CHAPTER IV

THE WIENER INTEGRAL

The purpose of this chapter is to extend the measure in chapter III to the measure on the Carathéodory extension of $\mathcal{J}$ in the usual way. With this Carathéodory extension measurable functionals on C may be defined and their integration on C may be considered.

Definition 4.1. The outer measure of an arbitrary set $\Gamma \subseteq C$ is defined to be

$$W_c^*(\Gamma) = \inf \sum_{k=1}^{\infty} W_c(I_k)$$

where $\{I_k\}$ ranges over all sequences from $\mathcal{J}$ such that $\Gamma \subseteq \bigcup_{k=1}^{\infty} I_k$.

Definition 4.2. A set $\Gamma \subseteq C$ is called Wiener measurable if for every set $A \subseteq C$ we have

$$W_c^*(A) = W_c^*(A - \Gamma) + W_c^*(A \cap \Gamma).$$

The collection $\mathcal{J}_{W_c}$ of all Wiener measurable sets, the Carathéodory extension of $\mathcal{J}$, is a σ-algebra containing $\mathcal{J}$ and if we define for each $c > 0$

$$\bar{W}_c(\Gamma) = W_c^*(\Gamma) \qquad \text{for } \Gamma \in \mathcal{J}_{W_c},$$

then $\bar{W}_c$ is a measure on $\mathcal{J}_{W_c}$. Let us denote this measure space by $(C, \mathcal{J}_{W_c}, \bar{W}_c)$.

Theorem 4.3. The σ-algebra $\sigma[\mathcal{J}]$ generated by $\mathcal{J}$ is the collection $\mathcal{B}(C)$ of all Borel sets of C.

Proof. We divide the proof into 4 steps :

Step 1. We show that $\sigma[\mathcal{J}]$ contains all closed balls. To prove this let $r > 0$ be any real number and let $r_1, r_2, \ldots$ be an enumeration of rational numbers in $[0,1]$. Then for an arbitrary fixed $z_0 \in C$,

$$\{x \in C : \|x-z_0\| \leq r\} = \bigcap_{n=1}^{\infty} \{x \in C : -r+z_0(r_n) \leq x(r_n) \leq r+z_0(r_n)\} .$$

Since $\{x \in C : -r+z_0(r_n) \leq x(r_n) \leq r+z_0(r_n)\} \in \mathcal{J}^{\circ} \subset \mathcal{J}$, it follows that $\{x \in C : \|x-z_0\| \leq r\} \in \sigma[\mathcal{J}^{\circ}] \subseteq \sigma[\mathcal{J}]$.

Step 2. Since C with sup-norm is separable we have that every open set in C is a countable union of open balls and hence every closed set is a countable intersection of closed balls. But then, step 1 implies that $\sigma[\mathcal{J}]$ is a σ-algebra containing all closed sets and must therefore contain the collection $\mathcal{B}(C)$ of all Borel sets, since $\mathcal{B}(C)$ is the smallest σ-algebra containing closed sets.

Step 3. We show that if $E \subseteq R^n$ is open, then the quasi-interval I defined by $I = \{x \in C : (x(t_1),\ldots,x(t_n)) \in E,\ 0 < t_1 < \ldots < t_n \leq 1\}$ is open. Let $z^* \in I$. Then $(z^*(t_1),\ldots,z^*(t_n)) \in E$. By letting $z^*(t_j) = \xi_j^*$, we have $\xi^* = (\xi_1^*,\ldots,\xi_n^*) \in E$. Since E is open, there exists a real number $r > 0$ such that $B(\xi^*,r) \subset E$ where $B(\xi^*,r) = \{\xi \in R^n : |\xi - \xi^*| < r\}$. Claim that $B(z^*,r) \subset I$ where

$B(z^*,r) = \{ x \in C : \|x-z^*\| < r \}$. To see this, let $x \in B(z^*,r)$ then $\|x-z^*\| < r$ and hence $\max_{0 \leq t \leq 1} |x(t)-z^*(t)| < r$. In particular, $\max_{1 \leq j \leq n} |x(t_j)-z^*(t_j)| < r$. Thus $(x(t_1),\ldots,x(t_n)) \in B(\xi^*,r) \subset E$. Therefore $x \in I$.

