CHAPTER III

INTEGRAL GEOMETRY OVER SETS OF POINTS, SETS OF STRAIGHT LINES, SETS OF PAIRS OF POINTS, SETS OF PAIRS OF STRAIGHT LINES AND SETS OF KINEMATICS.

Section 3.1 Integral Geometry Over Sets of Points.

3.1.1 Density and measure for sets of points.

Let (X, Φ) be a fixed rectangular cartesian coordinate neighborhood with (x,y) as its coordinate functions. We can write a density ω in the form

$$\omega = f(x,y)dx\,dy \qquad \text{where } f : \Phi(X) \longrightarrow \mathbb{R}$$

We want to determine all continuous densities ω which will make $m(A)$ invariant under the group of Euclidean motion $\mathfrak{M}$. Let $g \in \mathfrak{M}$ be represented by the equations

$$\varphi(x,y) = x^* = a + x\cos\theta + y\sin\theta$$

$$\psi(x,y) = y^* = b - x\sin\theta + y\cos\theta$$

i.e we want to find ω such that

$$\int_A \omega = \int_{g(A) = A^*} \omega \qquad \forall g \in \mathfrak{M}$$

i.e $$\iint_{\Phi(A)} f(x,y)dx\,dy = \iint_{\Phi(A^*)} f(x^*,y^*)\,dx^*dy^*$$

To find this, we have

$$(3.1) \quad \iint\limits_{\Phi(A^*)} f(x^*,y^*)dx^*dy^* = \iint\limits_{\Phi(A)} f(\varphi(x,y), \psi(x,y)) \left| \frac{\partial(x^*,y^*)}{\partial(x,y)} \right| dx\, dy$$

$$\text{but} \quad \left| \frac{\partial(x^*,y^*)}{\partial(x,y)} \right| = \begin{vmatrix} \frac{\partial x^*}{\partial x} & \frac{\partial x^*}{\partial y} \\ \frac{\partial y^*}{\partial x} & \frac{\partial y^*}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$(3.2) \quad \text{i.e} \quad \left| \frac{\partial(x^*,y^*)}{\partial(x,y)} \right| = 1$$

From (3.1) and (3.2) we get

$$\iint\limits_{\Phi(A)} f(x,y)\, dx\, dy = \iint\limits_{\Phi(A)} f(\varphi(x,y), \psi(x,y))\, dx\, dy \quad \forall \text{domain } \Phi(A)$$

and in order that this equality hold for any domain $\Phi(A)$ it must be true that

$$f(x,y) = f(\varphi(x,y), \psi(x,y)) \quad \forall g \in \mathfrak{M}$$

Since the point x,y can be transformed by a motion into any other point x^*, y^*, the last equality signifies that $f(x,y)$ takes the same value for all points of the plane. Thus

$$f(x, y) = \text{constant}$$

We normalize the situation by choosing the constant to be 1, so we have

$$m(A) = \iint\limits_{\Phi(A)} dx\, dy$$

Up to a constant factor , this measure is the only one which is invariant under the group of Euclidean motions in the plane. We denote this density dx dy by dP.

3.1.2 Theorem : (Crofton). Let C be a closed convex plane curve having a tangent at every point and P be the exterior point of C. Then

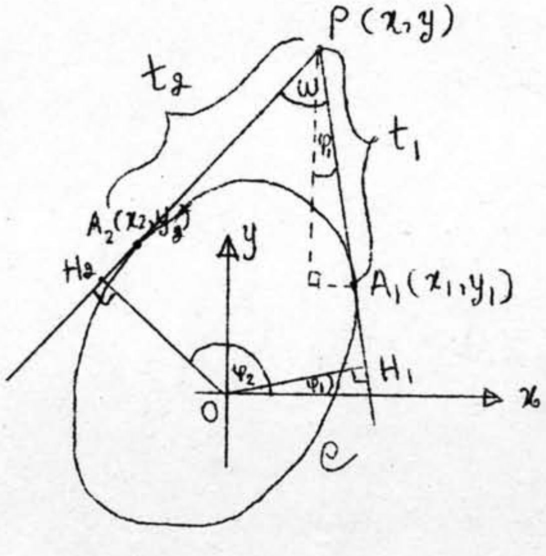
$$\int_{\sigma} \frac{\sin \omega}{t_1 t_2} \, dP = 2\pi^2$$

P is the exterior point of C

Where t_1 and t_2 represent the length of the tangents to C from P and ω is the angle of these two tangents at P.

Proof :

Figure 3



From each point P exterior to C, there can be drawn two tangents to C, say PA_1 and PA_2 (see Figure 3). To each of these tangents corresponds an angle φ_1 (and φ_2) formed by the perpendicular OH_1 (and OH_2) with a fixed direction OX. Conversely, the two angles φ_1, φ_2 determine the point P. We want to express the density dP in terms of the coordinates φ_1 and φ_2.

Let x_1, y_1 be the coordinates of the point of tangency A_1 and x, y, the coordinates of P. Let D be the exterior of C. We see that slope of $PA_1 = \tan(90 + \varphi_1) = -\cot\varphi_1$

The equation of PA_1 is $\frac{y-y_1}{x-x_1} = -\cot\varphi_1$ where $\begin{cases} \varphi_1 : D \to [0, 2\pi] \\ x_1 : D \to \mathbb{R} \\ y_1 : D \to \mathbb{R} \end{cases}$

we multiply both sides by $(x-x_1)\sin\varphi_1$, we get

$$\frac{y-y_1}{x-x_1} \cdot (x-x_1)\sin\varphi_1 = -\frac{\cos\varphi_1}{\sin\varphi_1} \cdot (x-x_1)\sin\varphi_1$$

This implies ;

$$\sin\varphi_1 (y-y_1) = -\cos\varphi_1 (x-x_1)$$

Adding $\cos\varphi_1(x-x_1)$ to both sides we get

$$\sin\varphi_1 (y-y_1) + \cos\varphi_1(x-x_1) = -\cos\varphi_1(x-x_1) + \cos\varphi_1(x-x_1)$$

$$(x-x_1)\cos\varphi_1 + (y-y_1)\sin\varphi_1 = 0 \quad \text{as functions on D}$$

By differentiation, we get

$$d\left[(x-x_1)\cos\varphi_1 + (y-y_1)\sin\varphi_1\right] = d(0) = 0$$

as diff. 1 - form on D

From Remark 2.30, we get

$$d\left[(x-x_1)\cos\varphi_1\right] + d\left[(y-y_1)\sin\varphi_1\right] = 0$$

$$(x-x_1)\, d(\cos\varphi_1) + \cos\varphi_1(dx - dx_1) + (y - y_1)d(\sin\varphi_1)$$
$$+ \sin\varphi_1(dy - dy_1) = 0$$

$$-(x-x_1)\sin\varphi_1\, d\varphi_1 + \cos\varphi_1 dx - \cos\varphi_1 dx_1 + \cos\varphi_1(y-y_1)d\varphi_1$$
$$+ \sin\varphi_1 dy - \sin\varphi_1 dy_1 = 0$$

(3.3) $$\cos\varphi_1 dx + \sin\varphi_1 dy - (\cos\varphi_1 dx_1 + \sin\varphi_1 dy_1) = \Big[(x-x_1)\sin\varphi_1 - (y-y_1)\cos\varphi_1\Big] d\varphi_1$$

but (x_1, y_1) is the coordinate of pt. on C, we can write

$$y_1 = f(x_1) \qquad \text{where } f : \mathbb{R} \longrightarrow \mathbb{R}$$

By differentiation we get

$$dy_1 = f'(x_1)\, dx_1$$
$$= -\cot\varphi_1 dx_1$$
$$= -\frac{\cos\varphi_1}{\sin\varphi_1} dx_1$$

(3.4) Thus ; $\cos\varphi_1 dx_1 + \sin\varphi_1 dy_1 = 0$

Let $(x-x_1)\sin\varphi_1 - (y-y_1)\cos\varphi_1 = 1$

Observe that $\sin\varphi_1 = \dfrac{x_1 - x}{t_1}$ and $\cos\varphi_1 = \dfrac{y - y_1}{t_1}$

therefore $(x - x_1)\left(\dfrac{x_1-x}{t_1}\right) - (y-y_1)\left(\dfrac{y-y_1}{t_1}\right) = 1$

$$-\frac{(x-x_1)^2}{t_1} - \frac{(y-y_1)^2}{t_1} = 1$$

$$-\left\{\frac{(x-x_1)^2 + (y-y_1)^2}{t_1}\right\} = 1$$

But $t_1^2 = (x-x_1)^2 + (y - y_1)^2$, we get

$$1 = -t_1$$

(3.5) i.e $(x-x_1)\sin\varphi_1 - (y-y_1)\cos\varphi_1 = -t_1$

Substituting (3.4) and (3.5) in (3.3) we get

(3.6) $$\cos\varphi_1 dx + \sin\varphi_1 dy = -t_1 d\varphi_1$$

Similarly the equation of PA_2 is $(x-x_2)\cos\varphi_2 + (y-y_2)\sin\varphi_2 = 0$

This equation gives

(3.7) $$\cos\varphi_2\, dx + \sin\varphi_2 dy = -t_2\, d\varphi_2$$

By exterior product we obtain from (3.6) and (3.7) the relation

$$(\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2)\, dx\, dy = t_1 t_2 d\varphi_1 d\varphi_2$$

$$\sin(\varphi_2 - \varphi_1)\, dx\, dy = t_1 t_2 d\varphi_1\, d\varphi_2$$

Furthermore, $\varphi_2 - \varphi_1 = \pi - \omega$. Consequently we have

$$dP = dx\, dy = \frac{t_1 t_2}{\sin\omega}\, d\varphi_1\, d\varphi_2$$

or $$\frac{\sin\omega}{t_1 t_2}\, dP = d\varphi_1\, d\varphi_2$$

Integrate both terms of this equality over all possible different values of the variables, we observe that P can vary over all points exterior to C, and φ_1, φ_2 can vary from 0 to 2π. However, if in each position we permute φ_1 and φ_2, we get the same point P; consequently, in order to count each point P only once, we must divide by 2. We have therefore

$$\int_{P \text{ is exterior to } C} \frac{\sin\omega}{t_1 t_2}\, dP = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} d\varphi_1\, d\varphi_2 = 2\pi^2$$

Q.E.D.

3.1.3 Remark : We see that this value of the integral does not depend on the convex curve C. So that this convex curve can be any size and shape.

Section 3.2 Integral Geometry Over Sets of Straight Lines.

3.2.1 Density and measure for sets of straight lines.

Choose a rectangular cartesian coordinate system (x, y) determined by two given orthogonal straight lines.. We denote by x^+ the positive part of the x - axis and y^+ the positive part of the y - axis. The point O is the origin and x^-, y^- are the negative parts of the x, y - axes respectively.

Let Y be the set of straight lines in the Euclidean Plane X. We want to give every straight line in Y a coordinate. To do this, we need 3 coordinate neighborhoods $\{(U_1, \Phi_1), (U_2, \Phi_2), (U_3, \Phi_3)\}$ to form the atlas such that

$U_1 = \{\text{st. lines } l \mid O \notin l \text{ and } l \text{ not } \perp \text{ with } x^+\text{- axis}\}$ where (x, y) are the coordinate functions of Φ_1.

