alpha} S_\beta = \bigcup_{\beta \in \alpha Y} S_\beta$, we have

3.9 Corollary. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. Assume $\alpha \in Y$, $a,b \in S$. If $(a\rho_{A_\alpha}, b\rho_{A_\alpha}) \in \mu(S/A_\alpha)$, then either $a,b \in A_\alpha$ or $a,b \in S_\beta$ and $(a,b) \in \mu(S_\beta)$ for some $\beta \in Y$, $\beta \not\leq \alpha$.

Proposition 3.8 gives the following proposition :

3.10 Proposition. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α and I be an ideal of Y. If S_α is fundamental for all $\alpha \in Y$, then A_I and the Rees quotient semigroup S/A_I are fundamental.

Proof : Assume S_α is fundamental for all $\alpha \in Y$. Let I be an ideal of Y. Since A_I is a semilattice I of inverse semigroups S_α, by Proposition 3.2, A_I is fundamental. Next, let $a,b \in S$ such that $(a\rho_{A_I}, b\rho_{A_I}) \in \mu(S/A_I)$. From Proposition 3.8, we have either $a,b \in A_I$ or $(a,b) \in \mu(S_\beta)$ for some $\beta \in Y \setminus I$. If $a,b \in A_I$, then $a\rho_{A_I} = b\rho_{A_I}$. Assume $(a,b) \in \mu(S_\beta)$. Since S_β is fundamental by assumption, $\mu(S_\beta)$ is the identity congruence on S_β, so $a = b$ and hence $a\rho_{A_I} = b\rho_{A_I}$.

Hence $\mu(S/A_I)$ is the identity congruence on S/A_I which implies S/A_I is fundamental. #

Hence the following corollary follows clearly :

3.11 Corollary. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of inverse semigroups S_α. If S_α is fundamental for all $\alpha \in Y$, then the Rees quotient semigroup S/A_α is fundamental for all $\alpha \in Y$.

Finally, a conclusion about semilattices of inverse semigroups relating the property of being fundamental should be given as follows:

3.12 Theorem. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice of inverse semigroups S_α. If S_α is fundamental for all $\alpha \in Y$, then we have

(1) S is fundamental,

(2) A_α is fundamental for all $\alpha \in Y$,

and (3) S/A_α is fundamental for all $\alpha \in Y$.