CHAPTER II

IDEALS AND REES QUOTIENT SEMIGROUPS

It is shown in this chapter that any ideal of a fundamental inverse semigroup S is a fundamental inverse subsemigroup of S. Properties of Rees quotient inverse semigroups relating to being fundamental are studied. A Rees quotient semigroup of a fundamental inverse semigroup is not necessarily fundamental. It is proved that if an ideal A of an inverse semigroup S and the Rees quotient semigroup S/A are fundamental, then S is fundamental.

First, we recall that in any inverse semigroup S, the maximum idempotent-separating congruence of S, $\mu(S)$ or μ, always exists and

$$\mu = \{(a,b) \in S \times S \mid aea^{-1} = beb^{-1} \text{ for all } e \in E(S)\} \quad ;$$

equivalently,

$$\mu = \{(a,b) \in S \times S \mid a^{-1}ea = b^{-1}eb \text{ for all } e \in E(S)\}$$

and it is contained in the Green's relation $\mathcal{H}$ of S. An inverse semigroup S is fundamental if and only if $\mu(S)$ is the identity congruence on S, if and only if $E(S) = C(E(S))$, the centralizer of $E(S)$ in S.

It was shown in the first chapter that an inverse subsemigroup of a fundamental inverse semigroup is not necessarily fundamental.

An ideal A of an inverse semigroup S is an inverse subsemigroup of S. To see this, let $a \in A$. Then $a^{-1} \in S$, so $a^{-1} = a^{-1}aa^{-1} \in A$. It is clear that A is a subsemigroup of S. Hence A is an inverse subsemigroup of S.

The first theorem shows that an ideal of a fundamental inverse semigroup is fundamental. The two following lemmas are required :

2.1 Lemma. Let T be an inverse subsemigroup of an inverse semigroup S. Then $\mu(S) \cap (T \times T) \subseteq \mu(T)$.

Proof : Let $(a,b) \in \mu(S) \cap (T \times T)$. Then $a,b \in T$ and $(a,b) \in \mu(S)$, so $aea^{-1} = beb^{-1}$ for all $e \in E(S)$. Since $E(T) \subseteq E(S)$, $afa^{-1} = bfb^{-1}$ for all $f \in E(T)$. Then $(a,b) \in \mu(T)$. Hence $\mu(S) \cap (T \times T) \subseteq \mu(T)$. #

2.2 Lemma. Let A be an ideal of an inverse semigroup S. Then A is an inverse subsemigroup of S and $\mu(A) = \mu(S) \cap (A \times A)$.

Proof : We have shown that an ideal of an inverse semigroup S is an inverse subsemigroup of S.

Next, we show that $\mu(A) = \mu(S) \cap (A \times A)$. By Lemma 2.1, $\mu(S) \cap (A \times A) \subseteq \mu(A)$. It remains to show that $\mu(A) \subseteq \mu(S) \cap (A \times A)$. Let $a,b \in A$ and $(a,b) \in \mu(A)$. Then

$$aea^{-1} = beb^{-1}$$

for all $e \in E(A)$; equivalently,

$$a^{-1}ea = b^{-1}eb$$

for all $e \in E(A)$. To show $(a,b) \in \mu(S)$, let $f \in E(S)$. Then

$$
\begin{aligned}
afa^{-1} &= a(a^{-1}afa^{-1}a)a^{-1} \\
&= b(a^{-1}afa^{-1}a)b^{-1} && \text{(because } a^{-1}afa^{-1}a \in E(A)) \\
&= bb^{-1}ba^{-1}afa^{-1}ab^{-1}bb^{-1} \\
&= ba^{-1}a(b^{-1}bfb^{-1}b)a^{-1}ab^{-1} \\
&= ba^{-1}b(b^{-1}bfb^{-1}b)b^{-1}ab^{-1} && \text{(because } b^{-1}bfb^{-1}b \in E(A)) \\
&= ba^{-1}(bfb^{-1})ab^{-1} \\
&= bb^{-1}(bfb^{-1})bb^{-1} && \text{(because } bfb^{-1} \in E(A)) \\
&= bfb^{-1}.
\end{aligned}
$$

Therefore $(a,b) \in \mu(S)$.

Hence, we have $\mu(A) = \mu(S) \cap (A \times A)$ as required. #

2.3 Theorem. An ideal of a fundamental inverse semigroup is fundamental.

Proof : Let A be an ideal of a fundamental inverse semigroup S. By Lemma 2.2, A is an inverse subsemigroup of S and $\mu(A) = \mu(S) \cap (A \times A)$. Because S is fundamental, $\mu(S)$ is the identity congruence on S, so $\mu(S) \cap (A \times A)$ is the identity congruence on A. Therefore $\mu(A)$ is the identity congruence on the ideal A. This shows that A is fundamental. #

If S is a semigroup and $a \in S$, it is clear that SaS is an ideal of S. Let A be an ideal of a semigroup S. Then A is called a principal ideal if and only if A can be generated by an element of S; equivalently, $A = S^1aS^1$ for some $a \in S$. If S is an inverse semigroup and $a \in S$, then $S^1aS^1 = SaS$. To show this, let $xay \in S^1aS^1$, $x,y \in S^1$. Since $x,y \in S^1$, $xaa^{-1} \in S$ and $a^{-1}ay \in S$. Then

$$xay = xaa^{-1}ay = xaa^{-1}aa^{-1}ay = (xaa^{-1})a(a^{-1}ay) \in SaS.$$

Hence, A is a principal ideal of an inverse semigroup S if and only if $A = SaS$ for some $a \in S$.