Step 4. Let n be any positive integer and let $0 < t_1 < \ldots < t_n \leq 1$. Define $\mathcal{A} = \{ E \in \mathcal{B}(R^n) : I = \{x \in C : (x(t_1),\ldots,x(t_n)) \in E\} \in \mathcal{B}(C) \}$. Then

(i). If E is an open set in R^n, then it follows from step 3 that $I \in \mathcal{B}(C)$.

(ii). If $E \in \mathcal{A}$, then $E \in \mathcal{B}(R^n)$ and $I = \{x \in C : (x(t_1),\ldots,x(t_n)) \in E\} \in \mathcal{B}(C)$. Since $\mathcal{B}(C)$ and $\mathcal{B}(R^n)$ are σ-algebras, $C - I \in \mathcal{B}(C)$ and $R^n - E \in \mathcal{B}(R^n)$. But then, by Lemma 2.10 (iv), $C - I = \{x \in C : (x(t_1),\ldots,x(t_n)) \in R^n - E\}$ and hence $R^n - E \in \mathcal{A}$.

(iii). If $E_i \in \mathcal{A}$ for $i = 1,2,\ldots$, then $E_i \in \mathcal{B}(R^n)$ and $I_i = \{x \in C : (x(t_1),\ldots,x(t_n)) \in E_i\} \in \mathcal{B}(C)$ for all i. Since $\mathcal{B}(C)$ and $\mathcal{B}(R^n)$ are σ-algebras, $\bigcup_{i=1}^{\infty} I_i \in \mathcal{B}(C)$ and $\bigcup_{i=1}^{\infty} E_i \in \mathcal{B}(R^n)$. But then, by Lemma 2.10 (i), $\bigcup_{i=1}^{\infty} I_i = \{x \in C : (x(t_1),\ldots, x(t_n)) \in \bigcup_{i=1}^{\infty} E_i\}$ and hence $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

From (i), (ii) and (iii) we have that $\mathcal{A}$ is a σ-algebra containing all open set in R^n and must therefore contain the collection $\mathcal{B}(R^n)$ of all Borel sets, since $\mathcal{B}(R^n)$ is the smallest σ-algebra containing open sets Hence $\mathcal{A} = \mathcal{B}(R^n)$. Since n is arbitrary, $\mathcal{J} \subseteq \mathcal{B}(C)$ and hence $\sigma[\mathcal{J}] \subseteq \mathcal{B}(C)$. It follows from step 2 that $\sigma[\mathcal{J}] = \mathcal{B}(C)$.

Q.E.D.

Remark 4.4. $\sigma[\mathcal{J}^\circ] = \sigma[\mathcal{J}] = \mathcal{B}(C)$.

Proof. Since $\mathcal{J}^\circ \subset \mathcal{J}$, $\sigma[\mathcal{J}^\circ] \subseteq \sigma[\mathcal{J}] = \mathcal{B}(C)$ and from the proof of steps 1 and 2 in the above theorem we also have that $\mathcal{B}(C) \subseteq \sigma[\mathcal{J}^\circ]$. Thus $\sigma[\mathcal{J}^\circ] = \sigma[\mathcal{J}] = \mathcal{B}(C)$.

Q.E.D.

