# CHAPTER II

## ON THE WIENER SPACE

In this chapter, we will define an algebra and a semi-algebra of subsets of the Wiener space.

Definition 2.1. The set of all real-valued continuous functions on an interval $[a,b]$, with the usual operations of addition of functions and scalar multiplication, forms a linear space which will be denoted by $\mathcal{C}^{\circ}[a,b]$. $\mathcal{C}^{\circ}[a,b]$ equipped with the norm $\|f\| = \max_{a \leq t \leq b} |f(t)|$, $f \in \mathcal{C}^{\circ}[a,b]$ is a separable Banach space.

Definition 2.2. The Wiener space C of functions of one variable is the collection of real-valued continuous functions x defined on $[0,1]$ and satisfying $x(0) = 0$.

Since for any $x, y \in C$ and any scalars $\alpha$ and $\beta$ we have $\alpha x + \beta y$ is continuous on $[0,1]$ and $(\alpha x+\beta y)(0) = \alpha x(0)+\beta y(0) = 0$, so that $\alpha x+\beta y \in C$ and clearly $0 \in C$. Therefore C is a subspace of $\mathcal{C}^{\circ}[0,1]$. Moreover, C is a closed subspace of $\mathcal{C}^{\circ}[0,1]$. In fact, if $x_o \in \mathcal{C}^{\circ}[0,1]$ is a limit point of C, then there exists a sequence $\{x_n\}$ in C such that $\|x_n - x_o\| \longrightarrow 0$ as $n \longrightarrow \infty$, i.e. $\{x_n\}$ converges uniformly to $x_o$ on $[0,1]$. Hence $x_o$ is continuous and in particular $x_o(0) = \lim_{n\to\infty} x_n(0)$. But $x_n(0) = 0$ for all n. Therefore $x_o(0) = 0$.

Thus $x_o \in C$. We have shown that C contains all its limit points. It follows that C is a closed subspace of $\mathscr{C}^{o}[0,1]$ and hence C is complete. Therefore C is a separable Banach space with the norm $\|\cdot\|$ defined by

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)| \quad , \quad x \in C .$$

Definition 2.3. Let $\{t_1,\ldots,t_n\}$ be a finite collection of numbers satisfying $0 < t_1 < \ldots < t_n \leq 1$ and let E be a Borel set of the n-dimensional Euclidean space $R^n$ ; i.e. $E \in \mathscr{B}(R^n)$. A subset I of C defined by

$$I = \{x \in C : (x(t_1),\ldots,x(t_n)) \in E\} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

will be called a quasi-interval in C. The points $t_1,\ldots, t_n$ and the set E will be called the restriction points and the restricting set of I.

In particular, if E is a rectangle in $R^n$, then I which we will denote by $I^o$ will be called an interval in C ; i.e.

$$I^o = \{x \in C : \alpha_i \leq x(t_i) \leq \beta_i \ , \ i = 1,\ldots,n\} \ldots\ldots\ldots\ldots (2)$$

or any set obtained by replacing any or all of the $\leq$ signs by $<$.

Definition 2.4. Let E be a set in $R^n$ and let k be an integer, $1 \leq k \leq n$. We define

$$E \underset{k}{\boxtimes} R = \{(\alpha_1,\ldots,\alpha_{k-1},\alpha^*,\alpha_k,\ldots,\alpha_n) \in R^{n+1} : (\alpha_1,\ldots,\alpha_{k-1},\alpha_k,\ldots,\alpha_n) \in E, \alpha^* \in R\}.$$

<u>Theorem 2.5</u>. If $E \in \mathcal{B}(R^n)$ then $E \underset{k}{\boxtimes} R \in \mathcal{B}(R^{n+1})$.

<u>Proof</u>. We divide the proof into 3 steps :

<u>Step 1</u>. We show that $(E \underset{k}{\boxtimes} R)' = E' \underset{k}{\boxtimes} R$,

where $(E \underset{k}{\boxtimes} R)' = R^{n+1} \setminus (E \underset{k}{\boxtimes} R)$ and $E' = R^n \setminus E$.