$U_2 = \{\text{st. lines } l \mid O' \notin l \text{ and } l \text{ not } \perp \text{ with } x'^+\text{- axis}\}$ where O' is the translations of O, one unit in the positive direction along x - axis and x' - axis and y' - axis are the rotation of the old x and y - axis by 90° in the counter clockwise direction

$U_3 = \{\text{st. lines } l \mid O'' \notin l \text{ and } l \text{ not } \perp \text{ with } x''^+\text{- axis}\}$

where O'' is the translations of O, one unit in the positive direction along y - axis and x'' and y'' - axis are the same direction as the old x and y - axis (see Figure 4)

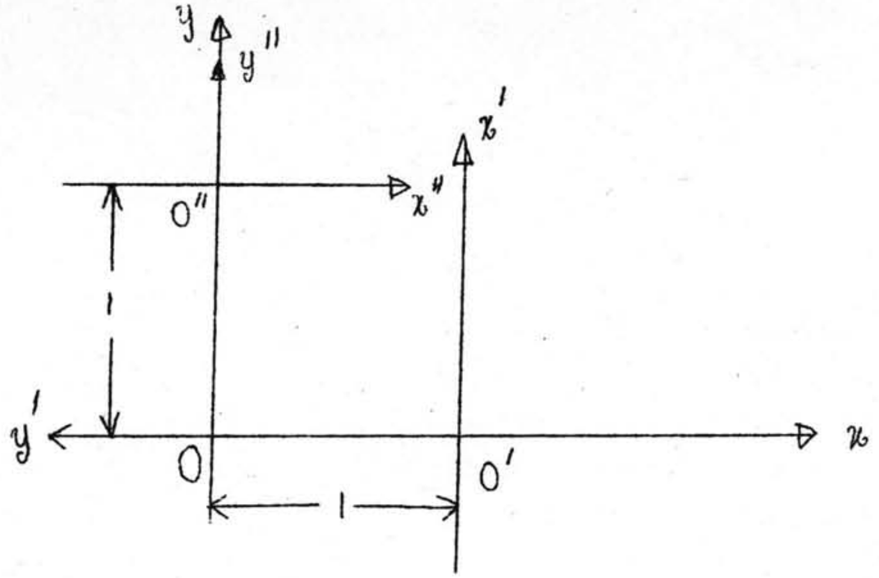


Figure 4

so that we can give every straight lines a coordinate (p, φ) such that $p > 0$ and $0 < \varphi < 2\pi$. We do not allow $O \in l$ because if $O \in l$ then φ is not defined and the range is not an open set. Similarly we do not allow $l \perp x^+$- axis because if $l \perp x^+$ - axis then φ can not be made continuous and the range is not open.

We will show that these three coordinate neighborhoods are c' - related. Firstly, we will show that (U_1, Φ_1) and (U_2, Φ_2) are c' - related. We see that $U_1 \cap U_2 \neq \emptyset$
Next, show that $(\Phi_1 \circ \Phi_2^{-1})$ and $(\Phi_2 \circ \Phi_1^{-1})$ are c' - maps.

There are 4 possible cases that a straight line can be lie in (U_1, Φ_1) and (U_2, Φ_2). Let (p^1, φ^1) be a coordinate of a straight line with respect to (U_2, Φ_2).

case 1

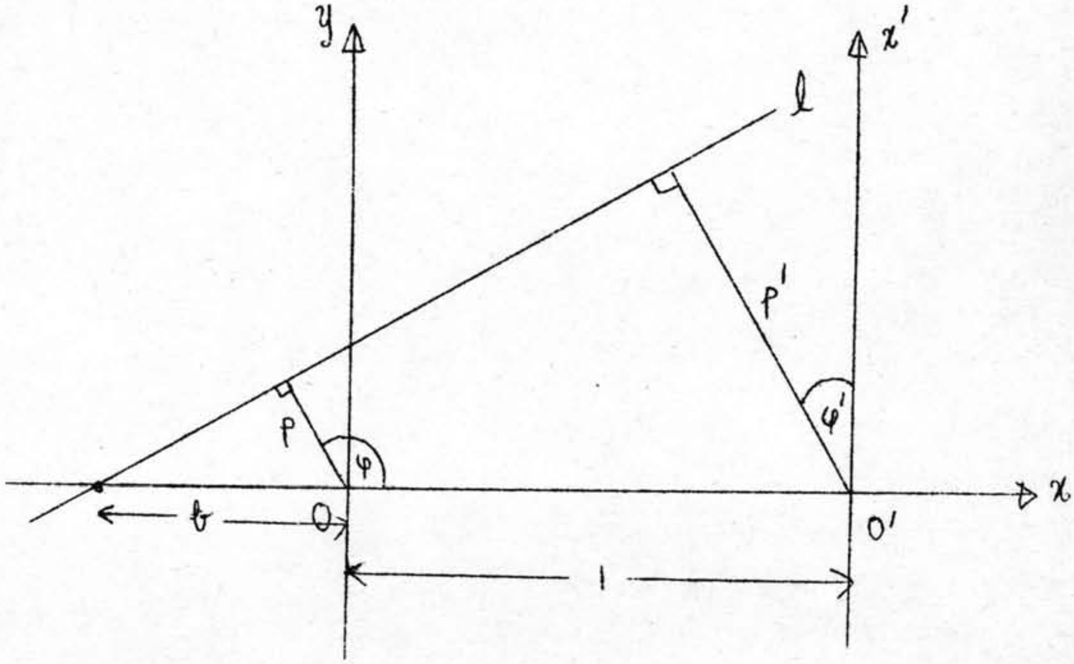


Figure 5

We see that $\varphi^1 = \varphi - \frac{\pi}{2}$

(3.8) and $\frac{p}{b} = \frac{p^1}{1+b}$ where b is shown in Figure 5.

but $\cos(\pi - \varphi) = \frac{p}{b}$

(3.9) so $b = \frac{p}{-\cos\varphi}$

From (3.8) and (3.9) we get

$$p' = \frac{p(1+b)}{b}$$

$$= p(1 + \frac{(-\cos\varphi)}{p})$$

$$= p - \cos\varphi$$

So, $\Phi_1 \circ \Phi_2^{-1} : \Phi_2(U_1 \cap U_2) \longrightarrow \Phi_1(U_1 \cap U_2)$ is C' - map.

and we have $p = p' + \cos\varphi$

$$\varphi = \varphi' + \frac{\pi}{2}$$

So, $\Phi_2 \circ \Phi_1^{-1} : \Phi_1(U_1 \cap U_2) \longrightarrow \Phi_2(U_1 \cap U_2)$ is C' - map.

Observe that

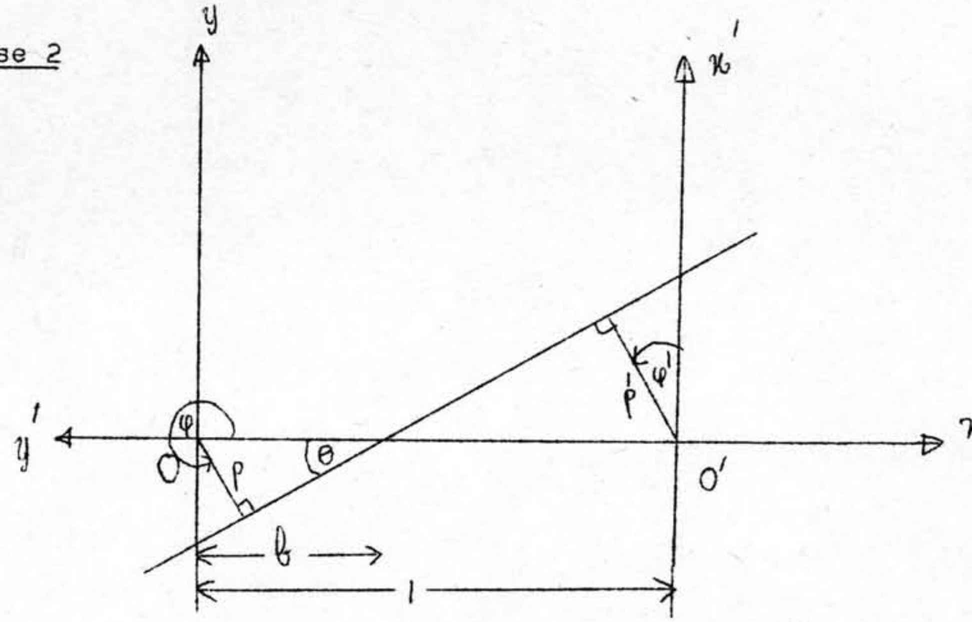
$$\text{Jac}(\Phi_1 \circ \Phi_2^{-1}) = \frac{\partial(p', \varphi')}{\partial(p, \varphi)} = \begin{vmatrix} \frac{\partial p'}{\partial p} & \frac{\partial p'}{\partial \varphi} \\ \frac{\partial \varphi'}{\partial p} & \frac{\partial \varphi'}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \sin\varphi \\ 0 & 1 \end{vmatrix} = 1$$

and $$\text{Jac}(\Phi_2 \circ \Phi_1^{-1}) = \frac{\partial(p, \varphi)}{\partial(p', \varphi')} = \begin{vmatrix} \frac{\partial p}{\partial p'} & \frac{\partial p}{\partial \varphi'} \\ \frac{\partial \varphi}{\partial p'} & \frac{\partial \varphi}{\partial \varphi'} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Figure 6

case 2



We see that $\varphi' = \varphi - \frac{3\pi}{2}$ and

$$\sin \theta = \frac{p}{b} = \frac{p'}{1 - b}$$ where θ and b are shown in Figure 6.

(3.10) this implies, $p' = p(\frac{1}{b} - 1)$

but $\cos (2\pi - \varphi) = \frac{p}{b}$

(3.11) so $b = \frac{p}{\cos \varphi}$

From (3.10) and (3.11) we get

$$p' = p(\frac{\cos \varphi}{p} - 1)$$

$$= \cos \varphi - p$$

and also $p = \cos \varphi - p'$

$$\varphi = \varphi' + \frac{3\pi}{2}$$

So $\Phi_1 \circ \Phi_2^{-1}$ and $\Phi_2 \circ \Phi_1^{-1}$ are C^1 - maps.

Observe that $\text{Jac}\,(\Phi_1 \circ \Phi_2^{-1}) = \begin{vmatrix} -1 & -\sin\varphi \\ 0 & 1 \end{vmatrix} = -1$

and $\text{Jac}\,(\Phi_2 \circ \Phi_1^{-1}) = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1$

case 3

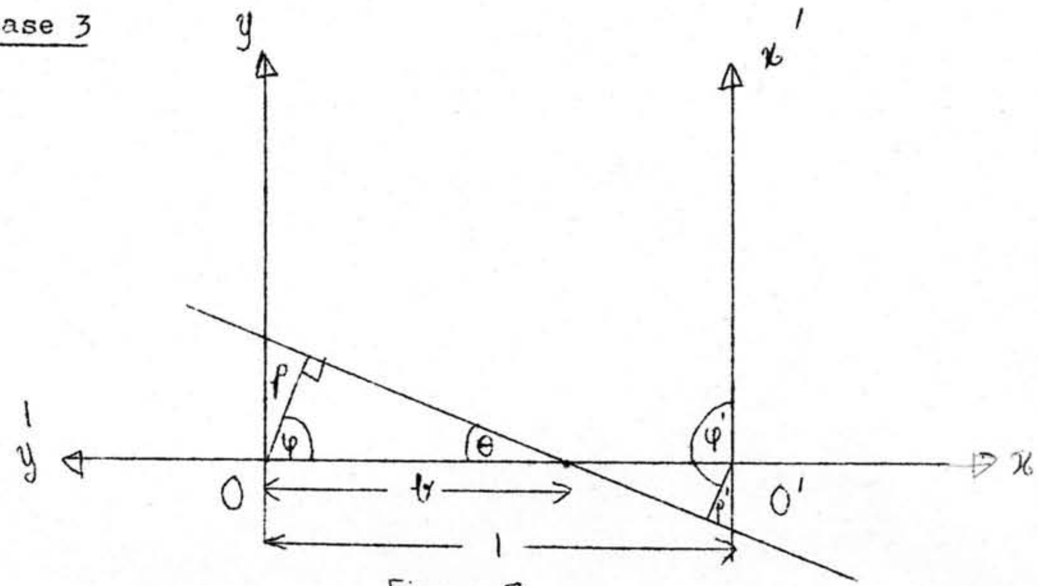


Figure 7

We see that $\varphi' = \varphi + \frac{\pi}{2}$ and

$$\sin\theta = \frac{p}{b} = \frac{p'}{1-b} \quad \text{this implies}$$

(3.12) $$p' = p\left(\frac{1}{b} - 1\right)$$

and $\cos\varphi = \frac{p}{b}$

(3.13) $$b = \frac{p}{\cos\varphi}$$

From (3.12) and (3.13) we get

$$p' = \cos\varphi - p$$

and also $$p = \cos\varphi - p'$$

$$\varphi = \varphi' - \frac{\pi}{2}$$

So, $\Phi_1 \circ \Phi_2^{-1}$ and $\Phi_2 \circ \Phi_1^{-1}$ are C' - maps.