Since $\mathcal{J}^o$ is a semi-algebra of sets, if we define $W_c^o(I^o) = W_c(I^o)$ for every $I^o \in \mathcal{J}^o$ then according to the properties of W_c and the fact that $\mathcal{J}^o \subset \mathcal{J}$, we have that W_c^o satisfies the conditions in (1.9) and hence W_c^o has a unique extension to a measure on the algebra $\mathcal{Q}$ generated by $\mathcal{J}^o$. If we extend W_c^o on $\mathcal{Q}$ by the Carathéodory extension, we have (for each $c > 0$) a measure $\overline{W_c^o}$ on the σ-algebra $\mathcal{J}_{W_c^o}$ containing $\mathcal{J}^o$. Let us denote this measure space by $(C, \mathcal{J}_{W_c^o}, \overline{W_c^o})$. We want to show that $(C, \mathcal{J}_{W_c}, \bar{W}_c) = (C, \mathcal{J}_{W_c^o}, \overline{W_c^o})$. This will enable us to express any Wiener measurable set in terms of members of $\mathcal{J}^o$.

Lemma 4.5. For any $\Gamma \subseteq C$, define

$$W_c^{o*}(\Gamma) = \inf \sum_{k=1}^{\infty} W_c^o(I_k^o)$$

where $\{I_k^o\}$ ranges over all sequences from $\mathcal{J}^o$ such that $\Gamma \subseteq \bigcup_{k=1}^{\infty} I_k^o$. Then $W_c^{o*}(\Gamma) = W_c^*(\Gamma)$.

Proof. Since $\mathcal{J}^o \subset \mathcal{J}$, we have that $W_c^*(\Gamma) \leq W_c^{o*}(\Gamma)$ for any $\Gamma \subseteq C$. It remains to show that $W_c^{o*}(\Gamma) \leq W_c^*(\Gamma)$.

<u>Step 1</u>. We show that for any $I \in \mathcal{J}$,

$$W_c(I) = \inf\left\{\sum_{n=1}^{\infty} W_c(I_n^o) : I \subseteq \bigcup_{n=1}^{\infty} I_n^o,\ I_n^o \in \mathcal{J}^o\right\}.$$

Let $I \in \mathcal{J}$. Then there exists a finite collection of points $\{t_1,\ldots,t_m\}$ where $0 < t_1 < \ldots . < t_m \leq 1$ and a Borel set E in R^m such that $I = \{x \in C : (x(t_1),\ldots,x(t_m)) \in E\}$. Given $\epsilon > 0$ and let $\epsilon_0 = \epsilon . \{\pi^m c^m\, t_1(t_2 - t_1)\ldots(t_m - t_{m-1})\}^{1/2}$. ....(1)

Then according to (1.18) (i.e. (Leb.) $m^*(E) = \inf\left\{\sum_{n=1}^{\infty} m(E_n^o) : E \subseteq \bigcup_{n=1}^{\infty} E_n^o\right\}$ where the E_n^o are rectangles in R^m.), we have that there exists a sequence $\{E_n^o\}$ of rectangles in R^m which we may assume that E_n^o's are disjoint and such that $E \subseteq \bigcup_{n=1}^{\infty} E_n^o$ and $\sum_{n=1}^{\infty} m(E_n^o) < m^*(E) + \epsilon_0$.

Since E is measurable and E_n^o's are disjoint, it follows that $\bar{m}(\bigcup_{n=1}^{\infty} E_n^o - E) < \epsilon_0$. Let $I_n^o = \{x \in C : (x(t_1),\ldots,x(t_m)) \in E_n^o\}$, $n = 1,2,\ldots$. Then by Lemma 2.10 (ii), I_n^o's are disjoint sets in $\mathcal{J}^o$. According to Lemma 2.10 (i) and (iv) and the fact that $\bigcup_{n=1}^{\infty} E_n^o - E \in \mathcal{B}(R^n)$, we have $\bigcup_{n=1}^{\infty} I_n^o - I = \{x \in C : (x(t_1),\ldots,x(t_m)) \in \bigcup_{n=1}^{\infty} E_n^o - E\} \in \mathcal{J}$ and hence

$$W_c(\bigcup_{n=1}^{\infty} I_n^o - I) = \int_{\bigcup_{n=1}^{\infty} E_n^o - E} \cdots \int K\{t_1,\ldots,t_m, \xi_1,\ldots,\xi_m\}\, d\xi_1 \ldots d\xi_m \quad .....(2)$$

where $K\{t_1,\ldots,t_m, \xi_1,\ldots,\xi_m\} = \dfrac{1}{\sqrt{\pi^m c^m t_1(t_2-t_1)\ldots(t_m-t_{m-1})}}$

$$\cdot \exp\left\{\frac{-\xi_1^2}{ct_1} \cdots\cdots\cdots \frac{-(\xi_m-\xi_{m-1})^2}{ct_m - ct_{m-1}}\right\} .$$