If $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in (E \underset{k}{\boxtimes} R)'$, then $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \notin E \underset{k}{\boxtimes} R$, so that $\alpha_k \in R$ but $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{n+1}) \notin E$. Therefore $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in E'$ and hence $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in E' \underset{k}{\boxtimes} R$.
Conversely, if $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in E' \underset{k}{\boxtimes} R$, then $\alpha_k \in R$ and $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in E'$, so that $\alpha_k \in R$ but $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{n+1}) \notin E$. Therefore $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \notin E \underset{k}{\boxtimes} R$ and hence $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in (E \underset{k}{\boxtimes} R)'$.

<u>Step 2</u>. We show that $\bigcup_{i=1}^{\infty} (E_i \underset{k}{\boxtimes} R) = \bigcup_{i=1}^{\infty} E_i \underset{k}{\boxtimes} R$.

If $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in \bigcup_{i=1}^{\infty} (E_i \underset{k}{\boxtimes} R)$, then $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in E_i \underset{k}{\boxtimes} R$ for some i, so that $(\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in E_i$ and $\alpha_k \in R$. Therefore

$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in \bigcup_{i=1}^{\infty} E_i \boxtimes_k R$. Conversely, if

$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in \bigcup_{i=1}^{\infty} E_i \boxtimes_k R$, then

$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in E_i$ for some i and $\alpha_k \in R$, so that

$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in E_i \boxtimes_k R$ for some i. Therefore

$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n+1}) \in \bigcup_{i=1}^{\infty} (E_i \boxtimes_k R)$.

Step 3. Let $\mathcal{A} = \left\{ E \in \mathcal{B}(R^n) : E \boxtimes_k R \in \mathcal{B}(R^{n+1}) \right\}$. Then

(i). It is clear that if E is a rectangle in $R^n$, then $E \boxtimes_k R$ is also a rectangle in $R^{n+1}$. Therefore $E \boxtimes_k R \in \mathcal{B}(R^{n+1})$ and hence $E \in \mathcal{A}$.

(ii). If $E \in \mathcal{A}$, then $E \boxtimes_k R \in \mathcal{B}(R^{n+1})$. Since $\mathcal{B}(R^{n+1})$ is a $\sigma$-algebra, $(E \boxtimes_k R)' \in \mathcal{B}(R^{n+1})$. It follows from step 1 that $E' \in \mathcal{A}$.

(iii). If $E_i \in \mathcal{A}$ for $i = 1,2,\ldots$, then $E_i \boxtimes_k R \in \mathcal{B}(R^{n+1})$ for all i. Since $\mathcal{B}(R^{n+1})$ is a $\sigma$-algebra, $\bigcup_{i=1}^{\infty} (E_i \boxtimes_k R) \in \mathcal{B}(R^{n+1})$. It follows from step 2 that $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

From (i), (ii) and (iii) we have that $\mathcal{A}$ is a $\sigma$-algebra containing all rectangles in $R^n$ and must therefore contain the collection $\mathcal{B}(R^n)$ of all Borel sets, since $\mathcal{B}(R^n)$ is the smallest $\sigma$-algebra containing rectangles. Hence $\mathcal{A} = \mathcal{B}(R^n)$.

Q.E.D.

Remark 2.6. For a given collection of n restriction points $t_1, \ldots, t_n$ and a given restricting set E there is associated a unique subset I of C given by (1), but the converse is not true. For instance for the subset I defined by (1) we may throw in a few more points in $(0,1]$, so that there are m additional points in $(0,1]$, let the restriction at each of the additional restriction points $t_{k_j}$ be the trivial restriction $-\infty < x(t_{k_j}) < \infty$ and let the restricting set be $E \underset{k_1}{\boxtimes} R \underset{k_2}{\boxtimes} \ldots \underset{k_m}{\boxtimes} R$, a Borel set in $R^{n+m}$, where

$$E \underset{k_1}{\boxtimes} R \underset{k_2}{\boxtimes} \ldots \underset{k_m}{\boxtimes} R = \Big\{ (\alpha_1, \ldots, \alpha_{k_1-1}, \alpha_{k_1}, \alpha_{k_1+1}, \ldots, \alpha_{k_j-1}, \alpha_{k_j}, \alpha_{k_j+1}, \ldots, \alpha_{n+m}) \in R^{n+m} : (\alpha_1, \ldots, \alpha_{k_1-1}, \alpha_{k_1+1}, \ldots, \alpha_{k_j-1}, \alpha_{k_j+1}, \ldots, \alpha_{n+m}) \in E \text{ and } \alpha_{k_1}, \ldots, \alpha_{k_j}, \ldots, \alpha_{k_m} \in R \Big\}.$$

Then the subset of C defined with the (n+m) restriction points and the restricting set $E \underset{k_1}{\boxtimes} R \underset{k_2}{\boxtimes} \ldots \underset{k_m}{\boxtimes} R$ is identical with the one defined by (1).

