CHAPTER III

STATIONARY OPERATORS OVER $L^2(\mathbb{T})$

1. Definition. Let E be a linear space of complex valued functions defined on $\mathbb{T}$ such that for any function f in E, $U_h f$ is also in E. An operator $P: E \longrightarrow E$ is said to be stationary, or to commute with translations, if and only if $P(U_h f) = U_h(pf)$ for all h in $\mathbb{T}$.

A study of stationary operators is beyond the scope of this thesis, and we shall only give a few examples of such operators in the next chapter. However, there is one space over which the continuous stationary operators can be represented simply.

2. Theorem. If P is a stationary continuous linear operator on $L^2(\mathbb{T})$, then there exists a bounded sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$ of complex numbers with $c_n(Pf) = \lambda_n c_n(f)$ for all f in $L^2(\mathbb{T})$ and for all $n \in \mathbb{Z}$.

Conversely, given a bounded sequence $\{\lambda_n\}$ of complex numbers, there is a stationary continuous linear operator P over $L^2(\mathbb{T})$ such that $c_n(Pf) = \lambda_n c_n(f)$ for all f in $L^2(\mathbb{T})$ and for all $n \in \mathbb{Z}$.

Proof. Suppose that P is a stationary continuous linear operator on $L^2(\mathbb{T})$. We will show first that the characters E_n of $\mathbb{T}$, which defined on $\mathbb{T}$ such that $E_n : \dot{x} \longrightarrow e^{2\pi inx}$, are eigen vectors of P. Since

$U_{\dot{h}}(E_n)(\dot{x}) = E_n(\dot{x} + \dot{h}) = E_n(\dot{x})\, E_n(\dot{h})$, we get $U_{\dot{h}}(E_n) = E_n(\dot{h}).\, E_n$. By linearity and stationarity of P, we get $U_{\dot{h}}\big(P(E_n)\big) = P\big(U_{\dot{h}}(E_n)\big) = E_n(\dot{h})\, P(E_n)$. So that $U_{\dot{h}}\big(P(E_n)(\dot{x})\big) = E_n(\dot{h})\ \big[P(E_n)(x)\big]$. Since $P(E_n)(\dot{x} + \dot{h}) = E_n(\dot{h})\ \big[P(E_n)(\dot{x})\big]$, by setting $\dot{x} = o$ in the above equation, we get $P(E_n)(\dot{h}) = E_n(\dot{h})\, P(E_n)(\dot{o})$. So that $P(E_n) = \lambda_n\, E_n$ where $\lambda_n = P(E_n)(\dot{o})$. Hence E_n is an eigenvector of P with eigenvalue λ_n for all n. Due to linearity and continuity of P, we know that P is bounded. And so $|\lambda_n|\, \| E_n \|_2 = \| P(E_n) \|_2 \leq \| P \|\ \| E_n \|_2 = \| P \|$, which yield $|\lambda_n| \leq \| P \|$ for all n. Therefore $\{\lambda_n\}_{n \in \mathbb{Z}}$ is a bounded sequence of complex numbers. Finally, due to the continuity of P and the Riesz - Fisher Theorem, we get for any f in $L^2(\mathbb{T})$.

$$P(f) = P\Big(\lim_{n \to \infty} \sum_{k=-n}^{n} c_k(f)\, E_k\Big)$$

$$= \lim_{n \to \infty} \sum_{k=-n}^{n} c_k(f)\, P(E_k)$$

$$= \lim_{n \to \infty} \sum_{k=-n}^{n} c_k(f)\, \lambda_k\, E_k .$$

And so, due to the uniquences theorem, $c_k\,(P(f)) = \lambda_k\, c_k(f)$ for all $k \in \mathbb{Z}$.

Conversely, suppose that $\{\lambda_n\}$ is a bounded sequence of complex numbers. Let f be any function in $L^2(\mathbb{T})$; due to Riesz - Fisher Theorem $f \sim \sum_{n\in\mathbb{Z}} c_n(f)\, E_n$

Observe that $\sum_{n\in\mathbb{Z}} |\lambda_n\, c_n(f)|^2$ converges, which follows by considering any integer n,

$$\sum_{k=-n}^{n} |\lambda_k\, c_k(f)|^2 = \sum_{k=-n}^{n} |\lambda_k|^2 |c_k(f)|^2$$

$$\leq \sup_{k\in\mathbb{Z}} |\lambda_k|^2 \sum_{k=-n}^{n} |c_k(f)|^2$$

$$= \sup_{k\in\mathbb{Z}} |\lambda_k|^2 \, ||f||_2^{\,2} < +\infty$$

So that

$$\sum_{n\in\mathbb{Z}} |\lambda_n\, c_n(f)|^2 \leq \sup |\lambda_n|^2 \; ||f||_2^{\,2} < +\infty$$

Therefore $\sum_{n\in\mathbb{Z}} \lambda_n\, c_n(f)\, E_n$ is the Fourier series of a unique function in $L^2(\mathbb{T})$. Let us denote this function by Pf.

First, we will show that the mapping $f \mapsto Pf$ is linear. Since c_n is linear,

$$P(\alpha f + \beta g) \sim \sum \lambda_n\, c_n(\alpha f + \beta g)\, E_n$$

$$= \sum \lambda_n \big(\alpha\, c_n(f) + \beta\, c_n(g)\big)\, E_n$$

$$= \alpha \sum \lambda_n\, c_n(f)\, E_n + \beta \sum \lambda_n\, c_n(g)\, E_n$$

$$= \alpha\, Pf + \beta\, Pg$$

for any complex number α, β and for any f, g in $L^2(\mathbb{T})$. Next P is bounded; in fact.

$$\|Pf\|_2^{\,2} = \sum_{n \in \mathbb{Z}} |\lambda_n|^2 |c_n(f)|^2$$

$$\leq (\sup_n |\lambda_n|^2) \sum_n |c_n(f)|^2$$

$$\leq (\sup_n |\lambda_n|^2) \|f\|_2^{\,2} ,$$

and so

$$\|Pf\|_2 \leq \sup_n |\lambda_n| \|f\|_2 ,$$

which proves the boundedness and therefore, the continuity of P.

Now, we will show that P is stationary.

We have $\quad f \sim \sum c_n(f) E_n$

$$U_h f \sim \sum c_n (U_h f) E_n$$

$$= \sum c_n(f) E_n(h) E_n .$$

This last equality follows from $c_n(U_h f) = \langle U_h f , E_n \rangle = E_n(h) c_n(f)$. By continuity and linearity of P,

we have

$$P(U_h f) \sim \sum \lambda_n c_n(U_h f) E_n$$

$$= \sum \lambda_n c_n(f) E_n(h) E_n$$

$$U_h (Pf) = \sum c_n(Pf) E_n(h) E_n$$

$$= \sum \lambda_n c_n(f) E_n(h) E_n$$

Hence $\quad P(U_h f) = U_h (Pf) .$

Therefore P is a stationary continuous linear operator over $L^2(\Gamma)$, such that $P(U_{\dot{h}} f) = U_{\dot{h}}(Pf)$ for all $\dot{h} \varepsilon \Gamma$. Thus the Theorem is now proved.

For two distinct bounded sequences, we get two distinct operators. Thus we have exhibited a bijection between the set of all linear stationary continuous operators on $L^2(\Gamma)$ and the set of all bounded infinite sequences of complex numbers.