Since $K\{t_1,\ldots,t_m, \xi_1,\ldots,\xi_m\} \leq \{\pi^m c^m t_1(t_2-t_1)\ldots(t_m-t_{m-1})\}^{-1/2}$ for all $(\xi_1,\ldots,\xi_m) \in R^m$ and $\bar{m}(\bigcup_{n=1}^{\infty} E_n^o - E) < \epsilon_0$, it follows from (1) and (2) that $W_c(\bigcup_{n=1}^{\infty} I_n^o - I) \leq \{\pi^m c^m t_1(t_2-t_1)\ldots(t_m-t_{m-1})\}^{-1/2}$. $\cdot\, \bar{m}(\bigcup_{n=1}^{\infty} E_n^o - E) < \epsilon$. But I_n^o's are disjoint and W_c is countably additive and $E \subseteq \bigcup_{n=1}^{\infty} E_n$ implies $I \subseteq \bigcup_{n=1}^{\infty} I_n^o$, we have that $\sum_{n=1}^{\infty} W_c(I_n^o) - W_c(I) = W_c(\bigcup_{n=1}^{\infty} I_n^o - I) < \epsilon$. Since ϵ is arbitrary,

$$W_c(I) = \inf\left\{\sum_{n=1}^{\infty} W_c(I_n^o) : I \subseteq \bigcup_{n=1}^{\infty} I_n^o ,\ I_n^o \in \mathcal{J}^o\right\} .$$

Step 2. We show that $W_c^{o*}(\Gamma) \leq W_c^*(\Gamma)$, $\Gamma \subseteq C$. Let $\Gamma \subseteq C$. Then according to the definition of $W_c^*(\Gamma)$, there exists a sequence $\{I_k\}$ in $\mathcal{J}$ such that $\Gamma \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} W_c(I_k) < W_c^*(\Gamma) + \epsilon/2$ . .....(3)

Since $I_k \in \mathcal{J}$, it follows from step 1 that there exists a sequence $\{I_{kn}^o\}$ in $\mathcal{J}^o$ such that $I_k \subseteq \bigcup_{n=1}^{\infty} I_{kn}^o$ and $\sum_{n=1}^{\infty} W_c(I_{kn}^o) < W_c(I_k) + \epsilon/2^{k+1}$.

By (3),

$$\Gamma \subseteq \bigcup_{k=1}^{\infty} I_k \subseteq \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} I_{kn}^o \text{ and } \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} W_c(I_{kn}^o) < \sum_{k=1}^{\infty} W_c(I_k) + \epsilon/2$$

$$< W_c^*(\Gamma) + \epsilon .$$

But then, it follows from the definitions of W_c^o and W_c^{o*} that

$$W_c^{o*}(\Gamma) \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} W_c(I_{kn}^o) < W_c^*(\Gamma) + \epsilon .$$

Since ϵ is arbitrary, we have that $W_c^{o*}(\Gamma) \leq W_c^*(\Gamma)$.

Q.E.D.

Theorem 4.6. $(C, \mathcal{J}_{W_c^o}, \overline{W}_c^o) = (C, \mathcal{J}_{W_c}, \overline{W}_c)$.

Proof. It follows from Definition 4.2 and Lemma 4.5 that $\mathcal{J}_{W_c^o} = \mathcal{J}_{W_c}$ and hence $\overline{W}_c^o(\Gamma) = W_c^{o*}(\Gamma) = W_c^*(\Gamma) = \overline{W}_c(\Gamma)$ for $\Gamma \in \mathcal{J}_{W_c^o} = \mathcal{J}_{W_c}$.