Lemma 2.7. Let $I_1 = \{ x \in C : (x(t_1), \ldots, x(t_n)) \in E_1 \}$ and $I_2 = \{ x \in C : (x(s_1), \ldots, x(s_n)) \in E_2 \}$ be two quasi-intervals which have no trivial restriction. Suppose $I_1 = I_2$. Then $s_1 = t_1, \ldots, s_n = t_n$ and $E_1 = E_2$.

Proof. Assume $I_1 = I_2$.

Step 1. We show that $s_i = t_i$ for all $i = 1, \ldots, n$. Suppose on the contrary and let k be the first positive integer such that $s_k \neq t_k$.

Case 1. If $s_k = t_{k+\ell}$ for some $\ell \geqslant 1$. Then $t_k \neq s_i$ for any $i = 1,\dots,n$. Since $I_1$ has no trivial restriction at the restriction point $t_k$, there exist real numbers $\gamma_1,\dots,\gamma_k,\dots,\gamma_n$ and $\beta_k$ such that $(\gamma_1,\dots,\gamma_k,\dots,\gamma_n) \in E_1$ but $(\gamma_1,\dots,\beta_k,\dots,\gamma_n) \notin E_1$. Construct a function $x \in C$ such that $x(t_1) = \gamma_1,\dots,x(t_k) = \gamma_k,\dots,x(t_n) = \gamma_n$, so that $x \in I_1$. But then $x \in I_2$ and hence $(x(s_1),\dots, x(s_n)) \in E_2$. By letting $\alpha_1 = x(s_1),\dots,\alpha_n = x(s_n)$, we have $(\alpha_1,\dots,\alpha_n) \in E_2$ and it follows that

if $s_i = t_q$ for some q, then $\alpha_i = \gamma_q$ . ......................(3)

By virtue of (3) and the fact that $t_k \neq s_i$ for any $i = 1,\dots,n$ , we can construct a function $y \in C$ such that $y(t_1) = \gamma_1,\dots,y(t_{k-1}) = \gamma_{k-1}$ , $y(t_k) = \beta_k$ , $y(t_{k+1}) = \gamma_{k+1},\dots, y(t_n) = \gamma_n$ and $y(s_1) = \alpha_1,\dots,y(s_n) = \alpha_n$ , so that $y \in I_2$ but $y \notin I_1$, which is a contradiction.

Case 2. If $s_k \neq t_{k+\ell}$ for any $\ell \geqslant 1$. Then $s_k \neq t_i$ for any $i = 1,\dots,n$. Since $I_2$ has no trivial restriction at the restriction point $s_k$, there exist real numbers $a_1,\dots,a_k,\dots,a_n$ and $d_k$ such that $(a_1,\dots,a_k,\dots,a_n) \in E_2$ but $(a_1,\dots,d_k,\dots,a_n) \notin E_2$. Construct a function $x \in C$ such that $x(s_1) = a_1,\dots,x(s_k) = a_k,\dots,x(s_n) = a_n$, so that $x \in I_2$. But then $x \in I_1$ and hence $(x(t_1),\dots,x(t_n)) \in E_1$. By letting $b_1 = x(t_1),\dots,b_n = x(t_n)$, we have $(b_1,\dots,b_n) \in E_1$ and it follows that

if $s_i = t_q$ for some q, then $a_i = b_q$. . ......................(4)

By virtue of (4) and the fact that $s_k \neq t_i$ for any $i = 1,\ldots,n$, we can construct a function $y \in C$ such that $y(t_1) = b_1,\ldots,y(t_n) = b_n$ and $y(s_1) = a_1,\ldots,y(s_{k-1}) = a_{k-1}$, $y(s_k) = d_k$, $y(s_{k+1}) = a_{k+1},\ldots,y(s_n) = a_n$, so that $y \in I_1$ but $y \notin I_2$, which is a contradiction.

It follows from both cases that $s_1 = t_1,\ldots, s_n = t_n$ .