$$\text{Observe that Jac } (\Phi_1 \circ \Phi_2^{-1}) = \begin{vmatrix} -1 & -\sin\varphi \\ 0 & 1 \end{vmatrix} = -1$$

$$\text{and} \quad \text{Jac } (\Phi_2 \circ \Phi_1^{-1}) = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1$$

case 4

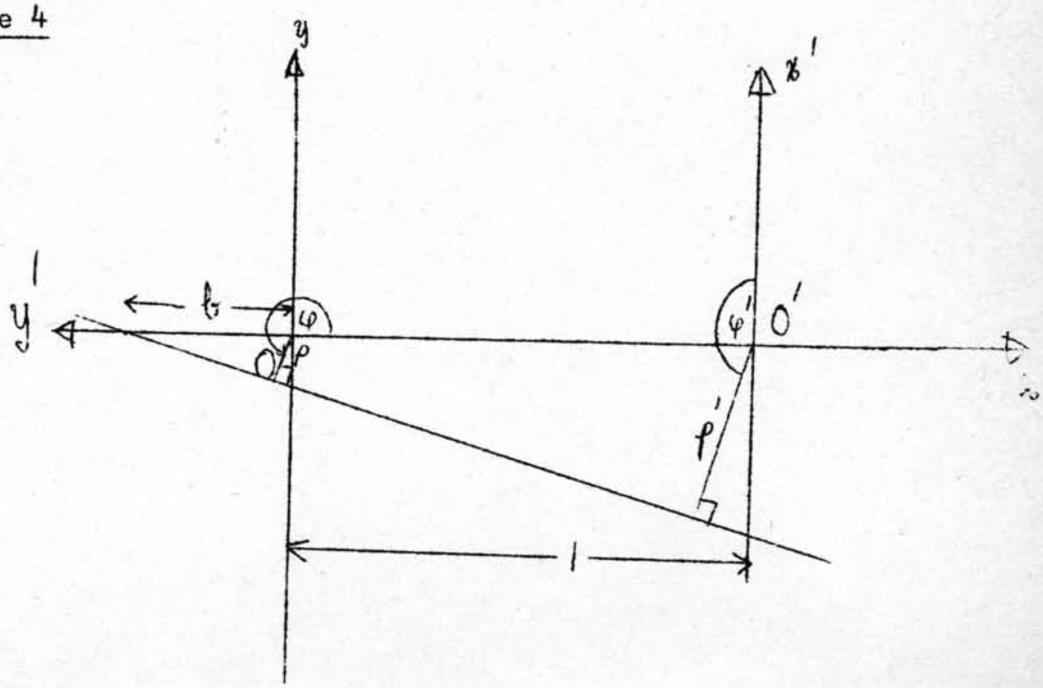


Figure 8

We see that $\varphi' = \varphi - \frac{\pi}{2}$ and

$$\frac{p}{b} = \frac{p'}{1+b} \quad \text{this implies}$$

(3.14) $$p' = p(\frac{1}{b} + 1)$$

$$\text{but } \cos(\varphi - \pi) = \frac{p}{b}$$

(3.15) so $$b = \frac{p}{-\cos\varphi}$$

From (3.14) and (3.15) we get

$$p' = -\cos\varphi + p$$

and also $p = p' + \cos\varphi$

$$\varphi = \varphi' + \frac{\pi}{2}$$

So, $\Phi_1 \circ \Phi_2^{-1}$ and $\Phi_2 \circ \Phi_1^{-1}$ are C' - maps

Observe that $\text{Jac}\,(\Phi_1 \circ \Phi_2^{-1}) = \begin{vmatrix} 1 & \sin\varphi \\ 0 & 1 \end{vmatrix} = 1$

and $\text{Jac}\,(\Phi_2 \circ \Phi_1^{-1}) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

From these cases we see that (U_1, Φ_1) and (U_2, Φ_2) are C' - related .

In the same way we can show that (U_1, Φ_1) and (U_3, Φ_3) are C' - related and (U_2, Φ_2) and (U_3, Φ_3) are C' - related.

We see that some $\text{Jac}\,(\Phi_1 \circ \Phi_2^{-1}) < 0$ so for sets of straight lines we need to distinguish between differential 2 - forms and densities. So for sets of straight lines we integrate densities not differential 2 - forms.

We form a new atlas by taking all coordinate neighborhoods (U, Φ) such that (U, Φ) is C' - related to (U_1, Φ_1), (U_2, Φ_2) and (U_3, Φ_3). When we change from one coordinate system to another. We allow only the coordinate systems which its coordinate neighborhood belong to the new atlas.

All definitions of differential form and density are the same as before. Now, we want to find a density ω which will make m(Y) invariant under the group of transformations $\mathfrak{M}$. Let $g \varepsilon \mathfrak{M}$ then in terms of rectangular cartesian coordinate on X, g can be represented by the equations

$$(3.16) \quad \begin{cases} x = a + x^* \cos \theta - y^* \sin \theta \\ y = b + x^* \sin \theta + y^* \cos \theta \end{cases}$$

The normal coordinate of straight line l is

$$(3.17) \quad x \cos \varphi + y \sin \varphi - p = 0$$

By the measure of a set Y of straight lines l, we shall understand an integral of the form

$$(3.18) \quad m(Y) = \iint_Y f(p, \varphi)\, dp\, d\varphi$$

with the condition that this integral must be invariant under the transformations of the group of Euclidean motions $\mathfrak{M}$.

By a motion (3.16) , the straight line (3.17) transforms into

$$(a + x^* \cos \theta - y^* \sin \theta) \cos \varphi + (b + x^* \sin \theta + y^* \cos \theta) \sin \varphi - p = 0$$

that is ,

$$x^* \cos(\varphi - \theta) + y^* \sin(\varphi - \theta) - (p - a\cos\varphi - b\sin\varphi) = 0 \tag{3.19}$$

Comparing this with (3.17), we see that under a motion (a,b,θ) the coordinates p, φ of 1 transform according to

$$\left\{ \begin{array}{lll} p^* = p - a\cos\varphi - b\sin\varphi & = & \alpha(p, \varphi) \\ \varphi^* = \varphi - \theta & = & \beta(p, \varphi) \end{array} \right. \tag{3.20}$$

If m(Y) is invariant, we must have

$$\iint_{Y^*} f(p^*, \varphi^*)\, dp^*\, d\varphi^* = \iint_{Y} f(p, \varphi)\, dp\, d\varphi$$

On the other hand, according to (3.20) we have

$$\iint_{Y^*} f(p^*, \varphi^*)\, dp^* d\varphi^* = \iint_{Y} f(\alpha(p, \varphi), \beta(p, \varphi))\, dp\, d\varphi \tag{3.21}$$

because $\left| \frac{\partial(p^*, \varphi^*)}{\partial(p, \varphi)} \right| = \begin{vmatrix} \frac{\partial p^*}{\partial p} & \frac{\partial p^*}{\partial \varphi} \\ \frac{\partial \varphi^*}{\partial p} & \frac{\partial \varphi^*}{\partial \varphi} \end{vmatrix}$

$$= \begin{vmatrix} 1 & a\sin\varphi - b\cos\varphi \\ 0 & 1 \end{vmatrix}$$

$$\text{i.e.} \quad \left| \frac{\partial(p^*, \varphi^*)}{\partial(p, \varphi)} \right| = 1 \tag{3.22}$$

From (3.21) and (3.22) we obtain

$$\iint_Y f(p, \varphi)\, dp\, d\varphi = \iint_Y f(\alpha(p,\varphi), \beta(p,\varphi))\, dp\, d\varphi$$

and in order that this equality hold for any domain Y, it must be true that

$$f(p, \varphi) = f(\alpha(p, \varphi), \beta(p, \varphi)) \quad \forall g \in \mathcal{M}.$$

Since any straight line $l(p, \varphi)$ can be transformed into any other $l(p^*, \varphi^*)$ by a motion, the last equality yields the result that $f(p, \varphi)$ must have the same value for any straight line of the plane ; that is, $f(p, \varphi)$ = constant. Taking this constant equal to 1, we have

$$m(Y) = \iint_Y dp\, d\varphi$$

Up to a constant factor, this measure is the only one which is invariant under the group of motion in the plane. We denote this density $dp\, d\varphi$ by dG.

3.2.2 <u>Theorem</u> : Let C be a fixed curve of length L which composed of a finite number of arcs with tangent at every point and l be a straight line which intersects C. Then

$$\int_{l \cap C \neq \phi} n\, dG = 2L$$

where n = number of intersection points of the line with the curve

<u>Proof</u> :

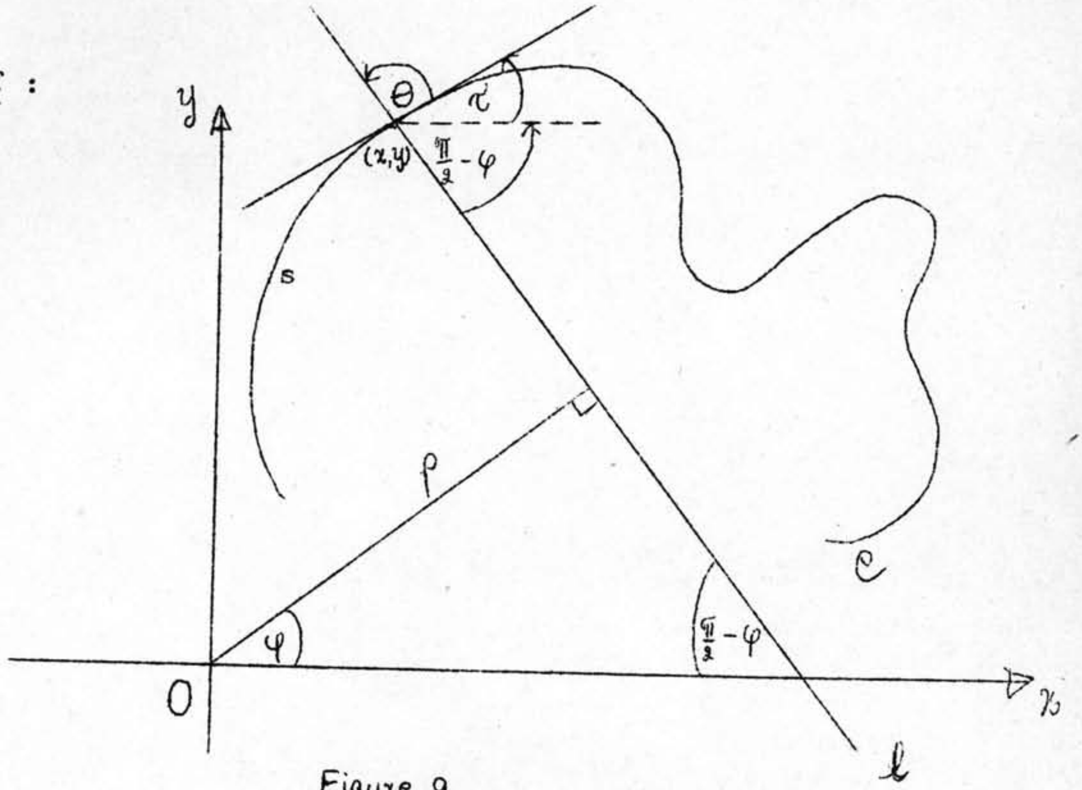


Figure 9

Let the equation of C be $x = x(s)$, $y = y(s)$ where the parameter s is the arc length. Let us consider a straight line l which intersects C at the point (x, y) and forms with the tangent at this point an angle θ. The length s corresponding to the point (x, y) and the angle θ determine the straight line l. We want to express the density dG in terms of the coordinate (s, θ) instead of (p, φ).

The normal coordinate of the straight line l is

$$x \cos \varphi + y \sin \varphi = p$$

and therefore

$$\begin{aligned} dp &= \cos \varphi \, dx - x \sin \varphi \, d\varphi + y \cos \varphi \, d\varphi + \sin \varphi \, dy \\ &= \cos \varphi \, dx + \sin \varphi \, dy + (- x \sin \varphi + y \cos \varphi) \, d\varphi \end{aligned}$$

Observe that $dx = \cos\tau \, ds$, $dy = \sin\tau \, ds$ where τ is the angle of tangent line and x^{+}- axis.

To see this, from the equation of C we have

$$x = x(s)$$

$$dx = d(x(s)) = x'(s)ds$$

Similarly, $dy = d(y(s)) = y'(s)ds$

Let g(s) be any curve in the plane we can write

$$g(s) = x(s)i + y(s)j$$

where i and j are unit vectors : Then we have seen that

$$g'(s) = x'(s)i + y'(s)j \quad \text{is a tangent vector.}$$

Let $g(s) = f(q(s))$ where $t = q(s)$

we have

$$g'(s) = f'(t)\frac{dt}{ds} = \frac{f'(t)}{\frac{ds}{dt}}$$

By definition of arc length we have

$$s = \int_{0}^{t} \left| f'(u) \right| du$$

$$\text{so} \quad \frac{ds}{dt} = \left| f'(t) \right|$$

$$\text{therefore } g'(s) = \frac{f'(t)}{\left| f'(t) \right|} = T$$

where T is a unit tangent vector.