Q.E.D.

Definition 4.7. Let $\mathcal{D}^o$ denote the collection of all countable union of elements of $\mathcal{J}^o$.

By $\mathcal{D}^{o\downarrow}$ we mean the collection of limits of decreasing sequences of members of $\mathcal{D}^o$.

Lemma 4.8. If $\Gamma \subseteq C$ is Wiener measurable, then we can write $\Gamma = G \sim N$ where $G \in \mathcal{D}^{o\downarrow}$ and $N \subseteq G$ with $\overline{W}_c(N) = 0$.

Proof. Let Γ be a Wiener measurable set. Then according to Theorem 4.6, we have that for any positive integer n there exists a sequence $\{I_{nk}^o\}$ in $\mathcal{J}^o$ such that

$$\Gamma \subseteq \bigcup_{k=1}^{\infty} I_{nk}^o \text{ and } \sum_{k=1}^{\infty} \overline{W}(I_{nk}^o) < \overline{W}_c(\Gamma) + \frac{1}{n} . \quad \text{................(4)}$$

Let $\Omega^\circ_n = \bigsqcup_{k=1}^{\infty} I^\circ_{nk}$. Then $\Omega_n \in \mathscr{D}^\circ$ for all n and

$$\Omega^\circ_1 \cap \Omega_2 = (\bigsqcup_{k=1}^{\infty} I^\circ_{1k}) \cap (\bigsqcup_{j=1}^{\infty} I^\circ_{2j}) = \bigsqcup_{k=1}^{\infty} \bigsqcup_{j=1}^{\infty} (I^\circ_{1k} \cap I^\circ_{2j}) \in \mathscr{D}^\circ,$$

because $I^\circ_{1k} \cap I^\circ_{2j} \in \mathscr{J}^\circ$. By induction, $\Omega^\circ_1 \cap \Omega^\circ_2 \cap \dots \cap \Omega^\circ_n \in \mathscr{D}^\circ$ for all n. Let $G_n = \bigcap_{\ell=1}^{n} \Omega^\circ_\ell$. Then $G_n \in \mathscr{D}^\circ$ for all n. Also $G_n \supseteq G_{n+1}$ and hence $\{G_n\}$ converges. Let $G = \lim_{n\to\infty} G_n = \bigcap_{n=1}^{\infty} G_n$. Then by Definition 4.7, $G \in \mathscr{D}^{\delta}$. Moreover, since $\Omega^\circ_\ell \supseteq \Gamma$ for all ℓ, $G_n \supseteq \Gamma$ for all n and hence $G \supseteq \Gamma$. Thus according to (4), we have

$$0 \leq \bar{W}_c(G \sim \Gamma) \leq \bar{W}_c(G_n \sim \Gamma) \leq \bar{W}_c(\Omega^\circ_n \sim \Gamma) = \bar{W}_c(\Omega^\circ_n) - \bar{W}_c(\Gamma) \leq$$

$$\sum_{k=1}^{\infty} W_c(I^\circ_{nk}) - \bar{W}_c(\Gamma) < \frac{1}{n}.$$

By letting $n \to \infty$, we have $\bar{W}_c(G \sim \Gamma) = 0$ and by taking $N = G \sim \Gamma$, it follows that

$$\Gamma = G \sim N, \quad N \subseteq G \text{ and } \bar{W}_c(N) = 0.$$

<u>Q.E.D.</u>

<u>Definition 4.9</u>. From now on, we shall simply write $W_c(\Gamma)$ instead of $\overline{W}_c(\Gamma)$ or $\overline{W^\circ_c}(\Gamma)$ even for set Γ in $\mathscr{J}_{W_c}$. W_c is called <u>Wiener measure in C</u>. In case c = 1, we will denote W_1 by W. The integral in C with respect to W_c is called <u>Wiener integral</u>. If F is a Wiener measurable functional on C, its integral will be denoted by $\int_C F(x)dW_c(x)$.