Step 2. It remains to show that $E_1 = E_2$ . Suppose on the contrary. Then either there exists a point $(\xi_1,\ldots, \xi_n) \in E_1$ but $(\xi_1,\ldots, \xi_n) \notin E_2$ or there exists a point $(\eta_1,\ldots, \eta_n) \in E_2$ but $(\eta_1,\ldots, \eta_n) \notin E_1$. Thus we can construct a function $z_1 \in C$ such that $z_1 \in I_1$ but $z_1 \notin I_2$ or a function $z_2 \in C$ such that $z_2 \in I_2$ but $z_2 \notin I_1$, which is a contradiction. Therefore $E_1 = E_2$ . Q.E.D.

Theorem 2.8. Let $I_1 = \{x \in C : (x(t_1),\ldots, x(t_n)) \in E_1\}$ and $I_2 = \{x \in C : (x(s_1),\ldots, x(s_m)) \in E_2\}$ be any two quasi-intervals which have no trivial restriction. Then $I_1 = I_2$ if and only if $n = m$, $s_1 = t_1,\ldots s_n = t_n$ and $E_1 = E_2$.

Proof. Clearly, if $n = m$, $s_1 = t_1,\ldots, s_n = t_n$ and $E_1 = E_2$, then $I_1 = I_2$ .

To prove the converse, assume $I_1 = I_2$. First, we want to show that $n = m$. Suppose $n < m$. Then there exists a point $s_j$ where $1 \leq j \leq m$ and $s_j \neq t_i$ for any $i = 1,\ldots, n$. Since $I_2$ has no trivial restriction at the restriction point $s_j$, there exist real numbers $\alpha_1,\ldots, \alpha_j,\ldots, \alpha_m$ and $\beta_j$ such that $(\alpha_1,\ldots, \alpha_j,\ldots, \alpha_m) \in E_2$ but $(\alpha_1,\ldots, \beta_j,\ldots, \alpha_m) \notin E_2$. Construct a function $x \in C$ such that $x(s_1) = \alpha_1,\ldots,x(s_j) = \alpha_j,\ldots,x(s_m) = \alpha_m$, so that $x \in I_2$. But then $x \in I_1$ and hence

$(x(t_1),\dots,x(t_n)) \in E_1$. By letting $\gamma_1 = x(t_1),\dots, \gamma_n = x(t_n)$, we have $(\gamma_1,\dots,\gamma_n) \in E_1$ and it follows that

if $s_k = t_\ell$ for some $\ell$, then $\alpha_k = \gamma_\ell$ . ................(5)

By virtue of (5) and the fact that $s_j \neq t_i$ for any $i = 1,\dots,n$, we can construct a function $y \in C$ such that $y(t_1) = \gamma_1,\dots, y(t_n) = \gamma_n$ and $y(s_1) = \alpha_1,\dots, y(s_{j-1}) = \alpha_{j-1}, y(s_j) = \beta_j, y(s_{j+1}) = \alpha_{j+1},\dots,y(s_m) = \alpha_m$, so that $y \in I_1$ but $y \notin I_2$, which is a contradiction. Therefore $n \geq m$. Similarly, we can show that $m \nless n$. Thus $n = m$. So

$I_1 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1\}$ and $I_2 = \{x \in C : (x(s_1),\dots,x(s_n)) \in E_2\}$ .

Then, it follows from Lemma 2.7 that $s_1 = t_1,\dots, s_n = t_n$ and $E_1 = E_2$ . Q.E.D.

Remark 2.9. It follows from the above theorem and Remark 2.6. that when we have two quasi-intervals in C, say $I_1 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1\}$ and $I_2 = \{x \in C : (x(s_1),\dots, x(s_m)) \in E_2\}$ we can make them comparable in terms of their restricting sets by using $\{t_1,\dots,t_n\} \cup \{s_1,\dots,s_m\}$ as their restriction points by means of adding trivial restriction at each newly restriction point and extending their restricting sets to two Borel sets of the same Euclidean space. By induction, this procedure is true for any finite collection of sets $I_1,\dots,I_n$ in C.

Lemma 2.10. Let $I_1 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1\}$ and $I_2 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_2\}$ be any two quasi-intervals in C. Then

(i) $I_1 \cup I_2 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1 \cup E_2\}$.

(ii) $I_1 \cap I_2 = \emptyset$ if and only if $E_1 \cap E_2 = \emptyset$.