We can shown $g'(s)$ by Figure 10

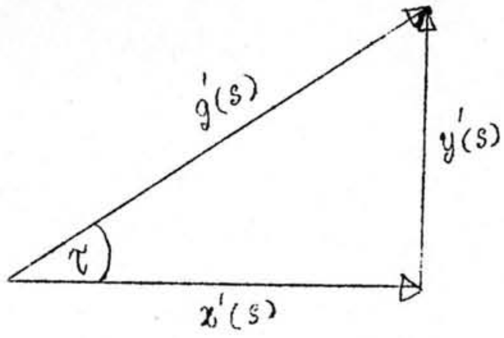


Figure 10

therefor $\quad \cos \tau = \dfrac{x'(s)}{g'(s)} = x'(s)$

and $\quad \sin \tau = \dfrac{y'(s)}{g'(s)} = y'(s)$

(3.23) Hence

$$dx = \cos \tau \, ds$$

$$dy = \sin \tau \, ds$$

From (3.23) we get

$$dp = \cos \varphi \cos \tau \, ds + \sin \varphi \sin \tau \, ds + (-x \sin \varphi + y \cos \varphi) \, d\varphi$$

$$= \cos (\varphi - \tau) \, ds + (-x \sin \varphi + y \cos \varphi) \, d\varphi$$

We have $\quad \varphi = \theta + \tau - \dfrac{\pi}{2}$

therefore $\quad d\varphi = d\theta + \tau' ds$

since τ is a function only of s.

Consequently we have

$$dG = dp\, d\varphi = \cos(\varphi - \tau)\, ds\, d\theta$$

$$= \cos(\theta - \frac{\pi}{2})\, ds\, d\theta$$

i.e

$$dG = |\sin\theta|\, ds\, d\theta$$

where we have written $|\sin\theta|$ because all densities are necessarily taken to be positive.

We integrate both sides of the last equality over all straight lines l which intersect C. We have

$$\int_{l \cap e \neq \phi} n\, dG = \int_0^L \int_0^\pi |\sin\theta|\, ds\, d\theta$$

On the left hand side we multiply by n where n is the number of intersection points of the line with the curve. Because each straight line l has been counted as many times as it has intersection points with C. To see this

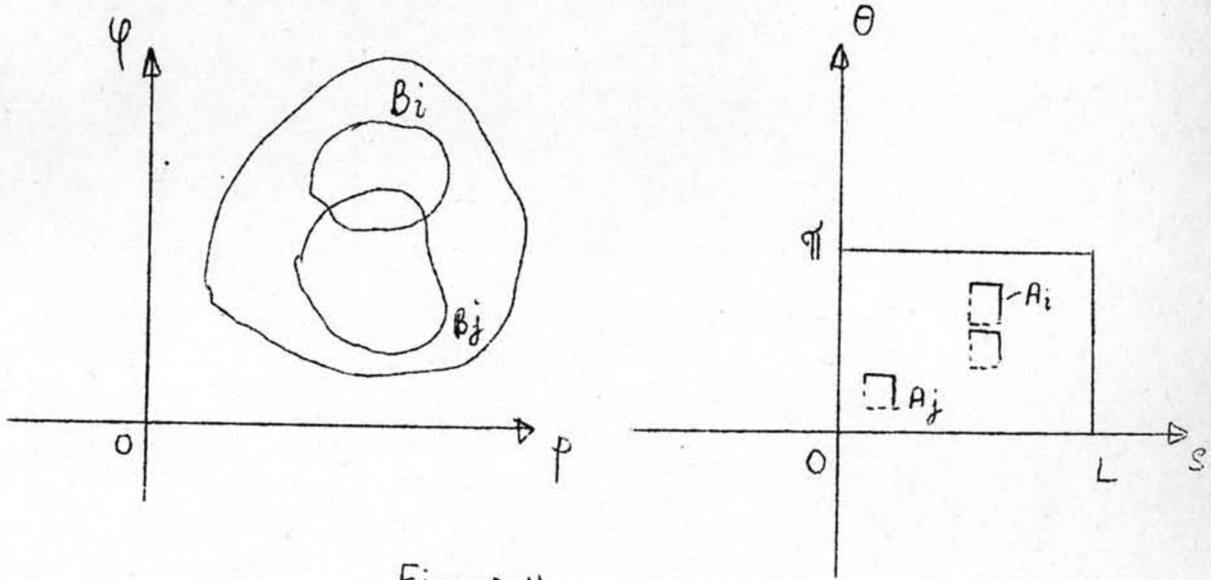


Figure 11

Cover $[0, L] \times [0, \pi]$ in the (s, θ) plane by a disjoint family of half - open rectangles so that in each half open rectangle different coordinates correspond to different lines. Write these half - open rectangles as A_i $(i = 1, 2, \ldots)$

Since each coordinate in A_i corresponds in a 1 - 1 way to a unique line (which has a unique (p, φ) coordinate) $\exists \; B_i \subset$ (p, φ) plane such that $\iint_{B_i} dp \, d\varphi = \iint_{A_i} |\sin \theta| \, ds \, d\theta$

Suppose we have an A_i, A_j such that $B_i \cap B_j \neq \emptyset$, therefore

$$\iint_{A_i \cup A_j} |\sin \theta| \, ds \, d\theta = \iint_{A_i} |\sin \theta| \, ds \, d\theta + \iint_{A_j} |\sin \theta| \, ds \, d\theta$$

$$= \iint_{B_i} dp \, d\varphi + \iint_{B_j} dp \, d\varphi$$

$$= \iint\limits_{B_i - B_i \cap B_j} dp\, d\varphi + \iint\limits_{B_i \cap B_j} dp\, d\varphi + \iint\limits_{B_j - B_i \cap B_j} dp\, d\varphi + \iint\limits_{B_i \cap B_j} dp\, d\varphi$$

$$= \iint\limits_{B_i - B_i \cap B_j} dp\, d\varphi + \iint\limits_{B_j - B_i \cap B_j} dp\, d\varphi + 2 \iint\limits_{B_i \cap B_j} dp\, d\varphi$$

$$= \iint\limits_{B_i \cup B_j} n\, dp\, d\varphi \qquad \text{where } n = \begin{cases} 1 & l \notin B_i \cap B_j \\ 2 & l \in B_i \cap B_j \end{cases}$$

Continue in this way we can prove that

$$\iint\limits_{l \cap e \neq \phi} n\, dp\, d\varphi = \int_0^L \int_0^\pi |\sin \theta|\, ds\, d\theta$$

$$= L\, (-\cos \theta)\Big|_0^\pi = 2L$$

Q.E.D.

3.2.3 <u>Corollary</u>: If C is a convex curve then the measure of the set of straight lines which intersect a convex curve is equal to its length i.e $\int\limits_{l \cap e \neq \phi} dG = L$

<u>Proof</u> : If C is a convex curve then the number of the intersection points is always 2.

From theorem 3.2.2 we get

$$\int\limits_{l \cap e \neq \phi} 2dG = 2L$$

therefore $\int\limits_{l \cap e \neq \phi} dG = L$

Q.E.D

3.2.4 Remark : We see that the measure of the set of straight lines which intersect a convex curve does not depend on shape of convex curve, it depends only length of the curve.

Section 3.3 Integral Geometry Over Sets of Pairs of Points.

3.3.1 Density for pairs of points.

A pair of distinct points P_1 (x_1, y_1), $P_2(x_2, y_2)$ may be determined by the four coordinates x_1, y_1, x_2, y_2. It may also be determined by the coordinate (p, φ) of the straight line l determined by P_1, P_2 together with the directed distances t_1, t_2 from P_1, P_2 to the foot of the perpendicular from the origin O to l. We want to expréss the product $dP_1dP_2 = dx_1dy_1dx_2dy_2$ by means of the coordinates p, φ, t_1, t_2.

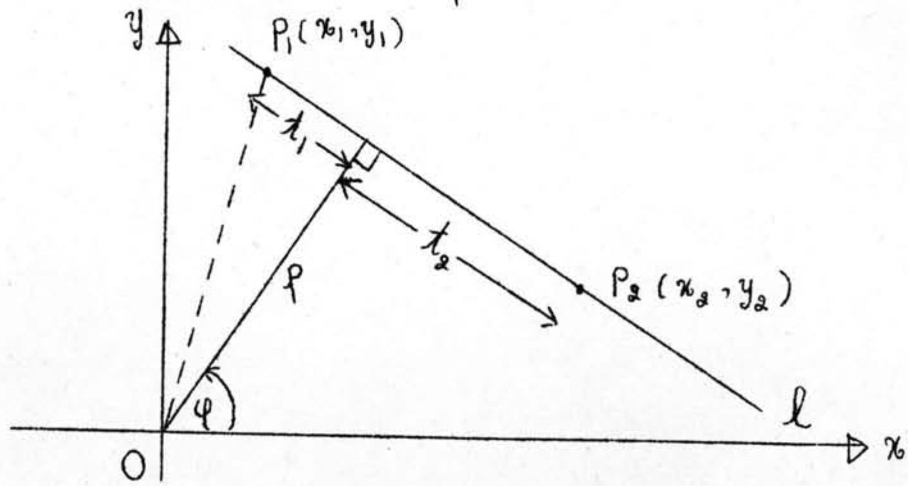


Figure 12

Let l be the straight line shown in Figure 12.

The normal coordinate of the straight line l is

$$x \cos \varphi + y \sin \varphi = p$$

since (x_1, y_1) is the pt. on l then we have

$$x_1 \cos \varphi + y_1 \sin \varphi = p$$

(3.24) $$x_1 = \frac{p - y_1 \sin \varphi}{\cos \varphi}$$

and we have

(3.25) $$x_1^2 + y_1^2 = p^2 + t_1^2$$

representing (3.24) in the last equality we get

$$\left(\frac{p - y_1 \sin \varphi}{\cos \varphi} \right)^2 + y_1^2 = p^2 + t_1^2$$

$$p^2 - 2py_1 \sin \varphi + y_1^2 \sin^2 \varphi + y_1^2 \cos^2 \varphi = p^2(1 - \sin^2 \varphi) + t_1^2 \cos^2 \varphi$$

$$y_1^2 - 2py_1 \sin \varphi + p^2 \sin^2 \varphi = t_1^2 \cos^2 \varphi$$

$$\text{therefore} \quad y_1 - p \sin \varphi = \pm t_1 \cos \varphi$$

First of all, we let

$$y_1 - p \sin \varphi = t_1 \cos \varphi$$

this implies,

(3.26) $$y_1 = p \sin \varphi + t_1 \cos \varphi$$

From (3.24) and (3.26) we get

$$x_1 = p \cos \varphi - t_1 \sin \varphi$$

By differentiation we get

$$dx_1 = - p \sin \varphi \, d\varphi + \cos \varphi \, dp - t_1 \cos \varphi \, d\varphi - \sin \varphi \, dt_1$$

$$= \cos \varphi \, dp - (p \sin \varphi + t_1 \cos \varphi) \, d\varphi - \sin \varphi \, dt_1$$

Similarly,

$$dy_1 = \sin \varphi \, dp + (p \cos \varphi - t_1 \sin \varphi) \, d\varphi + \cos \varphi \, dt_1$$

By exterior product we get

$$dx_1 dy_1 = (p \cos^2 \varphi - t_1 \sin \varphi \cos \varphi) \, dp \, d\varphi + \cos^2 \varphi \, dp \, dt_1 +$$
$$(p \sin^2 \varphi + t_1 \sin \varphi \cos \varphi) \, dp \, d\varphi - (p \sin \varphi \cos \varphi$$
$$+ t_1 \cos^2 \varphi) \, d\varphi \, dt_1 + \sin^2 \varphi \, dp \, dt_1 + (p \sin \varphi \cos \varphi - t_1 \sin^2 \varphi)$$
$$d\varphi \, dt_1$$

(3.27) i.e $$dx_1 dy_1 = p \, dp \, d\varphi + dp \, dt_1 - t_1 d\varphi \, dt_1$$

Similarly, we get

(3.28) $$dx_2 dy_2 = p \, dp \, d\varphi + dp \, dt_2 - t_2 \, d\varphi \, dt_2$$

By exterior product we obtain from (3.27) and (3.28) the relation

$$dx_1 dy_1 dx_2 dy_2 = - t_2 dp \, dt_1 d\varphi \, dt_2 - t_1 d\varphi \, dt_1 dp \, dt_2$$

$$= \left| t_2 - t_1 \right| dp \, d\varphi \, dt_1 dt_2$$

For the case $y_1 - p \sin \varphi = -t_1 \cos \varphi$

with the same method, we also get

(3.29) $dx_1 dy_1 dx_2 dy_2 \quad = \quad |t_2 - t_1| \quad dp\, d\varphi\, dt_1 dt_2$

So we can express the product $dP_1 dP_2 = dx_1 dy_1 dx_2 dy_2$ by means of the coordinates p, φ, t_1, t_2. We have taken the absolute value of $t_2 - t_1$ because all densities are positive.