<u>Theorem 4.10.</u> Let $0 < t_1 < \ldots < t_n \leq 1$ and $H(\xi_1, \ldots, \xi_n)$ be a Borel measurable function of n real variables $\xi_1, \ldots, \xi_n$. Then the functional $H(y(t_1), \ldots, y(t_n))$ defined on C is Wiener measurable and for each $c > 0$

$$\int_C H(y(t_1), \ldots, y(t_n))\, dW_c(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H(\xi_1, \ldots, \xi_n) K\{t_1, \ldots, t_n, \xi_1, \ldots, \xi_n\}\, d\xi_1 \ldots d\xi_n. \quad \ldots\ldots(5)$$

where $K\{t_1, \ldots, t_n, \xi_1, \ldots, \xi_n\}$ defined by (3.2) and the existence of one side implies that of the other and the validity of the equality.

<u>Proof.</u> Let $0 < t_1 < \ldots < t_n \leq 1$ be given and let $H(\xi_1, \ldots, \xi_n)$ be a Borel measurable function defined on R^n.

<u>Case 1.</u> If H is the characteristic function χ_E of a Borel set $E \subseteq R^n$. Let I be a quasi-interval in C defined by $I = \{x \in C : (x(t_1), \ldots, x(t_n)) \in E\}$. Then

$$\begin{aligned} H(y(t_1), \ldots, y(t_n)) &= \chi_E(y(t_1), \ldots, y(t_n)) \\ &= \begin{cases} 1 & (y(t_1), \ldots, y(t_n)) \in E \\ 0 & (y(t_1), \ldots, y(t_n)) \notin E \end{cases} \\ &= \begin{cases} 1 & y \in I \\ 0 & y \notin I \end{cases} \\ &= \chi_I(y) \end{aligned}$$

is Wiener measurable and for each $c > 0$,

$$W_c(I) = \int_I dW_c(y) = \int_C \chi_I(y)dW_c(y) = \int_C H(y(t_1),\ldots,y(t_n))\, dW_c(y) .$$

On the other hand, according to (3.1) and (3.2),

$$W_c(I) = \int_E \!\!\cdots\! \int K\{t_1,\ldots,t_n, \xi_1,\ldots,\xi_n\}\, d\xi_1\ldots d\xi_n$$

$$= \int_{-\infty}^{\infty} \!\!\cdots\! \int_{-\infty}^{\infty} \chi_E(\xi_1,\ldots,\xi_n) K\{t_1,\ldots,t_n,\xi_1,\ldots,\xi_n\}\, d\xi_1\ldots d\xi_n$$

$$= \int_{-\infty}^{\infty} \!\!\cdots\! \int_{-\infty}^{\infty} H(\xi_1,\ldots,\xi_n) K\{t_1,\ldots,t_n,\xi_1,\ldots,\xi_n\}\, d\xi_1\ldots d\xi_n .$$

Thus (5) holds for H in this case.

Case 2. If H is a simple function, i.e. if H is any finite linear combination of characteristic functions of disjoint Borel sets of R^n, then the functional $H(y(t_1),\ldots,y(t_n))$ is also a simple functional and hence (5) also holds.

Case 3. If H is a measurable extended non-negative valued function on R^n, then there exists a non-decreasing sequence of non-negative simple functions H_k which converges to H at each point in R^n. Hence the corresponding sequence of non-negative non-decreasing simple functionals $H_k(y(t_1),\ldots,y(t_n))$ converges to the functional $H(y(t_1),\ldots,y(t_n))$ at each $y \in C$. By case 2, (5) holds for each H_k. The limit may be passed under the integral sign by Lebesgue's Monotone Convergence Theorem and (5) holds for H in this case.

Case 4. Finally, if H is a measurable extended real-valued function on R^n, then by virtue of (1.23) we have $H = H^+ - H^-$. Hence by case 3, (5) holds for each of H^+ and H^-. In case the integrals of H^+ and H^- are not both infinite, the integral of H exists and (5) holds.