(iii) $I_1 \cap I_2 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1 \cap E_2\}$.

(iv) $I_1 - I_2 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1 - E_2\}$.

Proof. To prove (i), we let $I_3 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1 \cup E_2\}$. If $x \in I_1 \cup I_2$, then $x \in I_1$ or $x \in I_2$, so that $(x(t_1),\dots,x(t_n)) \in E_1$ or $(x(t_1),\dots,x(t_n)) \in E_2$. Therefore $(x(t_1),\dots,x(t_n)) \in E_1 \cup E_2$ and hence $x \in I_3$. Conversely, if $x \in I_3$, then $(x(t_1),\dots,x(t_n)) \in E_1 \cup E_2$, so that $(x(t_1),\dots,x(t_n)) \in E_1$ or $(x(t_1),\dots,x(t_n)) \in E_2$. Therefore $x \in I_1$ or $x \in I_2$ and hence $x \in I_1 \cup I_2$.

To prove (ii), we see that if $I_1 \cap I_2 \neq \emptyset$, then there exists $x \in C$ such that $x \in I_1$ and $x \in I_2$. But then $(x(t_1),\dots,x(t_n)) \in E_1$ and $(x(t_1),\dots,x(t_n)) \in E_2$. Therefore $E_1 \cap E_2 \neq \emptyset$. Conversely, if $E_1 \cap E_2 \neq \emptyset$, then there exists a point $(a_1,\dots,a_n) \in R^n$ such that $(a_1,\dots,a_n) \in E_1$ and $(a_1,\dots,a_n) \in E_2$. Thus, we can construct a function $x \in C$ such that $x(t_1) = a_1,\dots, x(t_n) = a_n$. Therefore $x \in I_1$ and $x \in I_2$. Hence $I_1 \cap I_2 \neq \emptyset$.

To prove (iii), we let $I_4 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1 \cap E_2\}$. By (ii), we may assume that $I_1 \cap I_2 \neq \emptyset$ (for otherwise $I_1 \cap I_2 = I_4 = \emptyset$) and hence $E_1 \cap E_2 \neq \emptyset$. Therefore $I_4 \neq \emptyset$. If $x \in I_1 \cap I_2$, then $x \in I_1$ and $x \in I_2$, so that $(x(t_1),\dots,x(t_n)) \in E_1$ and $(x(t_1),\dots,x(t_n)) \in E_2$.

Therefore $(x(t_1),\dots,x(t_n)) \in E_1 \cap E_2$ and hence $x \in I_4$. Conversely, if $x \in I_4$, then $(x(t_1),\dots,x(t_n)) \in E_1 \cap E_2$, so that $(x(t_1),\dots,x(t_n)) \in E_1$ and $(x(t_1),\dots,x(t_n)) \in E_2$. Therefore $x \in I_1$ and $x \in I_2$. Thus $x \in I_1 \cap I_2$.

To prove (iv), we let $I_5 = \{x \in C : (x(t_1),\dots,x(t_n)) \in E_1 - E_2\}$. By Theorem 2.8, we may assume that $I_1 \neq I_2$ and hence $E_1 \neq E_2$, so $I_5 \neq \emptyset$. If $x \in I_1 - I_2$, then $x \in I_1$ but $x \notin I_2$ and hence $(x(t_1),\dots,x(t_n)) \in E_1$ but $(x(t_1),\dots,x(t_n)) \notin E_2$, so $(x(t_1),\dots,x(t_n)) \in E_1 - E_2$. Therefore $x \in I_5$. Conversely, if $x \in I_5$, then $(x(t_1),\dots,x(t_n)) \in E_1 - E_2$, so that $(x(t_1),\dots,x(t_n)) \in E_1$ but $(x(t_1),\dots,x(t_n)) \notin E_2$. Therefore $x \in I_1$ but $x \notin I_2$ and hence $x \in I_1 - I_2$.

Q.E.D.