3.3.2 Theorem : Let C be a convex curve of length L and area F. Let σ be the length of the chord determined by the straight line l which is determined by the pair of points P_1, P_2 inside ~~or on~~ C. Consider

$$I_n = \int_{l \cap C \neq \phi} \sigma^n \, dG \quad \text{and} \quad J_n = \iint_{P_1, P_2 \in C} r^n \, dP_1 dP_2$$

where n is the positive integer and r is the distance between P_1 and P_2. Then

$$I_n = \frac{n(n-1)}{2} J_{n-3} \qquad (n \geq 2)$$

Proof :

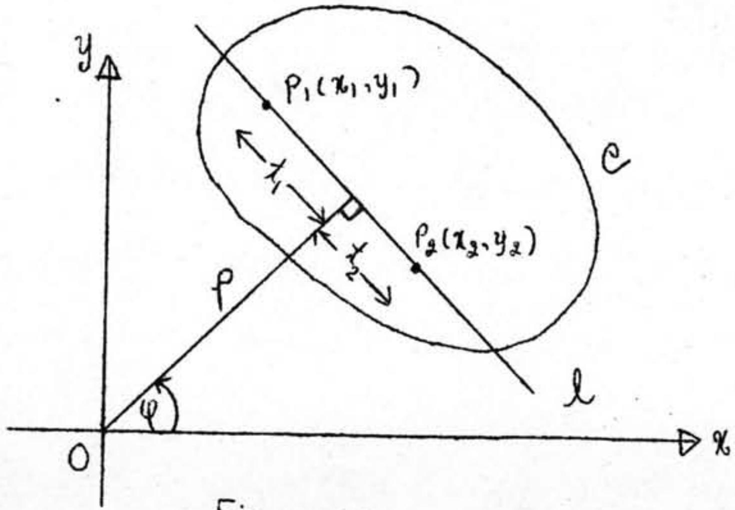


Figure 13

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be a pair of points shown in Figure 13. Observe that

$$r = |t_2 - t_1|$$

From (3.29) we get

$$J_n = \iint_{P_1, P_2 \in C} r^n \, dP_1 dP_2 = \iiint |t_2 - t_1|^{n+1} \, dG \, dt_1 dt_2$$

$$= \iint \left[\int_a^{t_1} (t_1 - t_2)^{n+1} dt_2 + \int_{t_1}^{b} (t_2 - t_1)^{n+1} dt_2 \right] dG \, dt_1$$

where b and a signifie the values of t corresponding to the end points of Δ, so that $b - a = \Delta$

therefore

$$J_n = \iint \left[- \frac{(t_1 - t_2)^{n+2}}{n+2} \Big|_a^{t_1} + \frac{(t_2 - t_1)^{n+2}}{n+2} \Big|_{t_1}^{b} \right] dG \, dt_1$$

$$= \iint \left[\frac{(t_1 - a)^{n+2}}{n+2} + \frac{(b - t_1)^{n+2}}{n+2} \right] dG \, dt_1$$

$$= \frac{1}{n+2} \int \left[\int_a^b \left\{ (t_1 - a)^{n+2} + (b - t_1)^{n+2} \right\} dt_1 \right] dG$$

$$= \frac{1}{n+2} \int \left[\frac{(t_1 - a)^{n+3}}{n+3} - \frac{(b - t_1)^{n+3}}{n+3} \Big|_a^b \right] dG$$

$$= \frac{1}{(n+2)(n+3)} \int \left[(b - a)^{n+3} + (b - a)^{n+3} \right] dG$$

$$= \frac{2}{(n+2)(n+3)} \int (b - a)^{n+3} \, dG$$

$$= \frac{2}{(n+2)(n+3)} \int \Delta^{n+3} \, dG$$

(3.30) $J_n = \frac{2}{(n+2)(n+3)} I_{n+3}$ which holds for any $n \geq 1$.

(3.30) can also be written as the equality

$$I_n = \frac{n(n-1)}{2} J_{n-3}$$ which holds for any $n \geq 2$.

Q.E.D.

3.3.3 Remark : From $I_n = \frac{n(n-1)}{2} J_{n-3}$

If n=0 $\quad I_0 = \int_{\ell \cap e \neq \phi} \sigma^0 \, dG = \int_{\ell \cap e \neq \phi} dG$

From Corollary 3.2.3 we get

$$I_0 = L$$

If n=1 $\quad I_1 = \int_{\ell \cap e \neq \phi} \sigma \, dG = \iint_{\ell \cap e \neq \phi} \sigma \, dp \, d\varphi$

To find I_1, we have 3 possible cases :

case 1 If the origin of the axes is in the convex curve C.

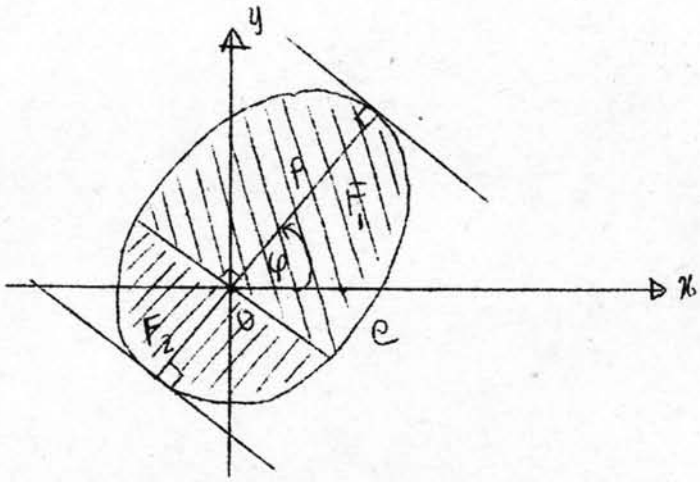


Figure 14

$$I_1 = \int_0^{\pi} \left[\int_0^{p(\varphi)} \sigma(p, \varphi) dp + \int_0^{p(\pi+\varphi)} \sigma(p, \pi+\varphi) \, dp \right] d\varphi$$

$= \int_0^{\pi} \left[F_1(\varphi) + F_2(\varphi) \right] d\varphi$ where F_1 and F_2 are areas such that $F_1 + F_2 = F$

$= \pi F$

<u>case 2</u> If the origin of the axes is on the convex curve C

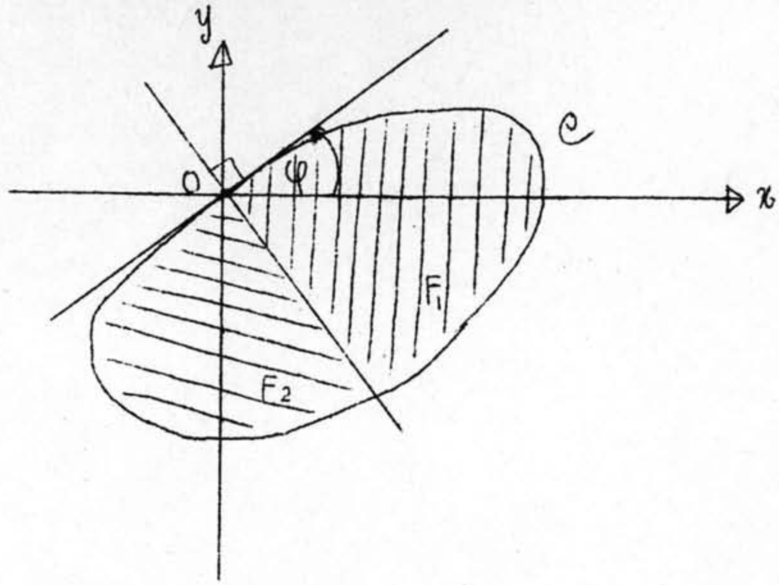


Figure 15

$$I_1 = \int_0^{\pi}\left[\int_0^{p(\varphi)} \sphericalangle(p,\ \varphi)\ dp + \int_0^{p(\pi+\varphi)} \sphericalangle(p,\ \pi+\varphi)\ dp\right] d\varphi$$

$$= \int_0^{\pi}\left[F_1(\varphi) + F_2(\varphi)\right] d\varphi \quad = \quad \pi F$$

<u>case 3</u> If the origin of the axes is outside the convex curve C.

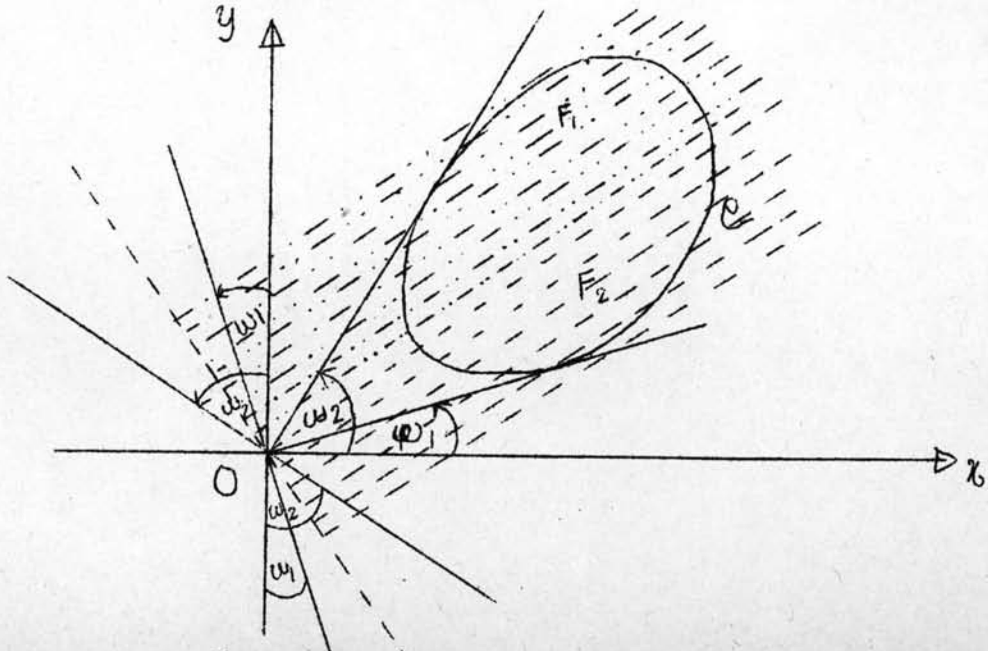


Figure 16

$$I_1 = \int_{0}^{\pi/2+\omega_1}\int_{p_1(\varphi)}^{p_2(\varphi)} \Delta(p, \varphi)\,dp\,d\varphi + \int_{\frac{\pi}{2}+\omega_1}^{\frac{\pi}{2}+\omega_2}\int_{p_3(\varphi)}^{p_4(\varphi)} \Delta(p, \varphi)\,dp\,d\varphi$$

$$+ \int_{\frac{3\pi}{2}+\omega_1}^{\frac{3\pi}{2}+\omega_2}\int_{p_5(\varphi)}^{p_6(\varphi)} \Delta(p, \varphi)\,dp\,d\varphi + \int_{\frac{3\pi}{2}+\omega_2}^{2\pi}\int_{p_7(\varphi)}^{p_8(\varphi)} \Delta(p, \varphi)\,dp\,d\varphi .$$

$$= (\frac{\pi}{2} + \omega_1)F + (\omega_2 - \omega_1)\,F_1 + (\omega_2 - \omega_1)\,F_2 + (\frac{\pi}{2} - \omega_2)F$$

$$= (\pi + \omega_1 - \omega_2)\,F + (\omega_2 - \omega_1)F$$

$$= \pi F$$

So we get

$$(3.31) \qquad I_1 = \int_{l\cap C\neq\phi} \Delta \; dG = \pi F$$

I_n and J_n are important geometric invariants of the curve like area and length and like area and length I_n and J_n are defined by integrals.

Section 3.4 Integral Geometry Over Sets of Pairs of Straight Lines.

3.4.1 Density for pair of straight lines.

A pair of lines $l_1(p_1, \varphi_1)$, $l_2(p_2, \varphi_2)$ can be determined by the coordinates p_i, φ_i $(i = 1, 2)$, it can also be determined by the coordinates x, y of their intersection point P together with the angles α_1, α_2 they form with a fixed direction in the plane, say the x - axis . We want now to express the product $dG_1 dG_2$ by means of the coordinates x, y, α_1, α_2.