By Theorem 2.3, the following corollary follows :

2.4 Corollary. Let S be a fundamental inverse semigroup. Then for each $a \in S$, the principal ideal SaS is fundamental.

The next proposition shows that the converse of Corollary 2.4 is true.

2.5 Proposition. Let S be an inverse semigroup. If every principal ideal of S is fundamental, then S is fundamental.

Proof : Let $(a,b) \in \mu(S)$. Since $\mu \subseteq \mathcal{H} \subseteq \mathcal{J}$, $(a,b) \in \mathcal{J}$. Thus $S^1aS^1 = S^1bS^1$, so $SaS = S^1aS^1 = S^1bS^1 = SbS$. Then $a,b \in SaS$. Since $(a,b) \in \mu(S)$,

$$a^{-1}ea = b^{-1}eb$$

for all $e \in E(S)$. But $E(SaS) \subseteq E(S)$, then

$$a^{-1}fa = b^{-1}fb$$

for all $f \in E(SaS)$. Hence $(a,b) \in \mu(SaS)$. By assumption, SaS is fundamental, so $\mu(SaS)$ is the identity congruence on SaS. Hence $a = b$.

This proves that $\mu(S)$ is the identity congruence on S, hence S is fundamental. #

Let S be a semigroup and A be an ideal of S. Let ρ_A be the

Rees congruence on S induced by the ideal A, that is,

$$a\rho_A = \begin{cases} \{a\} & \text{if } a \notin A, \\ A & \text{if } a \in A. \end{cases}$$

Then the quotient semigroup S/ρ_A is a semigroup with zero, and $a\rho_A$ is the zero of S/ρ_A if and only if $a \in A$. Recall that the semigroup S/ρ_A is called the Rees quotient semigroup of S induced by the ideal A, and denoted by S/A. Because a homomorphic image of an inverse semigroup is an inverse semigroup, if S is an inverse semigroup, then S/A is an inverse semigroup. By Introduction, page 5, if S is an inverse semigroup, then

$$E(S/\rho_A) = \{ e\rho_A \mid e \in E(S) \} ,$$

so

$$E(S/A) = \{e\rho_A \mid e \in E(S) \setminus A\} \cup \{a\rho_A\}$$

if $a \in A$.

Theorem 2.3 shows that an ideal A of a fundamental inverse semigroup S is fundamental, but the following example shows that the Rees quotient semigroup S/A need not be fundamental.

Example. Let $S = \{ I, K, E_{11}, E_{12}, E_{21}, E_{22}, 0 \}$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then under the usual matrix multiplication, we have the following table :

$\cdot$	I	K	E_{11}	E_{12}	E_{21}	E_{22}	0
I	I	K	E_{11}	E_{12}	E_{21}	E_{22}	0
K	K	I	E_{21}	E_{22}	E_{11}	E_{12}	0
E_{11}	E_{11}	E_{12}	E_{11}	E_{12}	0	0	0
E_{12}	E_{12}	E_{11}	0	0	E_{11}	E_{12}	0
E_{21}	E_{21}	E_{22}	E_{21}	E_{22}	0	0	0
E_{22}	E_{22}	E_{21}	0	0	E_{21}	E_{22}	0
0	0	0	0	0	0	0	0

Thus, S is a semigroup and $E(S) = \{I, E_{11}, E_{22}, 0\}$. From the table, we can see that any two idempotents of S commute. Moreover,

$$KKK = K,\ E_{12}E_{21}E_{12} = E_{12},\ E_{21}E_{12}E_{21} = E_{21}.$$

Then every element of S is regular. By Introduction, page 2, S is an inverse semigroup, and

$$K^{-1} = K,\ E_{12}^{-1} = E_{21},\ E_{21}^{-1} = E_{12},\ I^{-1} = I,\ E_{11}^{-1} = E_{11},\ E_{22}^{-1} = E_{22},$$
$0^{-1} = 0$.

To show that S is fundamental, it is equivalent to show $C(E(S)) = E(S)$. Since $E(S) \subseteq C(E(S))$, it suffices to show that for each $x \in S$, $x \notin E(S)$, there exists $e \in E(S)$ such that $ex \neq xe$, which implies $x \notin C(E(S))$.

All the nonidempotents of S are K, E_{12}, E_{21}. From the table, we have

$$
\begin{aligned}
KE_{22} &= E_{12} \neq E_{21} = E_{22}K , \\
E_{12}E_{22} &= E_{12} \neq 0 = E_{22}E_{12} , \\
E_{21}E_{22} &= 0 \neq E_{21} = E_{22}E_{21} .
\end{aligned}
$$

Therefore $C(E(S)) = E(S)$, so S is fundamental.