Example 2.11. Let $I_1 = \{x \in C : (x(\frac{1}{4}), x(\frac{2}{3})) \in E_1\}$ and $I_2 = \{x \in C : (x(\frac{1}{4}), x(\frac{1}{3}), x(\frac{1}{2})) \in E_2\}$ where $E_1 \in \mathcal{B}(R^2)$ and $E_2 \in \mathcal{B}(R^3)$. Then according to Remark 2.6, Remark 2.9 and Lemma 2.10, we have

$$I_1 \cup I_2 = \{x \in C : (x(\tfrac{1}{4}), x(\tfrac{1}{3}), x(\tfrac{1}{2}), x(\tfrac{2}{3})) \in (E_1 \boxtimes_2 R \boxtimes_3 R) \cup (E_2 \times R)\}.$$

$$I_1 \cap I_2 = \{x \in C : (x(\tfrac{1}{4}), x(\tfrac{1}{3}), x(\tfrac{1}{2}), x(\tfrac{2}{3})) \in (E_1 \boxtimes_2 R \boxtimes_3 R) \cap (E_2 \times R)\}.$$

$$I_1 - I_2 = \{x \in C : (x(\tfrac{1}{4}), x(\tfrac{1}{3}), x(\tfrac{1}{2}), x(\tfrac{2}{3})) \in (E_1 \boxtimes_2 R \boxtimes_3 R) - (E_2 \times R)\}.$$

Let $\mathcal{J}$ be the collection of all quasi-intervals defined by (1) and $\mathcal{J}^{\circ}$ be the collection of all intervals defined by (2).

Theorem 2.12. $\mathcal{J}$ is an algebra of sets and $\mathcal{J}^{\circ}$ is a semi-algebra of sets.

Proof. We first observe that $\emptyset = \{x \in C : (x(t_1),\ldots,x(t_n)) \in \emptyset\}$. Therefore $\emptyset$ belongs to $\mathcal{J}^{\circ}$ and $\mathcal{J}$. Also $C = \{x \in C : -\infty < x(t_1) < \infty, \ldots, -\infty < x(t_n) < \infty\}$. Therefore C belongs to $\mathcal{J}^{\circ}$ and $\mathcal{J}$.

From Remark 2.9 and Lemma 2.10 (i) and (iv) and from the fact that the collection $\mathcal{B}(R^n)$ of Borel sets is a $\sigma$-algebra of sets, it follows that $\mathcal{J}$ is an algebra of sets.

Next, we want to show that $\mathcal{J}^{\circ}$ is a semi-algebra of sets.

(i). From Remark 2.9 and Lemma 2.10 (iii) and from the fact that the intersection of any two rectangles in $R^n$ is also a rectangle in $R^n$, it follows that $I_1^{\circ} \cap I_2^{\circ} \in \mathcal{J}^{\circ}$ for any $I_1^{\circ}, I_2^{\circ} \in \mathcal{J}^{\circ}$.

(ii). We show that if $I^{\circ} \in \mathcal{J}^{\circ}$, then $C \sim I^{\circ}$ is a finite disjoint union of intervals in $\mathcal{J}^{\circ}$. To see this, let $I^{\circ} = \{x \in C : (x(t_1),\ldots,x(t_n)) \in E^{\circ}$, where $E^{\circ}$ is a rectangle in $R^n\}$ be any interval in $\mathcal{J}^{\circ}$. Since the collection of rectangles in $R^n$ is a semi-algebra of sets, $R \sim E^{\circ}$ is a finite disjoint union of rectangles, say $R \sim E^{\circ} = E_1^{\circ} \cup \ldots\ldots \cup E_m^{\circ}$ where $E_k^{\circ}$'s are disjoint rectangles. Let $I_k^{\circ} = \{x \in C : (x(t_1),\ldots,x(t_n)) \in E_k^{\circ}\}$, $k = 1,\ldots,m$. Since $E_k^{\circ}$'s are disjoint, it follows from Lemma 2.10 (ii) that $I_k^{\circ}$'s are disjoint. By using induction to Lemma 2.10 (i), we have

$\bigcup_{k=1}^{m} I_k^{\circ} = \{x \in C.(x(t_1),\ldots,x(t_n)) \in \bigcup_{k=1}^{m} E_k^{\circ}\}$. But $\bigcup_{k=1}^{m} E_k^{\circ} = R \sim E^{\circ}$,

therefore $\bigcup_{k=1}^{m} I_k^{\circ} = \{x \in C : (x(t_1),\ldots,x(t_n)) \in R \sim E^{\circ}\} = C \sim I^{\circ}$.

It follows from (i) and (ii) that $\mathcal{J}^{\circ}$ is a semi-algebra of sets.

Q.E.D.