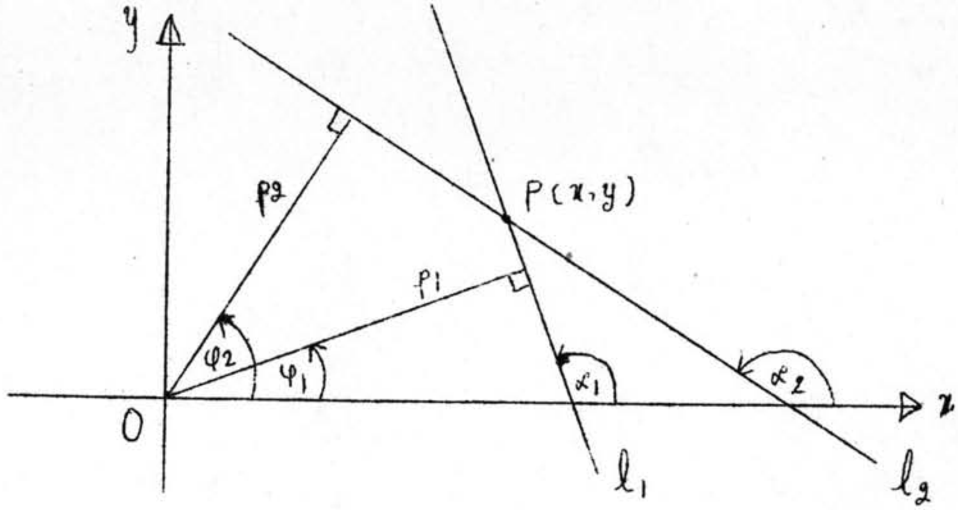


Figure 17

We have $\varphi_i = \alpha_i - \frac{\pi}{2}$ $(i = 1, 2)$

and consequently

$$p_i = x \cos \varphi_i + y \sin \varphi_i$$

$$= x \cos (\alpha_i - \frac{\pi}{2}) + y \sin (\alpha_i - \frac{\pi}{2})$$

$$= x \sin \alpha_i - y \cos \alpha_i$$

Hence :

$$d\varphi_i = d\alpha_i$$

$$dp_i = x \cos \alpha_i d\alpha_i + \sin \alpha_i dx + y \sin \alpha_i d\alpha_i - \cos \alpha_i dy$$

$$= \sin \alpha_i dx - \cos \alpha_i dy + (x \cos \alpha_i + y \sin \alpha_i) d\alpha_i$$

and by exterior product

$$dp_i d\varphi_i = \sin \alpha_i dx d\varphi_i - \cos \alpha_i dy d\varphi_i \qquad (i=1, 2)$$

From this we get

$$dp_1 d\varphi_1 dp_2 d\varphi_2 = (\sin\alpha_1 dx\, d\varphi_1 - \cos\alpha_1 dy\, d\varphi_1)(\sin\alpha_2 dx\, d\varphi_2 - \cos\alpha_2 dy\, d\varphi_2)$$

$$= (-\sin\alpha_1 \cos\alpha_2) dx\, d\varphi_1 dy\, d\varphi_2 - (\cos\alpha_1 \sin\alpha_2)\, dy\, d\varphi_1 dx\, d\varphi_2$$

$$= (\sin\alpha_1\cos\alpha_2 - \cos\alpha_1\sin\alpha_2)\, dx\, dy\, d\varphi_1 d\varphi_2$$

$$= \sin(\alpha_1 - \alpha_2)\, dx\, dy\, d\varphi_1 d\varphi_2$$

Consequently, taking absolute values, since densities must always be positive, we get

$$dG_1 dG_2 = \left|\sin(\alpha_1 - \alpha_2)\right| dP\, d\varphi_1\, d\varphi_2 \tag{3.32}$$

3.4.2 <u>Theorem</u> : (Crofton) Let C be a convex curve of length L and area F, l_1 and l_2 be a pair of straight lines which intersect C and P be the intersection point of these straight lines . Then

$$\int_{P \notin C} (\omega - \sin\omega)\, dP = \frac{L^2}{2} - \pi F$$

where ω is the angle formed by the tangents to C drawn from P.

Proof:

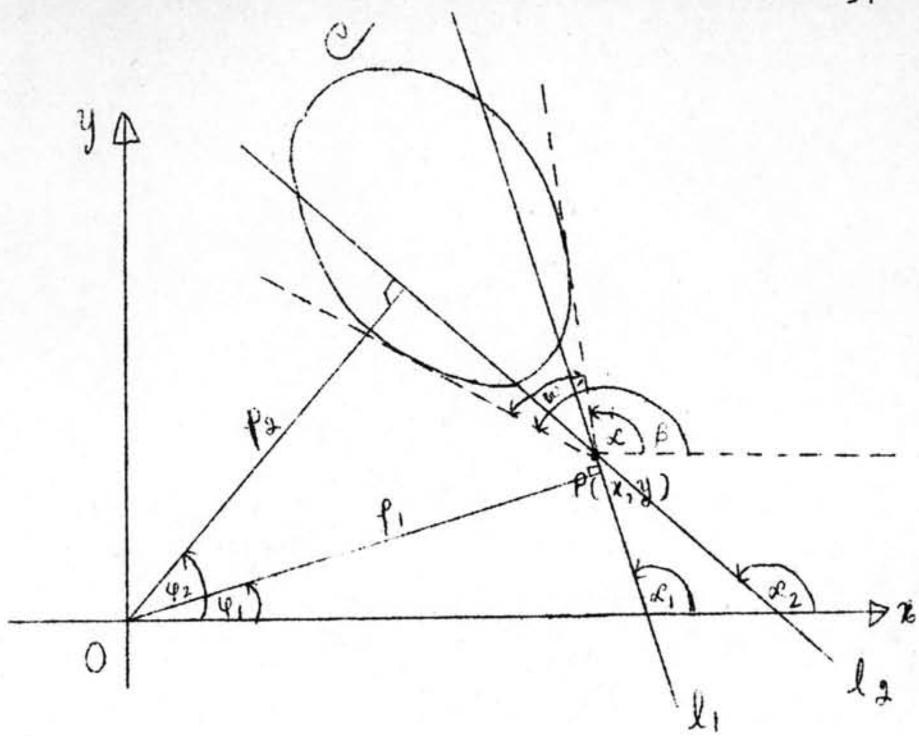


Figure 18

We have $dG_1 dG_2 = \left|\sin(\alpha_1 - \alpha_2)\right| dP\, d\alpha_1 d\alpha_2$

Integrate both sides, the left hand side gives, by Corollary 3.2.3

(3.33) $$\int_{l_1 \cap C \neq \phi} dG_1 \int_{l_2 \cap C \neq \phi} dG_2 = L \cdot L = L^2$$

The right side may be integrated first over the points P which are inside c ; that gives

(3.34) $$\int_{P \in C} dP \int_0^{\pi} \int_0^{\pi} \left|\sin(\alpha_1 - \alpha_2)\right| d\alpha_1\, d\alpha_2 = 2\pi F.$$

Because from advanced calculus [4], we have

$$\int_0^{\pi} \left|\sin^m(a - \theta)\right| d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{m}{2} + 1\right)}$$

$$\text{where } \Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, dx$$

Therefore

$$\int_{P \varepsilon C} dP \int_0^\pi \left[\int_0^\pi \left| \sin (\alpha_1 - \alpha_2) \right| d\alpha_2 \right] d\alpha_1$$

$$= F \int_0^\pi \frac{\Gamma(1)\sqrt{\pi}}{\Gamma(\frac{3}{2})} \, d\alpha_1$$

Since $\Gamma(1) = 1$ and

$$\Gamma(n) = (n-1)\,\Gamma(n-1)$$

we get

$$\int_{P \varepsilon C} dP \int_0^\pi \int_0^\pi \left| \sin (\alpha_1 - \alpha_2) \right| d\alpha_1 \, d\alpha_2 = F \int_0^\pi \frac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}} \, d\alpha_1$$

$$= 2\pi F$$

For the points P not contained in C, if α, β are the angles which the lines of support of C drawn from P form with the x^+- axis, we have

$$\int_\alpha^\beta \int_\alpha^\beta \left| \sin (\alpha_1 - \alpha_2) \right| d\alpha_1 d\alpha_2 = \int_\alpha^\beta d\alpha_1 \left[\int_\alpha^{\alpha_1} \sin (\alpha_1 - \alpha_2) \, d\alpha_2 + \int_{\alpha_1}^\beta \sin (\alpha_2 - \alpha_1) d\alpha_2 \right]$$

$$= \int_\alpha^\beta d\alpha_1 \left[\cos(\alpha_1 - \alpha_2) \Big|_\alpha^{\alpha_1} - \cos (\alpha_2 - \alpha_1) \Big|_{\alpha_1}^\beta \right]$$

$$= \int_\alpha^\beta d\alpha_1 \left[1 - \cos (\alpha_1 - \alpha) - \cos (\beta - \alpha_1) + 1 \right]$$

$$= \int_\alpha^\beta \left[2 - \cos (\alpha_1 - \alpha) - \cos (\beta - \alpha_1) \right] d\alpha_1$$

$$= 2\alpha_1 - \sin(\alpha_1 - \alpha) + \sin(\beta - \alpha_1) \Big|_{\alpha}^{\beta}$$

$$= 2(\beta - \alpha) - \sin(\beta - \alpha) - \sin(\beta - \alpha)$$

$$= 2(\beta - \alpha) - 2\sin(\beta - \alpha)$$

If we designate by $\omega = \beta - \alpha$ the angle formed by the tangents to C drawn from P, the integral of the right side of (3.32) extended over all points $P \notin C$, gives

$$\int_{P \notin C} dP \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \left|\sin(\alpha_1 - \alpha_2)\right| d\alpha_1 d\alpha_2 = 2 \int_{P \notin C} (\omega - \sin\omega)\, dP \tag{3.35}$$

The sum of (3.34) and (3.35) must be equal to (3.33) . We obtain

$$2 \int_{P \notin C} (\omega - \sin\omega)\, dP + 2\pi F = L^2$$

$$\int_{P \notin C} (\omega - \sin\omega) dP = \frac{L^2}{2} - \pi F$$

This formula holds for any convex curve C

Q.E.D.

Section 3.5 Integral Geometry Over Sets of Kinematics.

3.5.1 Sets of Congruent Figures

The position of a rigid figure K is determined on the plane by the position of a point P(x,y) of K and the angle φ formed by a direction PA fixed in K and a fixed direction OX of the plane. We say that x,y, φ are the coordinates of K.

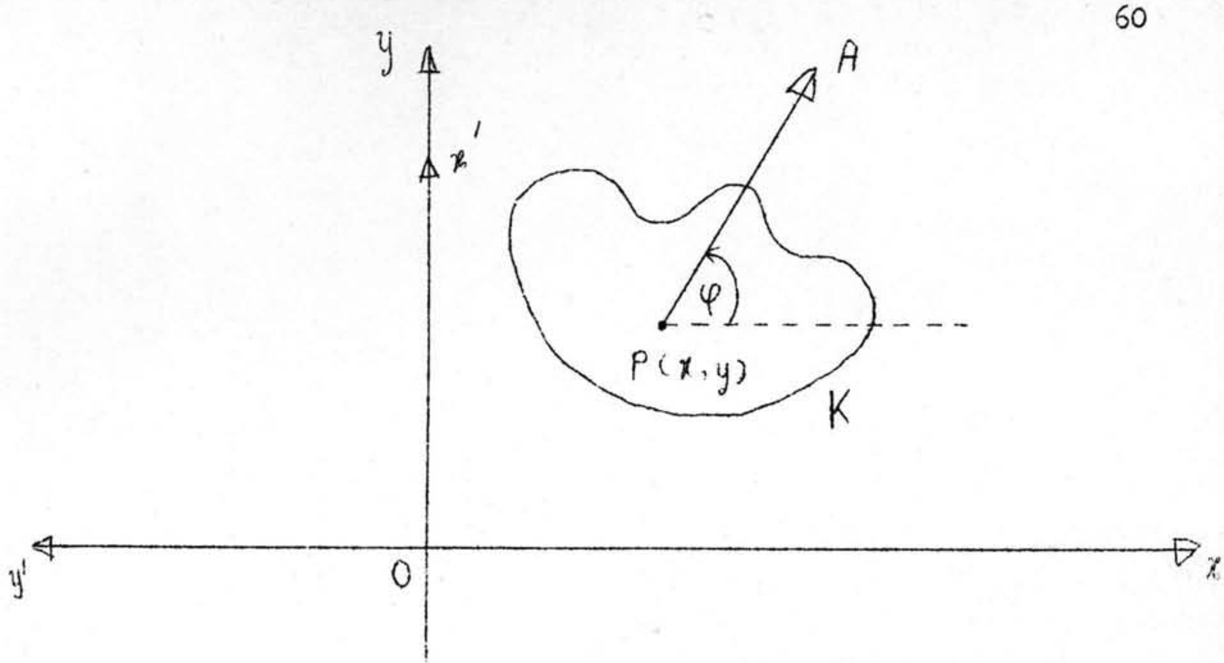


Figure 19

Let Z be the set of positions of a figure K in plane. We can give each position (x, y, φ) coordinates except those positions which have $\varphi = 0$ because we want the range to be an open set i.e. $\{(x, y, \varphi)/(x, y) \in \mathbb{R}^2, 0 < \varphi < 2\pi\}$. So to give every position in Z a coordinate, we need 2 coordinate neighborhoods (U_1, Φ_1) and (U_2, Φ_2) to form the atlas. If we let Φ_1 be a rectangular cartesian coordinate system with (x, y) as its coordinate functions and Φ_2 be the new coordinate system with (x', y') as its coordinate functions such that x' - axis is the rotation of the old x - axis 90° in the counter clockwise direction. From an atlas on Z by taking $\{(U_1, \Phi_1), (U_2, \Phi_2)\}$. These coordinate neighborhoods are C' - related because $U_1 \cap U_2 \neq \emptyset$ and

$$x' = x \cos \frac{\pi}{2} + y \sin \frac{\pi}{2} = y$$

and $$y' = - x \sin \frac{\pi}{2} + y \cos \frac{\pi}{2} = -x$$

these two functions are C' - maps.

