CHAPTER 0

INTRODUCTION

In this thesis, we are primarily concerned with some questions in Harmonic Analysis on $\mathbb{T}$, $\mathbb{Z}$ and $\mathbb{R}$, respectively the 1- dimensional torus, the integers and the reals. As is well known, Harmonic Analysis on $\mathbb{T}$ is just the usual Theory of Fourier Series while Harmonic Analysis on $\mathbb{R}$ is the ordinary Theory of Fourier Integrals. Of course we are considering $\mathbb{R}$ together with the Lebesgue measure and $\mathbb{T}$ together with the measure induced by the Lebesgue measure of $\mathbb{R}$ and $\mathbb{Z}$ together with the counting measure. Recall that if $f \in L^1(\mathbb{T})$, its Fourier transform $\hat{f}$ is defined to be the function

$$\hat{f}(n) = \int_{\mathbb{T}} f(\dot{t}) \overline{E_n(\dot{t})} \, d\dot{t} \qquad (n \in \mathbb{Z}).$$

The central problem is to determine whether, and in what sense, the Fourier series

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) E_k(\dot{x}) = \int_{\mathbb{Z}} \hat{f}(k) E_k(\dot{x}) \, dk$$

represents f.

If $f \in L^1(\mathbb{Z})$, we define, by analogy, its Fourier transform $\hat{f}$ by

$$\hat{f}(\dot{x}) = \int_{\mathbb{Z}} f(k) \overline{E_k(\dot{x})} \, dk$$

$$= \sum_{k=-\infty}^{\infty} f(k) \overline{E_k(\dot{x})} .$$

The problem is then to determine whether, and in what sense the integral

$$\int_{\mathbb{T}} \hat{f}(x)\, E_n(\dot{x})\, d\dot{x}$$

represents f.

If $f \in L^1(\mathbb{R})$, we again define its Fourier transform by

$$\hat{f}(t) = \int_{\mathbb{R}} f(x) e^{-ixt} dm(x)$$

where the measure is the Lebesgue measure divided by $\sqrt{2\pi}$. The problem is then to determine whether and in what sense the integral

$$\int_{\mathbb{R}} \hat{f}(t)\, e^{ixt} dm(t)$$

represents f.

Over $\mathbb{R}$, the answer to the problem is that the integral represents f a.e. . Over $\mathbb{Z}$, the answer is that the integral represents f everywhere.

Although satisfactory answers to the central problem are available in the case of $\mathbb{T}$, we have not dealt with these. Instead we intended to illustrate the importance and usefulness of the so-called stationary operators on homogeneous spaces. Let $\mathcal{C}$ be the set of all complex valued functions on $\mathbb{T}$. A normed subspace $H \subset \mathcal{C}$ is said to be <u>homogeneous</u> if

H_1 : $f \in H$ implies $U_{\dot{h}} f$ $(\dot{x} \to f(\dot{x} + \dot{h}))$ is in H, for all $\dot{h} \in \mathbb{T}$.

$$H_2 : \|U_{\dot{h}} f\| = \|f\| , \text{ for all } \dot{h} \in \mathbb{T}.$$

$$H_3 : \lim_{\dot{h} \to \dot{h}_0} \|U_{\dot{h}} f - U_{\dot{h}_0} f\| = 0 \text{ for any } \dot{h}_0 \in \mathbb{T}.$$

Examples of homogeneous spaces include the space $L^p(\mathbb{T})$ for $1 \leq p < +\infty$. $L^\infty(\mathbb{T})$ is not a homogeneous space. An operator P mapping a homogeneous space H into itself is said to be <u>stationary</u> if

$$P(U_{\dot{h}} f) = U_{\dot{h}}(Pf)$$

for all $\dot{h} \in \mathbb{T}$.

Although we had intended to illustrate the importance of stationary operators on homogeneous, we are short of our original intention. However, we do give a complete characterization of continuous linear stationary operator on $L^2(\mathbb{T})$: there are in one - to - one corresponden with the bounded sequences of complex numbers. As concrete examples of continuous linear stationary operators on other familiar homogeneous spaces, we offer the convolution operators.