Q.E.D.

Example 4.11. Let t and s be any two points in [0,1]. Then

$$\int_C (x(t) - x(s))^2 d\, W_c(x) = \frac{c}{2} |t-s| .$$

Solution. Assume $t < s$.

Case 1. If $t = 0$, then according to (5) we have

$$\int_C (x(s))^2 dW_c(x) = \frac{1}{\sqrt{\pi cs}} \int_{-\infty}^{\infty} \xi^2 \exp\left\{\frac{-\xi^2}{cs}\right\} d\xi . \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Let $\eta = \dfrac{\xi}{\sqrt{cs}}$. Then (6) becomes

$$\int_C (x(s))^2 dW_c(x) = \frac{cs}{\sqrt{\pi}} \int_{-\infty}^{\infty} \eta^2 \exp\{-\eta^2\} d\eta = \frac{cs}{\sqrt{\pi}} \int_0^{\infty} \eta \exp\{-\eta^2\} d\,\eta^2$$

$$= \frac{cs}{2} \qquad = \frac{c}{2} |t-s| ,$$

since $\displaystyle\int_0^{\infty} \eta \exp\{-\eta^2\} d\,\eta^2 = \frac{\sqrt{\pi}}{2}$ and $t = 0$.

Case 2. If $t \neq 0$, then according to (5) we have

$$\int_C (x(t)-x(s))^2 dW_c(x) = \frac{1}{\sqrt{\pi^2 c^2 t(s-t)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\xi_1 - \xi_2)^2 \exp\left\{\frac{-\xi_1^2}{ct} - \frac{(\xi_2-\xi_1)^2}{cs-ct}\right\} d\xi_1 d\xi_2 .$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Let $\eta_1 = \dfrac{\xi_1}{\sqrt{ct}}$, $\eta_2 = \dfrac{\xi_2 - \xi_1}{\sqrt{cs-ct}}$. Then

$$\frac{\partial(\eta_1,\eta_2)}{\partial(\xi_1,\xi_2)} = \begin{vmatrix} \frac{1}{\sqrt{ct}} & 0 \\ \frac{-1}{\sqrt{cs-ct}} & \frac{1}{\sqrt{cs-ct}} \end{vmatrix} = \frac{1}{\sqrt{c^2t(s-t)}}$$

and hence (7) becomes

$$\int_C (x(t)-x(s))^2 dW_c(x) = \pi^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\eta_2^2(cs-ct)\exp\{-\eta_1^2-\eta_2^2\}\,d\eta_1 d\eta_2$$

$$= \pi^{-1}(cs-ct)\int_{-\infty}^{\infty}\exp\{-\eta_1^2\}d\eta_1\int_{-\infty}^{\infty}\eta_2^2\exp\{-\eta_2^2\}\,d\eta_2 .$$

But then, since $\int_{-\infty}^{\infty}\exp\{-\eta_1^2\}\,d\eta_1 = \sqrt{\pi}$ and $\int_{-\infty}^{\infty}\eta_2^2\exp\{-\eta_2^2\}\,d\eta_2 = \frac{\sqrt{\pi}}{2}$, we have

$$\int_C (x(t)-x(s))^2 dW_c(x) = \frac{c}{2}(s-t) = \frac{c}{2}|t-s| .$$

<u>Ans</u>.

<u>Example 4.12</u>. $\int_C \left[\int_0^1 (x(t))^2 dt\right] d\,W_c(x) = \frac{c}{4}$.

<u>Solution</u>. According to the Fubini Theorem and Example 4.11,

$$\int_C \left[\int_0^1 (x(t))^2\,dt\right] dW_c(x) = \int_0^1 \left[\int_C (x(t))^2\,dW_c(x)\right] dt$$

$$= \int_0^1 \frac{c}{2}\,t\,dt = \frac{c}{4} .$$

<u>Ans</u>.