Now, we form a new atlas by taking all coordinate neighborhoods (U, Φ) such that (U, Φ) is C' - related to (U_1, Φ_1) and (U_2, Φ_2) with positive Jacobian.

All definitions of differential form and density are the same as set of points. Here, as in the case of the set of points we can take differential forms and not worry about densities because Jacobian > 0.

Similarly, we want to find a density which will make $m(Z)$ invariant under the group of transformations $\mathcal{M}$.

Let $g \varepsilon \mathcal{M}$ then g operates on the coordinates of the set of Kinematics by

(3.36) $$\left\{ \begin{aligned} x &= a + x^* \cos \theta - y^* \sin \theta \\ y &= b + x^* \sin \theta + y^* \cos \theta \\ \varphi &= \varphi^* + \theta \end{aligned} \right.$$

The condition that $m(Z)$ is invariant means that

(3.37) $$\iiint_Z f(x, y, \varphi)\, dx\, dy\, d\varphi = \iiint_{Z^*} f(x^*, y^*, \varphi^*)\, dx^*\, dy^*\, d\varphi^* \qquad \forall g \varepsilon \mathcal{M}$$

According to (3.36)

$$\frac{\partial (x, y, \varphi)}{\partial (x^*, y^*, \varphi^*)} = 1$$

and consequently

$$\iiint_{Z} f(x,y,\varphi)\, dx\, dy\, d\varphi = \iiint_{Z^*} f(x,y,\varphi)\, dx^*\, dy^*\, d\varphi^* \tag{3.38}$$

From (3.37) and (3.38) we deduce

$$\iiint_{Z^*} f(x^*, y^*, \varphi^*)\, dx^*\, dy^*\, d\varphi^*$$

$$= \iiint_{Z^*} f(x,y,\varphi)\, dx^*\, dy^*\, d\varphi^* \quad \forall \text{domain } z^*$$

In order that this equality hold for any domain z^*, it must be true that $f(x,y,\varphi) = f(x^*, y^*, \varphi^*)$ $\forall g \in \mathcal{M}$

Since by a motion we can transform any position (x, y, φ) into any other (x^*, y^*, φ^*) the function $f(x, y, \varphi)$ must take the same value for all positions of K ; thus it is a constant. Again, to normalize the situation we take this constant to be 1. We have

$$m(Z) = \iiint_{Z} dx\, dy\, d\varphi \tag{3.39}$$

Up to a constant factor, this measure is the only one that is invariant under the group of motions $\mathcal{M}$ in the plane. The differential form $dx\, dy\, d\varphi$ under the integral sign in (3.39), taken always in absolute value is called the <u>kinematic density</u> for the plane and we represent it by $dK = dx\, dy\, d\varphi$ and m(Z) is called the <u>kinematic measure</u> .

3.5.2 Another form for the kinematic density.

Instead of the coordinates x, y, φ for the figure K, we can choose other ones. For instance, the position of K may be determined by the straight line PA , which we call G(p, θ), and the distance t = HP from P to the foot H of the perpendicular drawn from O to G. (see Figure 20)

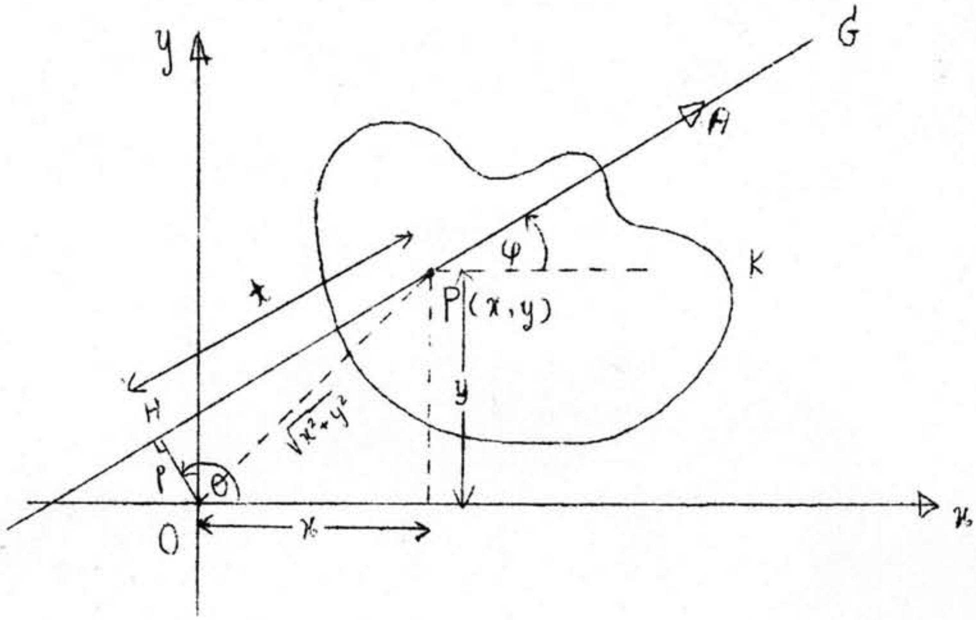


Figure 20

The normal coordinate of the straight line G is

$$x \cos \theta + y \sin \theta = p$$

(3.40) This implies $$y = \frac{p - x \cos \theta}{\sin \theta}$$

and we have $$x^2 + y^2 = p^2 + t^2$$

$$x^2 + \left(\frac{p - x \cos \theta}{\sin \theta} \right)^2 = p^2 + t^2$$

$$x^2 \sin^2\theta + p^2 - 2px \cos \theta + x^2 \cos^2\theta = (p^2 + t^2)(1 - \cos^2\theta)$$

$$x^2 - 2px\cos\theta + p^2\cos^2\theta = t^2\sin^2\theta$$

so $$x - p\cos\theta = \pm t\sin\theta$$

First of all, we will consider $x - p\cos\theta = t\sin\theta$

$$x = p\cos\theta + t\sin\theta$$

From (3.40) and the last equality we get

$$y = \frac{p-(p\cos\theta + t\sin\theta)\cos\theta}{\sin\theta}$$

$$= \frac{p-p\cos^2\theta - t\sin\theta\cos\theta}{\sin\theta}$$

$$= p\sin\theta - t\cos\theta$$

The transformation formulae are

$$x = p\cos\theta + t\sin\theta$$

$$y = p\sin\theta - t\cos\theta$$

$$\varphi = \theta - \frac{\pi}{2}$$

Hence $$\frac{\partial(x,y,\varphi)}{\partial(p,\theta,t)} = \begin{vmatrix} \cos\theta & -p\sin\theta + t\cos\theta & \sin\theta \\ \sin\theta & p\cos\theta + t\sin\theta & -\cos\theta \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -(-\cos^2\theta - \sin^2\theta) = 1$$

(3.41) and $$dK = dx\,dy\,d\varphi = \left|\frac{\partial(x,y,\varphi)}{\partial(p,\theta,t)}\right| \overrightarrow{dG}\,dt = \overrightarrow{dG}\,dt$$

We write $\vec{G}$ in order to indicate that G must be considered as oriented, because a change of orientation does not superpose K on itself.

For the case $x - p\cos\theta = -t\sin\theta$

with the same method, we also get

$$dK = dx\,dy\,d\varphi = d\vec{G}\,dt .$$

3.5.3 Theorem: Let K_0 be a fixed convex figure of length L_0 and area F_0 and let K be an oriented segment of length l which intersects K_0 . Letting G be the straight line that contains the segment K, and calling σ the length of the chord determined by G on K_0 . Then

$$\int_{K \cap K_0 \neq \phi} dK = 2\pi F_0 + 2l L_0$$

Proof :

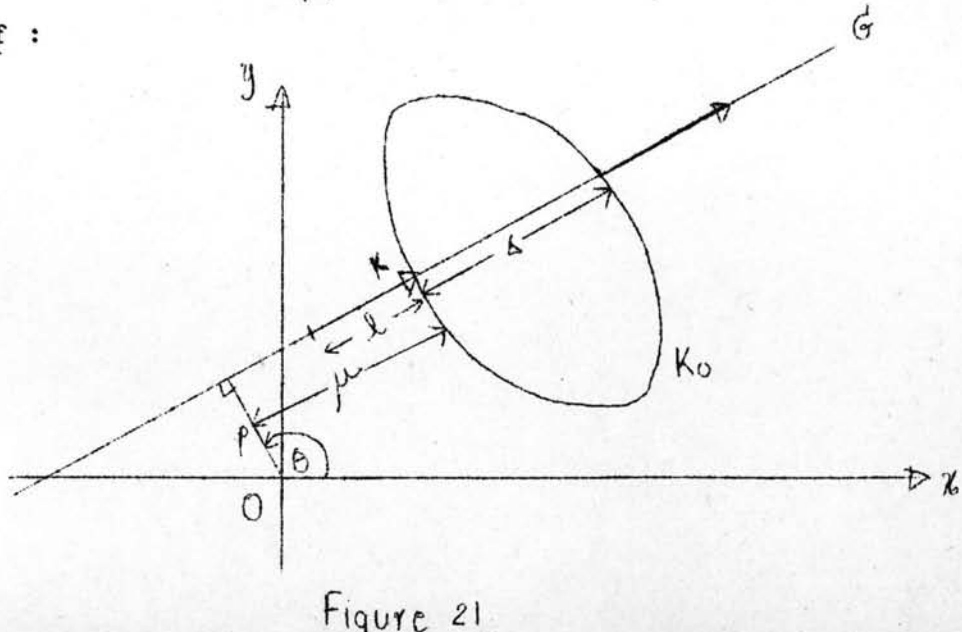


Figure 21

From (3.4) we have

$$\int\limits_{K \cap K_0 \neq \phi} dK = \int\limits_{G \cap K_0 \neq \phi} \left[\int_{\mu - l}^{\mu + \delta} dt \right] \vec{dG}$$ where μ and δ depend on G.

$$= \int\limits_{G \cap K_0 \neq \phi} (\delta + 1) \vec{dG}$$

Taking into account Corollary 3.2.3, (3.31), and the fact that each non-oriented line carries two oriented ones, we obtain

$$\int\limits_{K \cap K_0 \neq \phi} dK = 2 \int\limits_{G \cap K_0 \neq \phi} (\delta + 1) dG$$

$$= 2\pi F_0 + 2l L_0$$

Thus, the kinematic measure of all oriented segments of length l having a point in common with a convex figure of area F_0 and length L_0, is equal to $2\pi F_0 + 2l L_0$

Q.E.D.

3.5.4 <u>Corollary</u> : If we take the point P to be coincident with the origin of K. Then we have $\int\limits_{P \text{ exterior to } K_0} \omega \, dP = 2l L_0$

where ω is the angle formed by the tangents to K_0 drawn from P.