Let $A = \{E_{11}, E_{12}, E_{21}, E_{22}, 0\}$. From the table, A is an ideal of S. Let ρ_A be the congruence on S induced by the ideal A. Then

$$S/A = \{I\rho_A , K\rho_A , 0\rho_A\} ,$$

and $\quad E(S/A) = \{I\rho_A , 0\rho_A\} , \; (K\rho_A)^{-1} = K^{-1}\rho_A = K\rho_A .$

Because

$$
\begin{aligned}
(I\rho_A)(I\rho_A)(I\rho_A)^{-1} &= I\rho_A , \\
(K\rho_A)(I\rho_A)(K\rho_A)^{-1} &= I\rho_A , \\
(I\rho_A)(0\rho_A)(I\rho_A)^{-1} &= 0\rho_A ,
\end{aligned}
$$

and $\quad (K\rho_A)(0\rho_A)(K\rho_A)^{-1} = 0\rho_A ,$

it follows that $(I\rho_A, K\rho_A) \in \mu(S/A)$. But $I\rho_A \neq K\rho_A$. Then $\mu(S/A)$ is not the identity congruence on S/A. Hence S/A is not fundamental.

Since $E_{11} = IE_{11}I$, $E_{12} = IE_{11}E_{12}$, $E_{21} = E_{21}E_{11}I$, $E_{22} = E_{21}E_{11}E_{12}$, $0 = 0E_{11}0$, we have $A = SE_{11}S$, hence A is a principal ideal of S.

This also shows that the Rees quotient of a fundamental inverse semigroup S induced by a principal ideal of S is not necessarily fundamental.

Let S be a semigroup. An ideal A of S is called a <u>completely prime ideal</u> of S if for $a,b \in S$, $ab \in A$ imply $a \in A$ or $b \in A$.

The ideal A from the above example is completely prime by checking from the table of multiplication.

Hence, we have a remark that a Rees quotient of a fundamental inverse semigroup S induced by a completely prime principal ideal of S is still not necessarily fundamental.

It is obviously seen that if the Rees quotient semigroup of an inverse semigroup S is fundamental, the inverse semigroup S need not be fundamental. However, we have that if the Rees quotient semigroup of an inverse semigroup S induced by an ideal A is fundamental and the ideal A is fundamental, then so is S.

2.6 Theorem. Let A be an ideal of an inverse semigroup S. If A and the Rees quotient semigroup S/A are fundamental, then S is fundamental.

Proof : Assume that A and S/A are both fundamental inverse semigroups. To show S is fundamental, let $a,b \in S$ such that $(a,b) \in \mu(S)$. Then

$$aea^{-1} = beb^{-1}$$

for all $e \in E(S)$. To show that $a = b$, first we show that the case $a \in A$, $b \in S\setminus A$ and the case $b \in A$, $a \in S\setminus A$ cannot occur.

Suppose $a \in A$, $b \in S\setminus A$. Since $(a,b) \in \mu \subseteq \mathcal{H} \subseteq \mathcal{R}$, we have $a \mathcal{R} b$, so $aS = bS$. Then $b = ax$ for some $x \in S$. Because A is an ideal of S and $a \in A$, it follows that $b = ax \in A$ which is a contradiction. Therefore, it is impossible that $a \in A$ and $b \in S\setminus A$. Similarly, the case $b \in A$, $a \in S\setminus A$ cannot occur.

Hence, we have either $a,b \in A$ or $a,b \in S \setminus A$.

Assume $a,b \in A$. Since A is an ideal of S, $\mu(A) = \mu(S) \cap (A \times A)$ [Lemma 2.2]. Because $(a,b) \in \mu(S)$, $(a,b) \in \mu(A)$. The ideal A is fundamental by assumption, so $\mu(A)$ is the identity congruence on A. Hence $a = b$.

Assume $a,b \in S \setminus A$. Let ρ_A be the Rees congruence on S induced by the ideal A. Then $S / A = S/\rho_A$. Because $aea^{-1} = beb^{-1}$ for all $e \in E(S)$, we have, for each $e \in E(S)$,

$$\begin{aligned}(a\rho_A)(e\rho_A)(a\rho_A)^{-1} &= (a\rho_A)(e\rho_A)(a^{-1}\rho_A) \\ &= (aea^{-1})\rho_A \\ &= (beb^{-1})\rho_A \\ &= (b\rho_A)(e\rho_A)(b\rho_A)^{-1} .\end{aligned}$$

But $E(S/A) = E(S/\rho_A) = \{ e\rho_A \mid e \in E(S) \}$ [Introduction, page 5] . Then $(a\rho_A, b\rho_A) \in \mu(S/\rho_A) = \mu(S/A)$. Because S/A is fundamental, $\mu(S/A)$ is the identity congruence on S/A, so $a\rho_A = b\rho_A$. Since $a,b \in S \setminus A$, we have $a\rho_A = \{a\}$ and $b\rho_A = \{b\}$, hence $a = b$.

This proves that $\mu(S)$ is the identity congruence on S, and hence S is fundamental as required. #