Proof. We consider 2 cases :

case 1 For $P \varepsilon K_0$

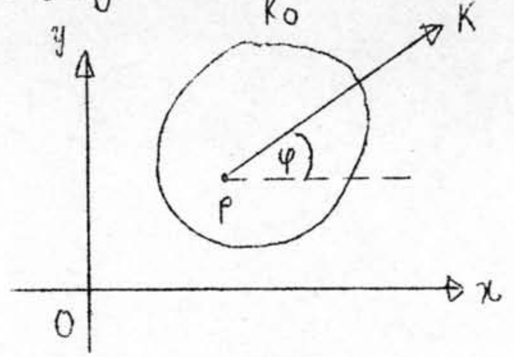


Figure 22

in this case φ varies from 0 to 2π .

case 2 For P exterior to K_0

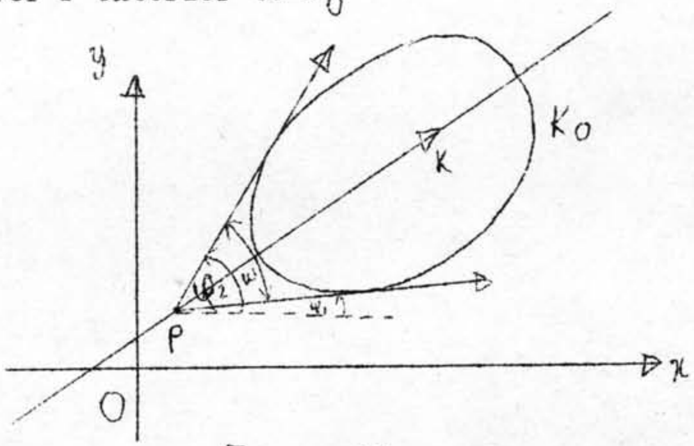


Figure 23

In this case φ varies from φ_1 to φ_2 where $\varphi_2 - \varphi_1 = \omega$

Consequently

$$\int_{K \cap K_0 \neq \phi} dK = \int_{P \varepsilon K_0} \left[\int_0^{2\pi} d\varphi \right] dP + \int_{P \text{ exterior to } K_0} \left[\int_{\varphi_1}^{\varphi_2} d\varphi \right] dP$$

$$= 2\pi F_0 + \int_{P \text{ exterior to } K_0} (\varphi_2 - \varphi_1)\, dP$$

From Theorem 3.5.3 we have

$$\int_{K \cap K_0 \neq \phi} dK = 2\pi F_0 + 2l L_0$$

Then we find the integral formula

$$\int_{P \text{ exterior to } K} \omega \, dP = 2l L_0$$

Q.E.D.

3.5.5 Theorem: Let K be the oriented segment of length l which intersects both sides of a given angle **A**. Let $\measuredangle$ be the chord cut by the angle **A** from the straight line G determined by K.

Then $$\int_{\substack{K \cap AB \neq \phi \\ K \cap AC \neq \phi}} dK = \frac{l^2}{2}\left[1 + (\pi - A)\cot A\right]$$ where AB and AC are sides of angle A.

Proof:

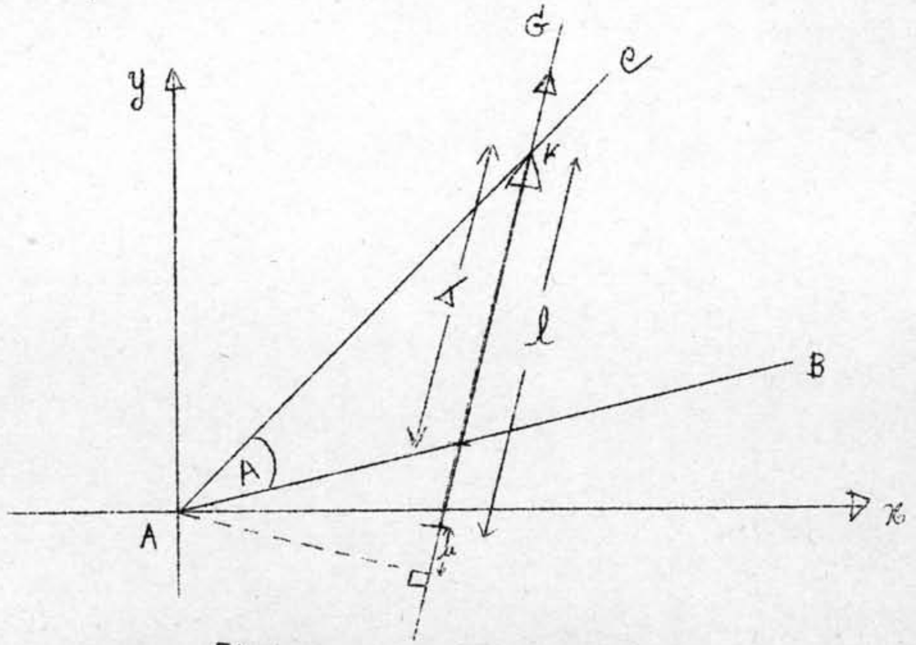


Figure 24

We have

$$\int\limits_{\substack{K\cap AB\neq\phi \\ K\cap AC\neq\phi}} dK = \iint \vec{dG}\, dt$$

$$= \int \left[\int_{\mu}^{\mu+l-\Delta} dt \right] \vec{dG} \qquad \text{where } \mu \text{ depends on G .}$$

$$= \int (\mu + l - \Delta - \mu)\, \vec{dG}$$

$$= 2 \int\limits_{\Delta \leq l} (l - \Delta)\, dG$$

To find $\int l dG$ and $\int \Delta\, dG$, we will draw a new figure 25 such that x - axis is AB .

Further $\int l dG = l \iint dp\, d\varphi = \int l . AE\, d\varphi$

$$= \int 2 . \frac{1}{2} . l . AE\; d\varphi$$

$$= 2 \int T\, d\varphi$$ where T is the area of the triangle AHM determined by a chord HM of length l normal to the direction φ .

On the other hand, we have

$$\int \Delta\, dG = \iint \Delta\, dp\, d\varphi = \int T\, d\varphi$$

Consequently

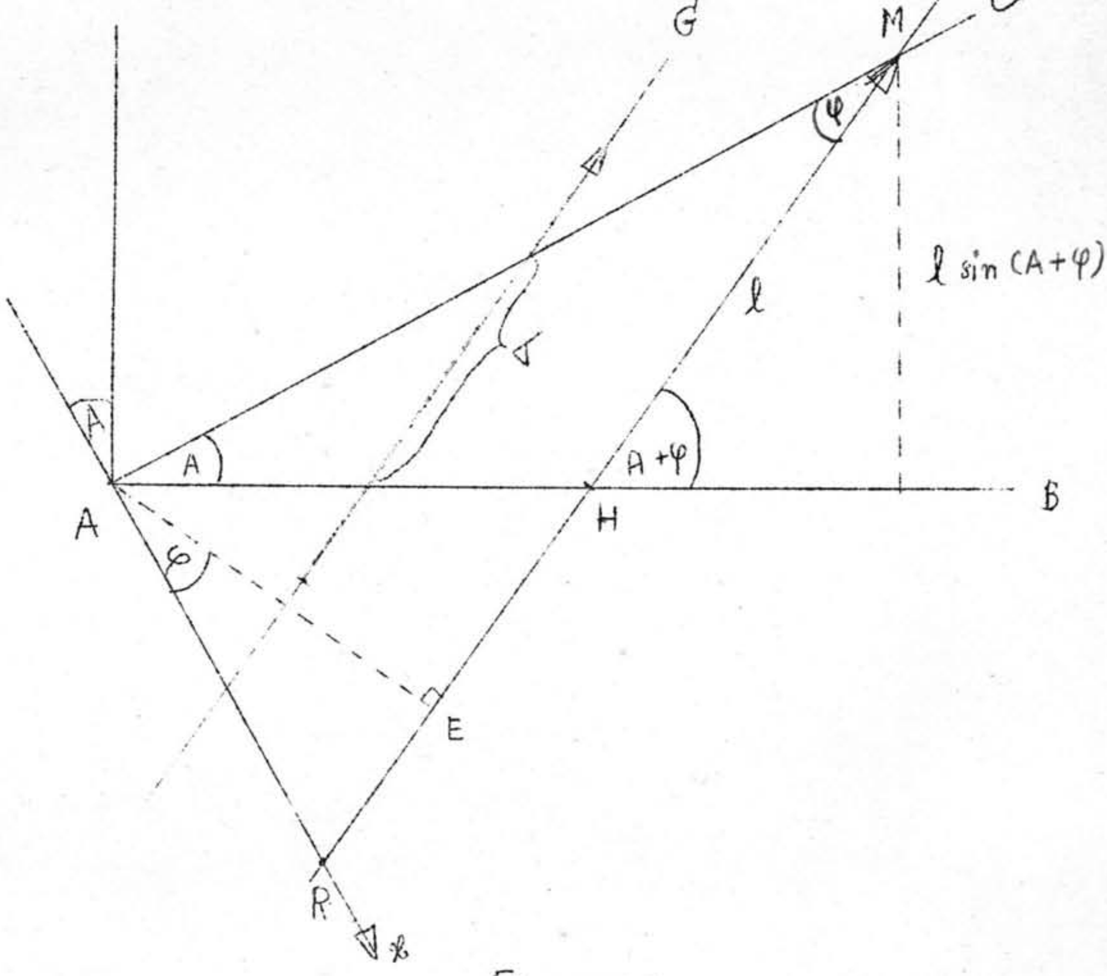


Figure 25

$$\int_{\substack{K\cap AB\neq\emptyset \\ K\cap AC\neq\emptyset}} dK = 2\left[2\int T\,d\varphi - \int T\,d\varphi\right] = 2\int T\,d\varphi$$

In order to evaluate this integral we observe that

$$2\,T = \frac{l^2}{\sin A} \sin\varphi \sin(A+\varphi)$$

To see this , we have

$$AM \sin A = l \sin(A+\varphi)$$

this implies

$$\sin(A + \varphi) = \frac{AM \sin A}{l}$$

Hence

$$\frac{l^2}{\sin A} \sin \varphi \sin(A + \varphi) = \frac{l^2}{\sin A} \sin \varphi \cdot \frac{AM \sin A}{l}$$

$$= l.AM. \sin \varphi$$

$$= l.A\,M. \frac{AE}{AM}$$

$$= l.AE$$

$$= 2(\frac{1}{2}\, l.AE)$$

$$= 2\,T$$

and consequently

$$\int\limits_{\substack{K \cap AB \neq \phi \\ K \cap AC \neq \phi}} dK = \frac{l^2}{\sin A} \int_0^{\pi - A} \sin \varphi \sin(A + \varphi)\, d\varphi$$

$$= \frac{l^2}{\sin A} \int_0^{\pi - A} \sin \varphi \left[\sin A \cos \varphi + \cos A \sin \varphi \right] d\varphi$$

$$= l^2 \left[\int_0^{\pi - A} \sin \varphi \cos \varphi\, d\varphi + \cot A \int_0^{\pi - A} \sin^2 \varphi\, d\varphi \right]$$

$$= l^2 \left[\int_0^{\pi - A} \sin \varphi\, d(\sin \varphi) + \cot A \int_0^{\pi - A} \frac{(1 - \cos 2\varphi)}{2}\, d\varphi \right]$$

$$= l^2 \left[\frac{\sin^2 \varphi}{2} + \cot A \left(\frac{\varphi}{2} - \frac{\sin 2\varphi}{4} \right) \Bigg|_0^{\pi - A} \right]$$

$$= \frac{1^2}{2}\left[\frac{1-\cos 2\varphi}{2} + \cot A\,(\varphi - \sin\varphi\cos\varphi)\Big|_0^{\pi - A} \right]$$

$$= \frac{1^2}{2}\left[\frac{1-\cos 2(\pi - A)}{2} + \cot A\,(\pi - A - \sin(\pi - A)\cos(\pi - A)) \right]$$

$$= \frac{1^2}{2}\left[\frac{1-\cos(2\pi - 2A)}{2} + \cot A\,(\pi - A) + \sin A\cos A) \right]$$

$$= \frac{1^2}{2}\left[\frac{1-\cos 2A}{2} + \cot A(\pi - A) + \cos^2 A \right]$$

$$= \frac{1^2}{2}\left[\frac{1}{2} - \frac{\cos 2A}{2} + \cot A\,(\pi - A) + \frac{\cos 2A}{2} + \frac{1}{2} \right]$$

$$= \frac{1^2}{2}\left[1 + (\pi - A)\cot A \right]$$

Q.E.D.